ON THE CONTINUITY OF THE TRACE OPERATOR IN $GSBV(\Omega)$ AND $GSBD(\Omega)$

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ABSTRACT. In this paper we present a new result of continuity for the trace operator acting on functions that might jump on a prescribed $(n-1)$-dimensional set Γ, with the only hypothesis of being rectifiable and of finite measure. We also show an application of our result in relation to the variational model of elasticity with cracks, when the associated minimum problems are coupled with Dirichlet and Neumann boundary conditions.

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1. INTRODUCTION

The space $SBV(\Omega)$ of special functions of bounded variation has been introduced to study the so called *free discontinuity problems* (see De Giorgi [7] and De Giorgi, Ambrosio [8] for the definition of such problems). It is composed of functions in $BV(\Omega)$ such that the singular part of their distributional gradient is concentrated on an $n-1$ dimensional set, called the jump set J_u. The prototype of the free discontinuity problems is the Mumford-Shah functional, whose definition for $u \in SBV(\Omega)$ is:

$$F(u) = \int_\Omega |\nabla u|^2 dx + \mathcal{H}^{n-1}(J_u) + \int_\Omega |u-g|^2 \; dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, ∇u denotes the density of the absolutely continuous part of the distributional gradient with respect to the n-dimensional Lebesgue measure $\mathcal{L}^n$, $\mathcal{H}^{n-1}$ indicates the $(n-1)$-dimensional Hausdorff measure, and g is some square integrable function. The study of some minimum problems for F, leads us to introduce the following subspace of $SBV(\Omega)$:

$$SBV_2^2(\Omega) := \{u \in SBV(\Omega) \mid u \in L^2(\Omega), \nabla u \in L^2(\Omega)\}.$$

When the function g appearing in (1.1) is bounded, a useful notion of convergence is the following:

$$\begin{cases} \sup_k \left(\|u_k\|_\infty + \|\nabla u_k\|_2 + \mathcal{H}^{n-1}(J_{u_k})\right) \le C \\ u_k \to u, \text{ in } L^1(\Omega) \\ \nabla u_k \rightharpoonup \nabla u, \text{ weakly in } L^1(\Omega, \mathbb{R}^n) \end{cases} \tag{1.2}$$

Indeed some compactness theorems (see for example [2, Theorem 4.8]) can be applied to obtain the convergence (in the sense of (1.2)) of suitable minimizing sequences in the Mumford-Shah minimization problems.

If we would like to consider some minimum problems of the Mumford-Shah with prescribed Dirichlet boundary condition, we have to study the behavior of the trace operator in the SBV context. When Ω is regular enough, the trace operator $Tr: BV(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1})$ is well defined, and continuous with respect to the strong topology in $BV(\Omega)$. Unfortunately, if we consider the space $SBV_2^2(\Omega) \subset BV(\Omega)$ then:

$$Tr: SBV_2^2(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1}),$$

is *not continuous* with respect to the convergence requirements in (1.2). This lack of continuity is due to the fact that a sequence in $SBV(\Omega)$ may have jump sets getting infinitesimally close to the boundary of Ω. Having this in mind, one can easily produce counterexamples to continuity

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which lead to a free discontinuity problem with no solution. For example, if one consider the Mumford-Shah functional in one dimension with Dirichlet boundary condition:

$$\min_{\substack{u\in SBV_2^2((0,1))\\ u(0)=\lambda}} \int_0^1 |u'|^2\, dx + \mathcal{H}^0(J_u) + \int_0^1 |u|^2\, dx, \tag{1.3}$$

it is easy to see that for sufficiently large value of λ, any admissible function pays strictly more then 1 in (1.3), while there exists a minimizing sequence for which the functional (1.3) converges to 1 in the limit.

To bypass this problem, it seems convenient to fix an $(n-1)$-dimensional set Γ, and to study the trace properties of functions whose jump sets are contained in Γ. So we introduce:

$$SBV(\Omega;\Gamma) := \{u \in SBV(\Omega) \mid J_u \subseteq \Gamma\},$$

and

$$SBV_p^p(\Omega;\Gamma) := \{u \in SBV(\Omega;\Gamma) \mid u \in L^p(\Omega), \nabla u \in L^p(\Omega)\},\ (p \geq 1).$$

More in generally in this paper we study the properties of the trace operator when $u \in GSBV(\Omega)$. These are all the $\mathcal{L}^n$-measurable functions such that at any level of truncation, the truncated functions belong to $SBV_{loc}(\Omega)$. This space has been introduced to guarantee existence of a solution to minimum problems which implies no bounds on the L^∞-norms of the minimizing sequences; for example when the function g appearing in (1.1) is only in $L^2(\Omega)$.

We can define the following spaces:

$$GSBV_p^p(\Omega) := \left\{u \in GSBV(\Omega) \mid u \in L^p(\Omega), \nabla u \in L^p(\Omega)\right\},\ (p \geq 1),$$

endowed with the following notion of convergence:

$$\begin{cases} \sup_k \left(\|u_k\|_p + \|\nabla u_k\|_p + \mathcal{H}^{n-1}(J_{u_k})\right) \leq C \\ u_k \to u, \text{ in } L^1(\Omega) \\ \nabla u_k \rightharpoonup \nabla u, \text{ weakly in } L^1(\Omega). \end{cases} \tag{1.4}$$

Note that the bound in the first line of (1.4), when $p > 1$, ensures compactness with respect to this notion of convergence. As it is shown for example in [2, Definition 4.30], in $GSBV(\Omega)$ there is still a notion of trace and of jump set J_u, that can be defined through the notion of *approximate limit*. As before, since we have no control on the distance of J_{u_k} from the boundary $\partial\Omega$, there are still no continuity results for the trace in $GSBV_q^p(\Omega)$ under conditions (1.4).

Then also in this case it seems convenient to study the properties of the trace in the space of functions that jumps on a prescribed $(n-1)$-dimensional set:

$$GSBV_p^p(\Omega;\Gamma) := \{u \in GSBV_p^p(\Omega) \mid Ju \subseteq \Gamma\},\ (p \geq 1).$$

The main results of the paper is that there exists a function Θ such that:

$$Tr: GSBV_p^p(\Omega;\Gamma) \to L^q(\partial\Omega, \Theta\mathcal{H}^{n-1}),\ \ (p > 1), \tag{1.5}$$

is continuous for every $1 \leq q < p$ when we consider the strong topology on $L^q(\partial\Omega, \Theta\mathcal{H}^{n-1})$, and also for $q = p$ when we consider the weak topology on $L^q(\partial\Omega, \Theta\mathcal{H}^{n-1})$. Θ is a weight function that depends *only on the geometry* of Γ *and is* $\mathcal{H}^{n-1}$*-a.e. strictly positive* (see theorem 5.1 and remark 5.2). We have also showed that $q = p$ cannot be reached in (1.5) when one considers the strong topology, by exhibiting a counterexample.

When Γ is a compact subset of Ω then $GSBV_p^p(\Omega;\Gamma)$ is equivalent to the Sobolev space $W^{1,p}(\Omega \setminus \Gamma)$. Moreover, if Γ is regular enough, the Sobolev embedding holds and in particular $u \in L^{p^*}(\Omega)$. If Γ is not regular, we cannot deduce that $u \in GSBV_p^p(\Omega;\Gamma)$ implies $u \in L^{p^*}(\Omega)$, but if we assume $u \in L^{p^*}(\Omega)$, then we can improve our summability results on q appearing in (1.5), and say that the trace operator is continuous:

$$Tr: GSBV_p^p(\Omega;\Gamma) \cap L^{p*}(\Omega) \to L^q(\partial\Omega, \Theta\mathcal{H}^{n-1})\ (p > 1),$$

for every $1 \leq q < p(n-1)/(n-p)$ when we consider the strong topology on $L^q(\partial\Omega, \Theta\mathcal{H}^{n-1})$, and also for $q = p(n-1)/(n-p)$ when we consider the weak topology on $L^q(\partial\Omega, \Theta\mathcal{H}^{n-1})$. Notice that $p(n-1)/(n-p)$ is the usual critical exponent for the trace of Sobolev functions in $W^{1,p}(\Omega)$.

Looking at the definition of Θ, it is easy to see that when $\overline{\Gamma} \subset\subset \Omega$ then $\Theta \geq \operatorname{dist}(\Gamma, \partial\Omega) > 0$. In the paper we give a finer property for Γ, that is an adaptation of the classical *cone condition*, in such a way to guarantee that $\operatorname{ess\,inf}_{\partial\Omega}\Theta > 0$, and to deduce the classical continuity properties of the trace without the use of weights (see Proposition 3.15 and Remark 3.16).

An alternative way to obtain a trace estimate without weight on $\partial\Omega$ is to consider a suitable weight Ψ defined on Ω. More precisely we have proved that there exists Ψ such that, if in addition to the convergence conditions in (1.4) we add the uniform bound on the $L^p(\Omega,\psi\mathcal{L}^n)$ norm, we have the continuity:

$$Tr\colon GSBV_p^p(\Omega;\Gamma)\cap L^p(\Omega,\Psi\mathcal{L}^n)\to L^q(\partial\Omega,\mathcal{H}^{n-1})\ (p>1), \tag{1.6}$$

for $1\le q<p$ if we consider the strong topology on $L^q(\partial\Omega,\mathcal{H}^{n-1})$, and also for $q=p$ if we consider the weak topology on $L^q(\partial\Omega,\mathcal{H}^{n-1})$; here Ψ is a weight function defined on Ω, locally integrable, and that depends only on the geometry of Γ (see Theorem 5.1 and Remark 5.2). A refined version of this result allows us to prove the following inclusions (see Theorem 3.17 and Remark 3.19):

$$GSBV_p^p(\Omega;\Gamma)\cap L^p(\Omega,\Psi\mathcal{L}^n)\subset SBV_p^p(\Omega;\Gamma),\ (p>1), \tag{1.7}$$

which can be considered as an improvement of the obvious inclusions $GSBV_p^p(\Omega;\Gamma)\cap L^\infty(\Omega)\subset SBV_p^p(\Omega;\Gamma)$.

All the results mentioned above are true in the context of vector fields having bounded deformation $BD(\Omega)$, and moreover, not only for the trace of u on the boundary of Ω, but also for both traces $u^\pm$ on Γ. Since the proofs in this context present more technical difficulties, we decide to prove our theorems with all the details in this case. Actually we deal with $GSBD(\Omega)$, the space of generalized special vector fields having *bounded deformation*. This space has been introduced in Dal Maso [4] to solve some variational problems coming from the theory of linearly elastic fracture mechanics, and is a generalization of $SBD(\Omega)$, the space of special vector fields of bounded deformation. At this point we would like to mention only that $SBD(\Omega)$ are the integrable vector fields such that the singular part of their *symmetric distributional gradients* Eu, as measure, are concentrated on an $n-1$ rectifiable set Ju:

$$Eu=\mathcal{E}u\,\mathcal{L}^n+([u]\odot\nu)\mathcal{H}^{n-1}\llcorner J_u,$$

where $\mathcal{E}u$ is the density of the absolutely continuous part of Eu with respect to the n-dimensional lebesgue measure $\mathcal{L}^n$, and $[u]\odot\nu$ denotes the symmetric tensor product between the jump $[u]=u^+-u^-$ of u and the orientation ν. For $BD(\Omega)$ we refer to Temam [14] for its functional properties and to Ambrosio, Coscia, Dal Maso [1] for the fine properties of BD functions.
The reason why we studied the trace operator in these spaces (more precisely in $GSBD_p^p(\Omega;\Gamma)$), comes from the theory of elasticity with cracks, when we consider a traction applied to some part of the boundary $\partial_N\Omega\subseteq\partial\Omega$. This leads to a linear term of the form:

$$\int_{\partial_N\Omega}F\cdot u\;d\mathcal{H}^{n-1},$$

in the weak formulation of the problem, where F represents the traction force acting on the Neumann part of the boundary. Hence asking about the continuity of this linear form is equivalent to ask about the continuity of the trace operator acting on all the admissible $u\in GSBD_2^2(\Omega;\Gamma)$. In the last section of this paper we propose a way to solve this problem: the idea is to restrict our attention among all the traction forces F, living in the dual of the Hilbert space $L^2(\partial_N\Omega,\Theta\mathcal{H}^{n-1})$.

In the literature, the problem of the integrability of the trace in $BV(\Omega)$ has been studied for example by Maz'ja in [12, Chapter 6], where the trace was defined for open set Ω of finite perimeter. The main results were obtained under the assumption of connectedness of Ω and that normals in the sense of Federer exist almost everywhere on the boundary.Then generalized to the class of open and connected sets Ω with the only hypothesis that its topological boundary is an $n-1$ rectifiable set, by Burago, Kosovski in [3]. Both works rely on the fact that for u the *Coarea Formula* holds true, and so the distributional gradient of u, as measure, can be reconstructed by averaging the perimeter of each level sets of u. In this case, under some *more* regularity conditions on the boundary, one can control the L^1 norm of the trace of u with the *full* norm in BV times a constant that depends only on Ω (see [12, Section 6.6.4.]).

In Temam [15] some continuity properties of the trace operator are studied in the space $BD(\Omega)$, with $\Omega\subset\mathbb{R}^n$ open set with smooth boundary. Here $BD(\Omega)$ is endowed with the norm given by the *total variation* of the symmetric distributional derivative. In this case, he introduces a notion of convergence, where morally our hypothesis of fixing the jump sets of some sequences $(u_k)_k \subset GSBD(\Omega)$, is substituted by asking that the total variation of the symmetric distributional gradient $\|Eu_k\|(\Omega)$ converges to the total variation $\|Eu\|(\Omega)$ of the limit u. Under this notion of convergence, it is possible to show the continuity of the trace in $L^1(\partial\Omega, \mathcal{H}^{n-1})$.

To make a parallel with the papers mentioned above, we have to notice that our results hold true in particular in the SBD and SBV cases. We work with notion of convergence that do not take care of the jump part of the total variation measure $|u^+ - u^-| \cdot \mathcal{H}^{n-1} \llcorner J_u$, while we fix a jump set Γ. On one side this leads us to introduce proper weights in order to have continuity results of the trace, but on the other side we do not make any regularity assumptions on Γ neither on Ω (except to be respectively $n-1$-rectifiable with finite $\mathcal{H}^{n-1}$-measure, and to be an open set of finite perimeter). Moreover, we can develop a theory in the SBD (even $GSBD$) context, where any kind of Coarea formula seems not to be true.

2. Notation and results in $GBD(\Omega)$

For the space $GBD(\Omega)$ we always refer to the seminal paper [4]. For convenience of the reader we will recall some useful notations and results.

For every $\xi \in \mathbb{S}^{n-1} = \{\xi \in \mathbb{R}^n \mid |\xi| = 1\}$ let $\Pi^\xi := \{y \in \mathbb{R}^n \mid y \cdot \xi = 0\}$ to be the hyperplane orthogonal to ξ passing through the origin, and let $\pi^\xi : \mathbb{R}^n \to \Pi^\xi$ be the orthogonal projection. For every set $B \subset \mathbb{R}^n$ and for every $y \in \Pi^\xi$ we define

$$B^\xi_y := \{t \in \mathbb{R} \mid y + t\xi \in B\}.$$

Moreover, for every function $u : B \to \mathbb{R}^n$ we define the function $\hat{u}^\xi_y : B^\xi_y \to \mathbb{R}$ by

$$\hat{u}^\xi_y := u(y + t\xi) \cdot \xi.$$

If $u : B \to \mathbb{R}^n$ is $\mathcal{L}^n$-measurable, for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the jump set of $\hat{u}^\xi_y$ is denoted by $J_{\hat{u}^\xi_y}$. Moreover we set

$$J^1_{\hat{u}^\xi_y} := \{t \in J_{\hat{u}^\xi_y} \mid |(\hat{u}^\xi_y)^+(t) - (\hat{u}^\xi_y)^-(t)| \geq 1\},$$

where $(\hat{u}^\xi_y)^-(t)$ and $(\hat{u}^\xi_y)^+(t)$ are the approximate right and left limits of $\hat{u}^\xi_y$ at t.

If μ is a Borel measure on a Borel set $E \subset \mathbb{R}^n$, its total variation is denoted by $|\mu|$. If $A \subset E$ is a Borel set, the Borel measure $\mu \llcorner A$ is defined by $(\mu \llcorner A)(B) := \mu(A \cap B)$ for every Borel set $B \subset E$.

If $U \subset \mathbb{R}^n$ is an open set, $\mathcal{M}(U)$ is the space of all Radon measures on U, $\mathcal{M}_b(U) := \{\mu \in \mathcal{M}(U) \mid |\mu|(U) < +\infty\}$ is the space of all bounded Radon measures on U, and $\mathcal{M}^+_b(U) := \{\mu \in \mathcal{M}_b(U) \mid \mu(B) \geq 0$ for every Borel set $B \subset U\}$ is the space of all non negative bounded Radon measures on U.

Definition 2.1. Let A be an $\mathcal{L}^n$-measurable subset of $\mathbb{R}^n$, let $v : A \to \mathbb{R}^m$ be an $\mathcal{L}^n$-measurable function, let $x \in \mathbb{R}^n$ be such that

$$\limsup_{\rho \to 0^+} \frac{\mathcal{L}^n(A \cap B_\rho(x))}{\rho^n} > 0,$$

and let $a \in \mathbb{R}^m$. We say that a is the *approximate limit* of v as $y \to x$, and write

$$\operatorname{ap}\lim_{y \to x} v(y) = a \tag{2.1}$$

if

$$\lim_{\rho \to 0^+} \frac{\mathcal{L}^n(\{y \in A \cap B_\rho(x) \mid |v(y) - a| > \epsilon\})}{\rho^n} = 0 \tag{2.2}$$

for every $\epsilon > 0$.

Remark 2.2. Let A, v, x and a be as in the previous definition, and let ψ be a homeomorphism between $\mathbb{R}^m$ and a bounded open subset of $\mathbb{R}^m$. It is easy to prove that (2.1) holds if and only if

$$\lim_{\rho \to 0^+} \frac{1}{\rho^n} \int_{A \cap B_\rho(x)} |\psi(v(y)) - \psi(a)| dy = 0.$$

In particular if v is $\mathcal{L}^n$-measurable , then $\mathcal{L}^n$-a.e. v admits an approximate limit.

Definition 2.3. Let U be an open set of $\mathbb{R}^n$. For every $\mathcal{L}^n$-measurable function $v : U \to \mathbb{R}^m$ we define the *approximate continuity set* as the set of points $x \in U$ for which there exists $a \in \mathbb{R}^m$ such that

$$\operatorname{ap}\lim_{y\to x} v(y) = a.$$

The vector a is uniquely determined and is denoted by $\tilde{v}(x)$. The *approximate discontinuity* set S_v is defined as the complement in U of the approximate continuity set.

Definition 2.4. Let U be an open subset of $\mathbb{R}^n$. For every $\mathcal{L}^n$-measurable function $v : U \to \mathbb{R}^m$ we define the *approximate jump set* J_v as the set of point $x \in U$ for which there exist $a, b \in \mathbb{R}^m$ with $a \neq b$, and $\nu \in \mathbb{S}^{n-1}$ such that

$$\operatorname{ap}\lim_{\substack{(y-x)\cdot\nu>0\\ y\to x}} v(y) = a \text{ and } \operatorname{ap}\lim_{\substack{(y-x)\cdot\nu<0\\ y\to x}} v(y) = b. \tag{2.3}$$

The triplet (a, b, ν) is uniquely determined up to a permutation of (a, b) and a change of sign of ν and is denoted by $(v^+(x), v^-(x),\ \nu_v(x))$. The *jump* of v is the function $[v] : J_v \to \mathbb{R}^m$ defined by $[v](x) := v^+(x) - v^-(x)$ for every $x \in J_v$. Finally we define

$$J_v^1 := \{x \mid |[v](x)\mid \ \geq 1\}. \tag{2.4}$$

Remark 2.5. By [4, Proposition 2.6] we have that S_v, J_v and J_v^1 are Borel sets and $\tilde{v} : U \backslash S_v \to \mathbb{R}^m$, defined as $\tilde{v}(x) = \operatorname{ap}\lim_{y\to x} v(y)$, is a Borel function.

Moreover, for every $x \in J_v$, we can choose the sign of $\nu(x)$ in such a way that $v^+ : J_v \to \mathbb{R}^m, v^- : J_v \to \mathbb{R}^m$, and $\nu_v : J_v \to \mathbb{S}^{n-1}$ are Borel functions.

Definition 2.6. We define $\mathcal{T}$ as the space of all functions τ of class C^1, defined on the real line $\mathbb{R}$, such that $-\frac{1}{2} < \tau < \frac{1}{2}$ and with bounded derivative $|\tau'| < 1$.

Following [4, Definition 4.1], we are now in position to define the space $GBD(\Omega)$. In what follows Ω is an open set of $\mathbb{R}^n$.

Definition 2.7. The space $GBD(\Omega)$ of *generalised functions of bounded deformation* is the space of all $\mathcal{L}^n$-measurable functions $u : \Omega \to \mathbb{R}^n$ with the following property: there exists $\lambda \in \mathcal{M}_b^+(\Omega)$ such that the following equivalent (see [4, Theorem 3.5]) conditions hold for every $\xi \in \mathbb{S}^{n-1}$:

(a) for every $\tau \in \mathcal{T}$ the partial derivative $D_\xi(\tau(u\cdot\xi))$ belongs to $\mathcal{M}_b(\Omega)$ and its total variation satisfies

$$|D_\xi(\tau(u\cdot\xi))|(B) \leq \lambda(B) \tag{2.5}$$

for every Borel set $B \subset \Omega$;

(b) for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the function $\hat{u}_y^\xi$ belongs to $BV_{loc}(\Omega_y^\xi)$ and

$$\int_{\Pi^\xi} (|D\hat{u}_y^\xi|(B_y^\xi \smallsetminus J^1_{\hat{u}_y^\xi}) + \mathcal{H}^0(B_y^\xi \cap J^1_{\hat{u}_y^\xi}))d\mathcal{H}^{n-1}(y) \leq \lambda(B) \tag{2.6}$$

for every Borel set $B \subset \Omega$.

Remark 2.8. Following [4, Definition 4.16] and [4, Proposition 4.17], for every $u \in GBD(\Omega)$, there exists a measure $\hat{\mu}_u \in \mathcal{M}_b^+$ that it is the smallest measure λ that satisfies (a) and (b) of the previous definition.

Definition 2.9. The space $GSBD(\Omega)$ of *generalised function of bounded deformation* is the set of functions $u \in GBD(\Omega)$ such that for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ the function $\hat{u}_y^\xi$ belongs to $SBV_{loc}(\Omega_y^\xi)$

Remark 2.10. The spaces $GBD(\Omega)$ and $GSBD(\Omega)$ are actually vector spaces (see [4, Remark 4.6]).

Now we want to recall some results about the space $GBD(\Omega)$. Let us start with the trace on regular submanifold.

Theorem 2.11. *(Traces on regular submanifold) Let $u \in GBD(\Omega)$ and let $M \subset \Omega$ be a C^1 submanifold of dimension $n-1$ with unit normal ν. Then for $\mathcal{H}^{n-1}$ -a.e. $x \in M$ there exist $u_M^+(x)$, $u_M^-(x) \in \mathbb{R}^n$ such that*

$$\operatorname{ap}\lim_{\substack{\pm(y-x)\cdot\nu(x)>0\\ y\to x}} u(y) = u_M^\pm(x). \tag{2.7}$$

Moreover for every $\xi \in \mathbb{S}^{n-1}$ *and for* $\mathcal{H}^{n-1}$*-a.e.* $y \in \Pi^\xi$ *we have*

$$u_M^{\pm}(y+t\xi)\cdot\xi = \underset{\substack{\sigma_y^\xi(t)(s-t)>0\\ s\to t}}{\text{ap lim}}\ \hat{u}_y^\xi(s) \text{ for every } t \in M_y^\xi, \tag{2.8}$$

where $\sigma : M \to \{-1,+1\}$ *is defined by* $\sigma(x) := sign(\xi\cdot\nu(x))$*. Finally, the functions* $u_M^{\pm} : M \to \mathbb{R}^n$ *are* $\mathcal{H}^{n-1}$*-measurable.*

Proof. See [4, Theorem 5.2] for a detailed proof. □

Definition 2.12. Let $u \in GBD(\Omega)$ and let $M \subset \Omega$ be a C^1-manifold of dimension $n-1$ oriented by ν. The $\mathbb{R}^n$-valued $\mathcal{H}^{n-1}$-measurable functions u_M^+ and u_M^-, defined $\mathcal{H}^{n-1}$-a.e. on M and satisfying (2.7) , are called the *traces* of u on the two sides of M.

Just for convenience of the reader we recall the definition of rectifiable set.

Definition 2.13. We say that $\Gamma \subset \mathbb{R}^n$ is a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set (according to [10, Definition 3.54]) if Γ is $\mathcal{H}^{n-1}$-measurable and

$$\Gamma \subseteq \bigcup_{i=1}^{\infty}\Gamma_i \cup \Gamma_0, \tag{2.9}$$

where $\mathcal{H}^{n-1}(\Gamma_0) = 0$, and there exists a sequence of lipschitz functions $(f_i)_{i=1}^\infty$ such that $\Gamma_i \subseteq f_i(\mathbb{R}^n)$ for each $i \geq 1$.

The following proposition will be useful later on.

Proposition 2.14. $\Gamma \subset \mathbb{R}^n$ *is countably* $(\mathcal{H}^{n-1}, n-1)$*-rectifiable set if and only if there exists a sequence of bounded open sets of finite perimeter* $(U_i)_{i=1}^\infty$ *such that*

$$\mathcal{H}^{n-1}(\Gamma \smallsetminus \bigcup_{i=1}^{\infty}\mathcal{F}U_i) = 0, \tag{2.10}$$

where $\mathcal{F}U_i$ *denotes the reduced boundary of* U_i *(see* [2, Definition 3.54]*).*

Proof. Using [10, Theorem 3.2.29] we know that $\mathcal{H}^{n-1}$ almost all of Γ is contained in a countably union of $(n-1)$-submanifold of $\mathbb{R}^n$ of class C^1. So we can reduce ourselves to prove the statement for a single $(n-1)$-submanifold M of class C^1; moreover by basic fact about differential geometry we have that M can be covered by countably many graphs of maps from $\mathbb{R}^{n-1}$ to $\mathbb{R}$ of class C^1. So for our purpose it is enough to prove the proposition for a $(n-1)$-submanifold of the form $M \subseteq \text{graph}(f)$ where $f \in C^1(\mathbb{R}^{n-1})$.

To prove this last assertion we can consider a countable measurable partition of $\mathbb{R}^{n-1}$ made for example by open cubes $(Q_i)_{i=0}^\infty$. For every $i \in \mathbb{N}$, up to a translation on M, we may assume that $\inf_{Q_i} f > 0$. Finally we define:

$$U_i := \{(y,t) \mid y \in Q_i,\ 0 < t < f(y)\}.$$

Clearly each U_i is an open set of finite perimeter such that:

$$\text{graph}(f \llcorner Q_i) \subset \mathcal{F}U_i,$$

and

$$\mathcal{H}^{n-1}(M \smallsetminus \bigcup_i \mathcal{F}U_i) \leq \mathcal{H}^{n-1}\big(M \smallsetminus \bigcup_i \text{graph}(f \llcorner Q_i)\big) = 0.$$

□

Definition 2.15. (Orientation) Let $\Gamma \subset \mathbb{R}^n$ be a countably $(\mathcal{H}^{n-1},\ n-1)$-rectifiable set. We call an *orientation* of Γ any map $\nu : \Gamma \to \mathbb{S}^{n-1}$ which is $\mathcal{H}^{n-1}$-measurable and such that $\nu(x)$ is orthogonal to the tangent space of Γ at x for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$.

Here we recall a fundamental theorem about the jump set of a $GBD(\Omega)$ function (see [4, Theorem, 8.1]). In particular this result tells us that the jump set can be reconstructed by the jump points of the one dimensional slices.

Theorem 2.16. *(Slicing of the jump set). Let* $u \in GBD(\Omega)$ *, then* J_u *is a countably* $(\mathcal{H}^{n-1}, n-1)$*-rectifiable set. Moreover let* $\xi \in \mathbb{S}^{n-1}$ *and let*

$$J_y^\xi := \{x \in J_u \mid (u^+(x) - u^-(x))\cdot\xi \neq 0\}. \tag{2.11}$$

Then for $\mathcal{H}^{n-1}$*-a.e.* $y \in \Pi^\xi$ *we have*

$$(J_u^\xi)_y^\xi = J_{\hat{u}_y^\xi}, \tag{2.12}$$

$$u^\pm(y+t\xi)\cdot\xi = (\hat{u}_y^\xi)^{\pm\sigma_y^\xi(t)}(t) \text{ for every } t \in (J_u)_y^\xi, \tag{2.13}$$

where $\sigma : M \to \{-1,+1\}$ *is defined by* $\sigma(x) := \operatorname{sign}(\xi\cdot\nu_u(x))$*, and* $\nu_{\hat{u}_y^\xi} = 1$.

Remark 2.17 (Integrable jump implies BD)*.* The previous theorem says that the jump set J_u can be reconstructed through the jump points of the one-dimensional restriction $J_{\hat{u}_y^\xi}$ for every direction ξ in $\mathbb{S}^{n-1}$. In particular if $u \in GBD(\Omega)$ has integrable jump, i.e. $[u] \in L^1(J_u, \mathcal{H}^{n-1})$, then u is actually a function in $BD(\Omega)$. Indeed, by definition of $BD(\Omega)$ (see [1]), we need only to check that for every $\xi \in \mathbb{S}^{n-1}$:

$$\int_{\Pi^\xi} |D\hat{u}_y^\xi|(\Omega_y^\xi)\ d\mathcal{H}^{n-1}(y) < \infty.$$

But relation (2.13) implies in particular that $[u\cdot\xi](y+t\xi) = [\hat{u}_y^\xi](t)$ for every $t \in J_{\hat{u}_y^\xi}$ and $\mathcal{H}^{n-1}$-a.e. y, so that we can write:

$$\begin{aligned}\int_{\Pi^\xi} |D\hat{u}_y^\xi|(\Omega_y^\xi)\ d\mathcal{H}^{n-1}(y) &\le \int_{\Pi^\xi} |D\hat{u}_y^\xi|(\Omega_y^\xi \setminus J_{\hat{u}_y^\xi}) + \sum_{t\in J_{\hat{u}_y^\xi}} |[u\cdot\xi](y+t\xi)|\ d\mathcal{H}^{n-1}(y)\\ &\le \lambda(\Omega\setminus J_u) + \int_{J_u} |[u]|\ d\mathcal{H}^{n-1},\end{aligned}$$

and we are done.

Every $u \in GBD(\Omega)$ admits an *approximate symmetric gradient* $\mathcal{E}u$ $\mathcal{L}^n$-almost everywhere, which is a map $\mathcal{E}u : \Omega \to \mathbb{M}^{n\times n}_{sym}$ such that

$$\operatorname*{ap\,lim}_{y\to x} \frac{\big(u(y)-u(x)-\mathcal{E}u(x)(y-x)\big)\cdot(y-x)}{|y-x|^2} = 0. \tag{2.14}$$

Formula (2.14) says that the approximate symmetric gradient is unique. The following theorem proves that $\mathcal{E}u$ is an L^1-function.

Theorem 2.18. *Let* $u \in GBD(\Omega)$ *. Then there exists a function* $\mathcal{E}u \in L^1(\Omega;\mathbb{M}^{n\times n}_{sym})$ *such that* (2.14) *holds for* $\mathcal{L}^n$*-a.e.* $x \in \Omega$*. Moreover for every* $\xi \in \mathbb{R}^n \setminus \{0\}$ *and for* $\mathcal{H}^{n-1}$*−a.e.* $y \in \Pi^\xi$ *we have (see* [4, Theorem, 9.1]*)*

$$(\mathcal{E}u)_y^\xi \xi\cdot\xi = \nabla\hat{u}_y^\xi, \tag{2.15}$$

$\mathcal{L}^1$*-a.e. on* Ω_y^ξ.

When Ω is an open set of finite perimeter and $u \in GBD(\Omega)$, it is possible to extend u to a vector field defined on the whole of $\mathbb{R}^n$ which belongs to $GBD(\mathbb{R}^n)$. Before doing this, we need the following proposition concerning an extension property of BV functions in one variable:

Proposition 2.19. *Let* $E = \bigcup_{k=1}^M I_k$ *where* $I_k = (a_k,\ b_k) \subset \mathbb{R}$ *are open intervals (possibly unbounded) and pairwise disjoints. If* $u \in BV(E)$ *then the function defined by:*

$$v(t) := \begin{cases} u(t) & \text{if } t \in E,\\ 0 & \text{otherwise.}\end{cases}$$

belongs to $BV(\mathbb{R})$*. Moreover*

$$Dv = \sum_{k=0}^M (u^-(b_k)\delta_{b_k} - u^+(a_k)\delta_{a_k}) + Du(E), \tag{2.16}$$

where $\delta_{(\cdot)}$ *denotes the Dirac's delta, and*

$$|Dv| = \sum_{k=0}^M (|u^-(b_k)| + |u^+(a_k)|) + |Du|(E). \tag{2.17}$$

Proof. It is a simple application of the theory of BV functions in one variable. ☐

Proposition 2.20. *(Extension of GBD functions) Let $\Omega \subset \mathbb{R}^n$ be an open set of finite perimeter (see* [2, Definition 3.35]*) and let $u \in GBD(\Omega)$. If we define:*

$$\underline{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{otherwise,} \end{cases}$$

then $\underline{u} \in GBD(\mathbb{R}^n)$. Moreover if we denote the reduced boundary of Ω as $\mathcal{F}\Omega$, we have:

(a) $J_{\underline{u}} \subset J_u \cup \mathcal{F}\Omega$;

(b) for every Borel set $B \subset \mathbb{R}^n$ and every $\xi \in \mathbb{S}^{n-1}$ the following inequality holds true:

$$\int_{\Pi^\xi} \left(|D\hat{\underline{u}}^\xi_y|(B^\xi_y \setminus J^1_{\hat{\underline{u}}^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{\hat{\underline{u}}^\xi_y}) \right) d\mathcal{H}^{n-1}(y) \le \hat{\mu}_u(B) + \mathcal{H}^{n-1}(\mathcal{F}\Omega \cap B) \ , \tag{2.18}$$

where $\hat{\mu}_u$ is the smallest measure relative to u that satisfies conditions (2.5) *and* (2.6) *(see Remark 2.8);*

(c) the approximate symmetric gradient of $\underline{u}$ is such that:

$$\mathcal{E}\underline{u}(x) = \begin{cases} \mathcal{E}u(x) & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{2.19}$$

(d) if $u \in GSBD(\Omega)$ then $\underline{u} \in GSBD(\mathbb{R}^n)$.

Proof. First we show that (2.5) holds true. Fix $\xi \in \mathbb{S}^{n-1}$ and $\tau \in \mathcal{T}$. By [2, Theorem 3.103] we have

$$|D_\xi \tau(\underline{u} \cdot \xi)|(B) = \int_{\Pi^\xi} |D(\tau(\underline{u} \cdot \xi)^\xi_y)|(B^\xi_y) dy, \tag{2.20}$$

for any Borel set $B \subset \mathbb{R}^n$, and

$$\int_{\Pi\xi} |D\mathbb{1}_{\Omega^\xi_y}|(\mathbb{R}) \ dy = |D_\xi \mathbb{1}_\Omega|(\mathbb{R}^n) \le \mathcal{H}^{n-1}(\mathcal{F}\Omega) < \infty. \tag{2.21}$$

It follows that for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, Ω^ξ_y has finite perimeter. By the characterization of sets of finite perimeter in $\mathbb{R}$, we know that for those $y \in \Pi^\xi$, Ω^ξ_y is equivalent to a finite union of open pairwise disjoint intervals. Notice that

$$\tau(\underline{u} \cdot \xi) = \tau(u \cdot \xi)\mathbb{1}_\Omega + \tau(0)\mathbb{1}_{\Omega^c}. \tag{2.22}$$

Now for each $y \in \Pi^\xi$ such that $|D\mathbb{1}_{\Omega^\xi_y}| < \infty$, we can apply Proposition 2.19 to the one dimensional sections $t \mapsto \tau(\hat{u}^\xi_y)\mathbb{1}_{\Omega^\xi_y} + \tau(0)\mathbb{1}_{(\Omega^c)^\xi_y}$, and by using also (2.20) and the Coarea formula we have that:

$$\begin{aligned} |D_\xi \tau(\underline{u} \cdot \xi)|(B) &= \int_{\Pi^\xi} |D\big(\tau(\hat{u}^\xi_y)\mathbb{1}_{\Omega^\xi_y} + \tau(0)\mathbb{1}_{(\Omega^c)^\xi_y}\big)|(B^\xi_y) \ dy \\ &\le \int_{\Pi^\xi} \left(|D\tau(\hat{u}^\xi_y)|(\Omega^\xi_y \cap B^\xi_y) + \sum_{t \in \mathcal{F}\Omega^\xi_y \cap B^\xi_y} |\tau(\hat{u}^\xi_y(t))^{\sigma^\xi_y(t)} - \tau(0)| \right) dy \\ &\le |D_\xi \tau(u \cdot \xi)|(B \cap \Omega) + \mathcal{H}^{n-1}(\mathcal{F}\Omega \cap B) \\ &\le \hat{\mu}_u(B \cap \Omega) + \mathcal{H}^{n-1}(\mathcal{F}\Omega \cap B), \end{aligned} \tag{2.23}$$

for every Borel set $B \subseteq \mathbb{R}^n$, where $\sigma(x) = \operatorname{sign}(\nu_{\mathcal{F}\Omega}(x) \cdot \xi)$ and $\nu_{\mathcal{F}\Omega}$ denotes the measure theoretic inner unit normal. Let $\eta := \hat{\mu}_u + \mathcal{H}^{n-1} \llcorner \mathcal{F}\Omega$ then

$$|D_\xi \tau(\mathbb{1}_\Omega u \cdot \xi)|(B) \le \eta(B), \tag{2.24}$$

for every $\tau \in \mathcal{T}$ and for every $\xi \in \mathbb{S}^{n-1}$. This is exactly (2.5), and we deduce that $\underline{u} \in GBD(\mathbb{R}^n)$.

Point (a) can be deduced simply by Theorem 2.16.

To show estimate (2.18) it is enough to notice that the two definitions of $GBD(\Omega)$ are equivalent (see Definition 2.7).

Point (c) follows from the characterization of the symmetric approximate gradient given by the formula (2.14).

Finally (d) follows from Proposition 2.19 using the same argument as above. □

Remark 2.21. Under the assumptions of Proposition 2.20 let $\mathcal{F}\Omega$ be oriented by its measure theoretic inner unit normal. Then the extended function $\underline{u}$ of the previous proposition, is such that $\underline{u}^- = 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{F}\Omega$. Roughly speaking, $\underline{u}$ has almost everywhere zero trace from the complement of Ω. Indeed we can consider a finite measurable partition of $\mathcal{F}\Omega$, say $(\Sigma_i)_{i=1}^N$.

To each Σ_i there exists an orthonormal basis of $\mathbb{R}^n$ $\{\xi_1, \dots, \xi_n\}$ such that $\nu(x) \cdot \xi_i \neq 0$ for every $x \in \Sigma_i$ and for every $i = 1, \dots, n$ (see Remark 3.6). If we call $\sigma(x) = \text{sign}(\nu_{\mathcal{F}\Omega}(x) \cdot \xi)$, it is easy to see that for any $i = 1, \dots, n$, it holds

$$(\underline{\hat{u}}_y^{\xi_i})^{-\sigma_y^\xi(t)}(t) = 0, \text{ for every } t \in J_{\underline{\hat{u}}_y^{\xi_i}}, \text{ and for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^{\xi_i}. \tag{2.25}$$

Since $\mathcal{F}\Omega$ can be covered by countably many submanifold of dimension $(n-1)$ and class C^1, using Theorem 2.11 and $\nu(x) \cdot \xi_i \neq 0$, we can conclude

$$\underline{u}_{\mathcal{F}\Omega}^-(x) = 0 \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Sigma_i. \tag{2.26}$$

Because of the fact that $(\Sigma_i)_{i=1}^N$ is a measurable partition of $\mathcal{F}\Omega$ we have

$$\underline{u}_{\mathcal{F}\Omega}^-(x) = 0 \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \mathcal{F}\Omega, \tag{2.27}$$

which is the desired result.

3. Integrability of the trace in $GBD(\Omega)$

Given $\Gamma \subset \Omega$ countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable with finite measure $(\mathcal{H}^{n-1}(\Gamma) < \infty)$, we want to introduce a family of functions $(\theta^\xi)_{\xi \in \mathbb{S}^{n-1}}$, $\theta^\xi : \mathbb{R}^n \to \mathbb{R}^+$, called *one sectional distance*, which will play a fundamental role in the integrability of the trace of a GBD function. Before doing this, let us recall a property of rectifiable sets with finite measure.

Remark 3.1. Let $\Gamma \subset \mathbb{R}^n$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with finite measure. Choose any $\xi \in \mathbb{S}^{n-1}$ then

$$\mathcal{H}^0(\Gamma_y^\xi) < \infty \text{ for a.e. } y \in \Pi^\xi. \tag{3.1}$$

This fact is a simply consequence of the Coarea formula applied to the projection map π^ξ from $\mathbb{R}^n$ onto Π^ξ restricted on Γ.

Definition 3.2. (One sectional distance) Let $\Gamma \subset \mathbb{R}^n$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with finite $\mathcal{H}^{n-1}$ measure, and let $\xi \in \mathbb{S}^{n-1}$. Writing $x \in \mathbb{R}^n$ as $x = y + t\xi$ (for $(y,t) \in \Pi^\xi \times \mathbb{R}$), we define $\theta^\xi : \mathbb{R}^n \to \mathbb{R}^+$ in such a way that:

$$\theta^\xi(y + t\xi) = \begin{cases} |t_{i+1} - t_i| \wedge 1 & \text{if } 1 < \mathcal{H}^0(\Gamma_y^\xi) < \infty \text{ and } t \in (t_i, t_{i+1}) \\ 1 & \text{otherwise,} \end{cases} \tag{3.2}$$

where $(t_i)_{i=1}^{\mathcal{H}^0(\Gamma_y^\xi)}$ are the elements of the set Γ_y^ξ ordered so that $t_1 < \dots < t_i < \dots < t_{\mathcal{H}^0(\Gamma_y^\xi)}$.

Proposition 3.3. *Let $\Gamma \subset \mathbb{R}^n$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with finite measure, and choose $\xi \in \mathbb{S}^{n-1}$. Then the function θ^ξ of Definition 3.2 is $\mathcal{L}^n$-measurable.*

Proof. By [10, Theorem 3.2.29] Γ is contained in a countably union of C^1 submanifolds of $\mathbb{R}^n$ say $(M_k)_{k \in \mathbb{N}}$ up to a $\mathcal{H}^{n-1}$-negligible set. If we define $\Gamma^\xi := \{x \in \Gamma \mid \nu_\Gamma(x) \cdot \xi \neq 0\}$ and $M_k^\xi := \{x \in M_k \mid \nu_{M_k}(x) \cdot \xi \neq 0\}$, where $\nu_\Gamma(\cdot)$ and $\nu_{M_k}(\cdot)$ are respectively an orientation of Γ and of M_k in the sense of Definition 2.15, then

$$\mathcal{H}^{n-1}(\Gamma^\xi \setminus \bigcup_k M_k^\xi) = 0.$$

For each k, M_k^ξ can be covered by countably many $n-1$ dimensional submanifolds of class C^1, say $(\Sigma_{k,i})_{i \in \mathbb{N}}$, which are the graph of C^1 functions, say $(f_{k,i})_{i \in \mathbb{N}}$, defined on some open subset of Π^ξ (using Lindëlof property and the Implicit Function Theorem). Hence, possibly re-enumerating the $(\Sigma_{k,i})_{(k,i) \in \mathbb{N}^2}$ as $(\Sigma_i)_{i \in \mathbb{N}}$ (and respectively the $(f_{k,i})_{(k,i) \in \mathbb{N}^2}$ as $(f_i)_{i \in \mathbb{N}}$), we have

$$\mathcal{H}^{n-1}(\Gamma^\xi \setminus \bigcup_i \Sigma_i) = 0. \tag{3.3}$$

For any couple of indices $(i_1, i_2) \in \mathbb{N}^2$, define θ_{i_1,i_2}^ξ to be the one sectional distance relative to the rectifiable set $\Sigma_{i_1} \cup \Sigma_{i_2}$. Suppose for a moment that we already know that θ_{i_1,i_2}^ξ is $\mathcal{L}^n$-measurable for any (i_1, i_2) . In this case we can define

$$\tilde{\theta}_{i_1,i_2}^\xi(y + t\xi) := \begin{cases} \theta_{i_1,i_2}^\xi(y + t\xi) & \text{if } y \in \pi^\xi(\Gamma \cap \Sigma_{i_1}) \cap \pi^\xi(\Gamma \cap \Sigma_{i_2}) \\ 1 & \text{otherwise} \end{cases} \tag{3.4}$$

Clearly $\tilde{\theta}^{\xi}_{i_1,i_2}$ is $\mathcal{L}^n$-measurable because the set $\pi^{\xi}(\Gamma\cap\Sigma_{i_1})\cap\pi^{\xi}(\Gamma\cap\Sigma_{i_2})$ is $\mathcal{H}^{n-1}$-measurable and we use Fubini's theorem on the product space $\Pi^{\xi}\times\mathbb{R}$ (see [9, Section 1.4]) to deduce that the set $\big(\pi^{\xi}(\Gamma\cap\Sigma_{i_1})\cap\pi^{\xi}(\Gamma\cap\Sigma_{i_2})\big)\times\mathbb{R}$ is $\mathcal{L}^n$-measurable. With (3.4) it is easy to see that

$$\theta^{\xi}(x)=\inf_{(i_1,i_2)\in\mathbb{N}^2}\tilde{\theta}^{\xi}_{i_1,i_2}(x), \tag{3.5}$$

for any $x=y+t\xi$ such that $(\Gamma^{\xi})^{\xi}_y\subset\bigcup_{i=0}^{\infty}(\Sigma_i)^{\xi}_y$. Thanks to (3.3), the previous inclusion holds for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^{\xi}$, hence (3.5) holds for $\mathcal{L}^n$-a.e. $x\in\mathbb{R}^n$. This gives that θ^{ξ} is $\mathcal{L}^n$-measurable.

Finally it remains to prove the measurability of $\theta^{\xi}_{i_1,i_2}$. It is enough to notice that on the set of point where $f_{i_1}<f_{i_2}$:

$$\theta^{\xi}_{i_1,i_2}(y+t\xi)=\begin{cases}|f_{i_2}(y)-f_{i_1}(y)|\wedge 1 & \text{if } \ y\in\pi^{\xi}(\Sigma_{i_1})\cap\pi^{\xi}(\Sigma_{i_2})\ ,\ f_{i_1}(y)<t<f_{i_2}(y)\\ 1 & \text{otherwise,}\end{cases} \tag{3.6}$$

while on the set of points where $f_{i_1}>f_{i_2}$:

$$\theta^{\xi}_{i_1,i_2}(y+t\xi)=\begin{cases}|f_{i_2}(y)-f_{i_1}(y)|\wedge 1 & \text{if } \ y\in\pi^{\xi}(\Sigma_{i_1})\cap\pi^{\xi}(\Sigma_{i_2})\ ,\ f_{i_2}(y)<t<f_{i_1}(y)\\ 1 & \text{otherwise,}\end{cases} \tag{3.7}$$

□

Remark 3.4. The one sectional distance θ^{ξ} of a rectifiable set Γ with finite $\mathcal{H}^{n-1}$ measure, has finite total variation in the direction ξ. In fact it can be easily proved that:

$$|D_{\xi}\theta^{\xi}|(\mathbb{R}^n)\le\int_{\Gamma}|\nu(x)\cdot\xi|d\mathcal{H}^{n-1}(x)\le\mathcal{H}^{n-1}(\Gamma). \tag{3.8}$$

So given any countably $(\mathcal{H}^{n-1},n-1)$-rectifiable set $\Gamma\subset\mathbb{R}^n$ with finite measure, by [4, Theorem 5.1], we can talk about the trace of θ^{ξ} on the set $\{x\in E\mid\nu_E(x)\cdot\xi\neq 0\}$.

Definition 3.5. Let $\xi\in\mathbb{S}^{n-1}$ and let $0<L<1$.

We define the cone with axis ξ and opening L as

$$C(\xi,L):=\{x\in\mathbb{R}^n\setminus\{0\}\mid|\xi\cdot x|>L|x|\}.$$

We define the upper half cone with axis ξ and opening L as:

$$C^{+}(\xi,L):=\{x\in\mathbb{R}^n\setminus\{0\}\mid\xi\cdot x>L|x|\},$$

and analogously the lower half cone cone with axis ξ and opening L as:

$$C^{-}(\xi,L):=\{x\in\mathbb{R}^n\setminus\{0\}\mid\xi\cdot x<-L|x|\}.$$

Remark 3.6. Consider $\Xi:=\{\xi_1,\dots,\xi_n\}$ an orthonormal basis of $\mathbb{R}^n$ and let δ be a real number such that $0<\delta<1/\sqrt{n}$. Define:

$$C(\Xi,\ \delta):=\bigcap_{i=1}^{n}C(\xi_i,\ 1/\sqrt{n}-\delta)\cap\mathbb{S}^{n-1}. \tag{3.9}$$

Notice that $C(\Xi,\delta)$ is open in the relative topology of $\mathbb{S}^{n-1}$ and contains for example the vector $\sum_{i=1}^{n}\xi_i/\sqrt{n}$. This means that the family $\Lambda:=\{\ C(\Xi,\delta)\mid\Xi \text{ orthonormal basis}\}$ is an open covering of $\mathbb{S}^{n-1}$, and so by compactness we can always extract a finite subcovering from Λ.

We denote by $N(\delta)$ the minimum number of elements of Λ that needs to cover $\mathbb{S}^{n-1}$. $N(\delta)$ is a constant that depends only on the dimension n and on δ.

Let us introduce the space of vector fields that jump on a prescribed set:

Definition 3.7. Let $\Gamma\subset\Omega$ be a countably $(\mathcal{H}^{n-1},n-1)$-rectifiable set with finite measure and let $p\ge 1$. We define the following spaces:

$$GSBD^p_p(\Omega):=\{u\in GSBD(\Omega)\mid u\in L^p(\Omega),\mathcal{E}u\in L^p(\Omega)\}, \tag{3.10}$$

$$GBD(\Omega;\Gamma):=\{u\in GBD(\Omega)\mid J_u\subseteq\Gamma\}, \tag{3.11}$$

$$GSBD^p_p(\Omega;\Gamma):=\{u\in GSBD^p_p(\Omega)\mid J_u\subseteq\Gamma\}. \tag{3.12}$$

Remark 3.8. Actually one can show that $GSBD^p_p(\Omega;\Gamma)$ is a vector space, and that can be endowed with the norm $\|u\|_p+\|\mathcal{E}u\|_p$. Thanks to [4, Theorem 11.3], $GSBD^p_p$ with this norm $\|\cdot\|_p$ is a Banach space.

Now we want to extend the notion of trace operator for an arbitrary open set of $\mathbb{R}^n$ having finite perimeter:

Definition 3.9. (Trace operator in $GBD(\Omega)$). Let $\Omega \subset \mathbb{R}^n$ be an open set of finite perimeter, and let $u \in GBD(\Omega;\Gamma)$. We define the trace operator as:

$$Tr(u)(x) := \underline{u}^{+}_{\mathcal{F}\Omega}(x), \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \mathcal{F}\Omega, \tag{3.13}$$

where $\underline{u}$ is the function extended to 0 outside of Ω given in proposition 2.20, and the trace from above $\underline{u}^{+}$ is considered with respect to the inner measure theoretic unit normal $\nu_{\mathcal{F}\Omega}$ of the reduced boundary $\mathcal{F}\Omega$.

Moreover in order to simplify the notation, when there is no misunderstanding, we simply write:

$$u^{+}(x) = \begin{cases} u^{+}_{\Gamma}(x) & \text{if } x \in \Gamma, \\ Tr(u)(x) & \text{if } x \in \mathcal{F}\Omega, \end{cases} \tag{3.14}$$

and:

$$u^{-}(x) = \begin{cases} u^{-}_{\Gamma}(x) & \text{if } x \in \Gamma, \\ 0 & \text{if } x \in \mathcal{F}\Omega. \end{cases} \tag{3.15}$$

Remark 3.10 (Coincidence of Trace). When Ω is a lipschitz regular domain, our definition of trace coincides with the usual one in the space $BD(\Omega)$.

First of all in this case, the reduced boundary $\mathcal{F}\Omega$ coincides with the topological one. Moreover on the space of regular functions up to the boundary, our definition coincides with the restriction operator on $\partial\Omega$. Then using a density argument together with identities (5.3) and (5.5) in [4], we deduce the coincidence of our notion of trace with the usual one in $BD(\Omega)$.

Now we are in position to prove our main results about the integrability of the trace in $GBD(\Omega;\Gamma)$ and $GSBD^p_p(\Omega;\Gamma)$. As mentioned in the introduction, we will consider the trace on $\mathcal{F}\Omega$ and both traces $u^{\pm}$ on Γ. We decide to split our results into two theorems, the first concerns the case GBD:

Theorem 3.11. *(Trace inequality in $GBD(\Omega)$). Let $\Omega \subset \mathbb{R}^n$ be an open set of finite perimeter, and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1},\ n-1)$-rectifiable set, with $\mathcal{H}^{n-1}(\Gamma) < \infty$ and oriented by ν. Then there exist two $\mathcal{H}^{n-1}$-measurable functions $\Theta^{\pm} : \Gamma \cup \mathcal{F}\Omega \to \mathbb{R}^{+}$ depending only on the geometry of Γ, its orientation ν, and on Ω, such that denoting with $u^{\pm}$ the traces of u according to Definition 3.9 , we have*

(a) $\mathcal{H}^{n-1}\big(\{\Theta^{\pm} = 0\}\big) = 0$ and $\Theta^{\pm} \in L^{\infty}(\Gamma \cup \mathcal{F}\Omega,\ \mathcal{H}^{n-1})$ (in particular $\|\Theta^{\pm}\|_{\infty} \leq 1$);

(b) For every $u \in GBD(\Omega;\Gamma) \cap L^1(\Omega,\mathbb{R}^n)$ we have:

$$\int_{\Gamma\cup\mathcal{F}\Omega} |u^{\pm}(x)|\Theta^{\pm}(x)d\mathcal{H}^{n-1}(x) \leq C(n)\bigg(\hat{\mu}_u(\Omega \setminus J_u) + \int_{\Omega} |u(x)|dx\bigg). \tag{3.16}$$

Proof. Let $\underline{u} \in GBD(\mathbb{R}^n;\Gamma \cup \mathcal{F}\Omega)$ be the function extended to 0 outside of Ω as in proposition 2.20. In order to simplify the notation, we write Γ to denote $\Gamma \cup \mathcal{F}\Omega$, and ν to denote the orientation that coincides with the given orientation ν on Γ, and with $\nu_{\mathcal{F}\Omega}$ on $\mathcal{F}\Omega$. By our definition of $u^{\pm}$ (Definition 3.9) and by Proposition 2.20 (in particular point (b) tells us that $\hat{\mu}_{\underline{u}} \leq \hat{\mu}_u + \mathcal{H}^{n-1} \llcorner \mathcal{F}\Omega$), (3.16) can be rewritten as:

$$\int_{\Gamma} |\underline{u}^{\pm}|\Theta^{\pm}\ d\mathcal{H}^{n-1} \leq C(n)\bigg(\hat{\mu}_{\underline{u}}(\mathbb{R}^n \setminus J_{\underline{u}}) + \int_{\mathbb{R}^n} |\underline{u}(x)|dx\bigg). \tag{3.17}$$

So let us prove (3.17) for any function in the space $GBD(\mathbb{R}^n;\Gamma \cup \mathcal{F}\Omega)$.

Consider Λ the covering of $\mathbb{S}^{n-1}$ of Remark 3.6 and by compactness define $\big(C(\Xi_i,\delta)\big)_{i=1}^{N(\delta)}$ to be a subcovering of Λ. If we define for any $i = 1,\dots,N$, $\Gamma_i := \nu^{-1}\big(C(\Xi_i,\delta)\big)$ then $(\Gamma_i)_{i=1}^{N}$ is a finite measurable covering of Γ. By definition of Λ, for any $\xi \in \Xi_i$ and for every $x \in \Gamma_i$, we have $|\xi \cdot \nu(x)| > 1/\sqrt{n} - \delta$.

Now we fix i and $\xi \in \Xi_i$. We write the generic point $x \in \mathbb{R}^n$ as $(y,t) \in \Pi^{\xi} \times \mathbb{R}$, and from now on we will work on the set of points $y \in \pi^{\xi}(\Gamma_i)$ such that $\hat{\underline{u}}^{\xi}_{y} \in BV_{loc}(\mathbb{R})$ and $\mathcal{H}^0(\Gamma^{\xi}_y) < \infty$; from the Definition 2.7 of GBD and Remark 3.1 we already know that $\mathcal{H}^{n-1}$ almost all of y have these properties.

We call $(t_k)_{k=1}^{\mathcal{H}^0(\Gamma^{\xi}_y)}$ the point of the slicing Γ^{ξ}_y ordered such that $t_k < t_{k+1}$ for any k.

Let $\theta^\xi : E \to \mathbb{R}^+$ be the one sectional distance introduced in Definition 3.2. Thanks to Remark 3.4, for $x \in \Gamma$ we can consider $\theta^{\xi^\pm}(x)$ the trace respectively from above and from below on Γ_i.

By Theorem 2.16:

$$\underline{u}^+(y+t\xi)\cdot\xi = (\hat{\underline{u}}^\xi_y)^+(t) \text{ if } t \in (\Gamma_i)^\xi_y \text{ and } \nu(x)\cdot\xi > 0, \tag{3.18}$$

and

$$\underline{u}^+(y+t\xi)\cdot\xi = (\hat{\underline{u}}^\xi_y)^-(t) \text{ if } t \in (\Gamma_i)^\xi_y \text{ and } \nu(x)\cdot\xi < 0. \tag{3.19}$$

Since ξ has been fixed, in order to simplify the notation, we omit the dependence on ξ and write $\Gamma_i^+ := \Gamma_i \cap \{\nu\cdot\xi > 0\}$ and $\Gamma_i^- := \Gamma_i \cap \{\nu\cdot\xi < 0\}$. Let's focus for example on the set Γ_i^+:

$$\hat{\underline{u}}^\xi_y(t) - (\hat{\underline{u}}^\xi_y)^+(t_k) = \int_{t_k}^t dD\hat{\underline{u}}^\xi_y, \text{ for } t_k \in (\Gamma_i^+)^\xi_y \text{ and } t_k < t < t_{k+1}. \tag{3.20}$$

Now at fixed $y \in \pi^\xi(\Gamma_i^+)$ we can integrate again on $t \in (t_k,\ t_{k+1})$ to get

$$\begin{gathered}(t_{k+1}-t_k)|(\hat{\underline{u}}^\xi_y)^+(t_k)| \le (t_{k+1}-t_k)\int_{t_k}^{t_{k+1}} d|D\hat{\underline{u}}^\xi_y| + \int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(t)|dt,\\ \text{for } t_k \in (\Gamma_i^+)^\xi_y, \text{ and } t_{k+1}-t_k \le 1,\end{gathered} \tag{3.21}$$

and

$$\begin{gathered}|(\hat{\underline{u}}^\xi_y)^+(t_k)| \le \int_{t_k}^{1+t_k} d|D\hat{\underline{u}}^\xi_y| + \int_{t_k}^{1+t_k} |\hat{\underline{u}}^\xi_y(t)|\, dt,\\ \text{for } t_k \in (\Gamma_i^+)^\xi_y, \text{ and } t_{k+1}-t_k > 1.\end{gathered} \tag{3.22}$$

Using the fact that θ^{ξ^+} is equal to $t_{k+1}-t_k$ or 1 on the set $\{y+t\xi \mid t_k < t < t_{k+1}\}$, we sum on $t_k \in (\Gamma_i^+)^\xi_y$ to get

$$\begin{aligned}\sum_{t_k} |(\hat{\underline{u}}^\xi_y)^+(t_k)|\theta^{\xi^+}(y+t_k\xi) &\le \sum_{t_k}\left(\int_{t_k}^{t_k+\theta^{\xi^+}(y+t_k\xi)} d|D\hat{\underline{u}}^\xi_y| + \int_{t_k}^{t_k+\theta^{\xi^+}(y+t_k\xi)} |\hat{\underline{u}}^\xi_y(t)|\, dt\right)\\ &\le |D\hat{\underline{u}}^\xi_y|(\mathbb{R}\setminus J_{\hat{\underline{u}}^\xi_y}) + \sum_{t_k}\int_{t_k}^{t_k+\theta^{\xi^+}(y+t_k\xi)} |\hat{\underline{u}}^\xi_y(t)|\, dt.\end{aligned} \tag{3.23}$$

The first term in the left hand side of (3.23) is a measurable function of y. In fact thanks to theorem 2.16, $(\hat{\underline{u}}^\xi_y)^+(t_k)$ is the trace on Γ_i^+ of $u\cdot\xi$ hence $\mathcal{H}^{n-1}$-measurable, and θ^{ξ^+} is $\mathcal{H}^{n-1}$-measurable as well because trace of a measurable function. Then approximating $|\underline{u}^+\cdot\xi|\theta^{\xi^+}$ by simple functions $(s_m)_{m=0}^\infty$ and applying the Coarea formula with the projection map π^ξ on Γ_i^+, we have in particular that the maps:

$$y \mapsto \sum_{t_k\in(\Gamma_i^+)^\xi_y} (s_m)^\xi_y(t_k),$$

are $\mathcal{H}^{n-1}$-measurable for every $m \in \mathbb{N}$, hence we deduce directly that the term in the left hand-side of (3.23) is $\mathcal{H}^{n-1}$-measurable.

The term $|D\hat{\underline{u}}^\xi_y|(\mathbb{R}\setminus J_{\hat{\underline{u}}^\xi_y})$ is a measurable function of y just by definition of GBD, while the last term in the right hand-side of (3.23) is a measurable function of y once we show that the set:

$$\Lambda_i^{\xi^+} := \left\{(y,t) \in \pi^\xi(\Gamma_i^+)\times\mathbb{R} \mid t_k < t < t_k + \theta^{\xi^+}(y+t_k\xi),\ t_k \in (\Gamma_i^+)^\xi_y\right\},$$

is $\mathcal{L}^n$-measurable. To show this, we notice that since $\Gamma_i^+ \subset \{x \in \Gamma \mid \nu(x)\cdot\xi \ne 0\}$, then using the characterization of rectifiable set (as explained in proposition 3.3), $\mathcal{H}^{n-1}$-almost all of Γ_i^+ can be covered by countably many submanifold of class C^1 say $(\Sigma_j)_{j=0}^\infty$, which are graphs of C^1 functions $(f_j)_{j=0}^\infty$ defined on some open subset of Π^ξ. Clearly if we call $\Lambda_{i,j}^{\xi^+}$ the set of points $(y,t) \in \Pi^\xi\times\mathbb{R}$ such that :

$$y \in \pi^\xi(\Sigma_j\cap\Gamma_i^+) \text{ and } f_j(y) < t < f_j(y) + \theta^{\xi^+}(y+f_j(y)\xi),$$

then $\Lambda_{i,j}^{\xi^+}$ is $\mathcal{L}^n$-measurable, simply because both maps appearing in the left hand side and in the right hand side of the previous inequality are restriction of $\mathcal{H}^{n-1}$-measurable functions on a $\mathcal{H}^{n-1}$-measurable set. Finally we notice that:

$$\mathcal{L}^n(\Lambda_i^{\xi^+}\Delta\bigcup_{j=0}^\infty\Lambda_{i,j}^{\xi^+}) = 0,$$

and we are done.

So we can consider the integral on $\pi^\xi(\Gamma_i^+)$ on both sides of (3.23). By Theorem 2.16 $J_{\hat{u}_y^\xi} = (J_u)_y^\xi$ for a.e. y, so after integration we have:

$$\int_{\pi^\xi(\Gamma_i^+)} \sum_{t_k} |(\hat{\underline{u}}_y^\xi)^+(t_k)|\theta^{\xi^+}(y+t_k\xi)\, dy \le \int_{\pi^\xi(\Gamma_i^+)} |D\hat{\underline{u}}_y^\xi|(\mathbb{R}\setminus(J_{\underline{u}})_y^\xi)\, dy + \int_{\pi^\xi(\Gamma_i^+)} \left(\sum_{t_k}\int_{t_k}^{t_k+\theta^{\xi^+}(y+t_k\xi)} |\underline{u}(y+t\xi)|\, dt\right) dy. \tag{3.24}$$

Analogously we have the same inequality on the set where $\{\nu\cdot\xi<0\}$:

$$\int_{\pi^\xi(\Gamma_i^-)} \sum_{t_k} |(\hat{\underline{u}}_y^\xi)^-(t_k)|\theta^{\xi^+}(y+t_k\xi)\, dy \le \int_{\pi^\xi(\Gamma_i^-)} |D\hat{\underline{u}}_y^\xi|(\mathbb{R}\setminus(J_{\underline{u}})_y^\xi)\, dy + \int_{\mathbb{R}^n} |\underline{u}(x)|\, dx. \tag{3.25}$$

Summing the two inequality (3.24) and (3.25), by the relations between the trace of the function and the trace of its slicing (3.37) and (3.38), we have:

$$\int_{\pi^\xi(\Gamma_i)} \sum_{t_k\in(\Gamma_i)_y^\xi}^{\mathcal{H}^0((\Gamma_i)_y^\xi)} |\underline{u}^+(y+t_k\xi)\cdot\xi|\theta^{\xi^+}(y+t_k\xi)\, dy \le 2\int_{\pi^\xi(\Gamma_i)} |D\hat{\underline{u}}_y^\xi|(\mathbb{R}\setminus(J_{\underline{u}})_y^\xi)\, dy + 2\int_{\mathbb{R}^n} |\underline{u}(x)|\, dx. \tag{3.26}$$

Finally Coarea formula on the rectifiable set Γ_i applied to the projection π^ξ with the fact that $|\nu(x)\cdot\xi| > 1/\sqrt{n}-\delta$, allows us to write:

$$\begin{aligned}\frac{1-\sqrt{n}\delta}{\sqrt{n}}\int_{\Gamma_i} |\underline{u}^+(x)\cdot\xi|\theta^{\xi^+}(x)\, d\mathcal{H}^{n-1}(x) &\le \int_{\Gamma_i} |\underline{u}^+(x)\cdot\xi|\, |\nu(x)\cdot\xi|\theta^{\xi^+}(x)\, d\mathcal{H}^{n-1}(x)\\ &= \int_{\pi^\xi(\Gamma_i)}\left(\sum_{t_k\in(\Gamma_i)_y^\xi}^{\mathcal{H}^0((\Gamma_i)_y^\xi)} |\hat{\underline{u}}_y^\xi(t_k)|\theta^{\xi^+}(y,t_k)\right) dy\\ &\le 2\int_{\pi^\xi(\Gamma_i)} |D\hat{\underline{u}}_y^\xi|(\mathbb{R}\setminus(J_{\underline{u}})_y^\xi)dy + 2\int_{\mathbb{R}^n}|u(x)|dx\\ &\le 2\hat{\mu}_{\underline{u}}\big((\pi^\xi(\Gamma_i)\times\mathbb{R})\setminus J_{\underline{u}}\big) + 2\int_{\mathbb{R}^n}|\underline{u}(x)|dx.\end{aligned} \tag{3.27}$$

Repeating the same argument for every $\xi_j\in\Xi_i$ we may write:

$$\begin{aligned}\int_{\Gamma_i}\sum_{\xi_j\in\Xi_i} |\underline{u}^+(x)\cdot\xi_j|\theta^{\xi_j^+}(x)\, d\mathcal{H}^{n-1}(x) &\le \frac{2\sqrt{n}}{1-\sqrt{n}\delta}\sum_{\xi_j\in\Xi_i}\hat{\mu}_{\underline{u}}\big((\pi^{\xi_j}(\Gamma_i)\times\mathbb{R})\setminus J_{\underline{u}}\big)\\ &\quad + \frac{2n^{3/2}}{1-\sqrt{n}\delta}\int_{\mathbb{R}^n}|\underline{u}(x)|\, dx\\ &\le \frac{2n^{3/2}}{1-\sqrt{n}\delta}\left(\hat{\mu}_{\underline{u}}(\mathbb{R}^n\setminus J_{\underline{u}}) + \int_{\mathbb{R}^n}|\underline{u}(x)|\, dx\right).\end{aligned} \tag{3.28}$$

Now define $\Theta^+ : \Gamma_i \to \mathbb{R}^+$ as:

$$\Theta^+(x) := \min\{\theta^{\xi_j^+}(x) \mid \xi_j\in\Xi_i\} \text{ for } x\in\Gamma_i. \tag{3.29}$$

By construction for each $j=1,\dots,n$ the functions $\theta^{\xi_j^+}$ are strictly greater then zero $\mathcal{H}^{n-1}$-a.e. on Γ_i, hence $\Theta^+(x)>0$ for $\mathcal{H}^{n-1}$-a.e. $x\in\Gamma_i$ and this gives (b). So by inequality (3.28) and the definition of Θ^+ we can write

$$\int_{\Gamma_i} |\underline{u}^+(x)|\Theta^+(x)\, d\mathcal{H}^{n-1}(x) \le \frac{2n^{3/2}}{1-\sqrt{n}\delta}\left(\hat{\mu}_{\underline{u}}(\mathbb{R}^n\setminus J_{\underline{u}}) + \int_{\mathbb{R}^n}|\underline{u}(x)|\, dx\right). \tag{3.30}$$

Eventually redefining the $\big(C(\Xi_i,\delta)\big)_{i=1}^{N(\delta)}$, we may assume that $(\Gamma_i)_{i=1}^{N(\delta)}$ are pairwise disjoint; now summing the last inequality (3.30) for every i, together with the choice $\delta = 1/2\sqrt{n}$, we get:

$$\int_\Gamma |\underline{u}^+(x)|\Theta^+(x)\, d\mathcal{H}^{n-1}(x) \le 4n^{3/2}N(1/2\sqrt{n})\left(\hat{\mu}_{\underline{u}}(\mathbb{R}^n\setminus J_{\underline{u}}) + \int_{\mathbb{R}^n}|\underline{u}(x)|\, dx\right), \tag{3.31}$$

which is (3.17) for $\underline{u}^+$. Defining $\Theta^-:\Gamma_i\to\mathbb{R}^+$ as

$$\Theta^-(x):=\min\{\theta^{\xi_j^-}(x)\mid \xi_j\in\Xi_i\}\text{ for }x\in\Gamma_i, \tag{3.32}$$

using the same argument we can prove (a) for Θ^- and (3.16) for $\underline{u}^-$, and we conclude. □

The following is analogous of Theorem 3.11 in the case $GSBD^p_p$:

Theorem 3.12. *(Trace inequality in $GSBD^p_p(\Omega)$). Let Ω and Γ be as in Theorem 3.11. Then there exist two $\mathcal{H}^{n-1}$-measurable functions $\Theta^\pm:\Gamma\cup\mathcal{F}\Omega\to\mathbb{R}^+$ depending only on the geometry of Γ, its orientation ν, and on Ω, such that denoting with $u^\pm$ the traces of u according to Definition 3.9 , we have*

(a) $\mathcal{H}^{n-1}(\{\Theta^\pm=0\})=0$ and $\Theta^\pm\in L^\infty(\Gamma\cup\mathcal{F}\Omega,\mathcal{H}^{n-1})$ (in particular $\|\Theta^\pm\|_\infty\le 1$);

(b) For every $u\in GSBD^p_p(\Omega;\Gamma)$ $(p\ge 1)$ we have:

$$\int_{\Gamma\cup\mathcal{F}\Omega}|u^\pm|^p\Theta^\pm\,d\mathcal{H}^{n-1}\le C(n,p)\bigg(\int_\Omega|\mathcal{E}u|^p dx+\int_\Omega|u|^p\,dx\bigg), \tag{3.33}$$

where $C(n,p)$ is a constant depending only on n and p;

(c) Let $p^*=np/(n-p)$ be the usual critical Sobolev exponent, then we have:

$$\int_{\Gamma\cup\mathcal{F}\Omega}|u^\pm|^{\frac{p(n-1)}{n-p}}\Theta^\pm\,d\mathcal{H}^{n-1}\le C'(n,p)\bigg(\int_\Omega|\mathcal{E}u|^p\,dx+\int_\Omega|u|^{p^*}\,dx\bigg), \tag{3.34}$$

for every $u\in GSBD^p_p(\Omega;\Gamma)\cap L^{p*}(\Omega)$.

Proof. Let $\underline{u}\in GBD(\mathbb{R}^n;\Gamma\cup\mathcal{F}\Omega)$ be the function extended to 0 outside of Ω as in Proposition 2.20. In order to simplify the notation, we write Γ to denote $\Gamma\cup\mathcal{F}\Omega$, and ν to denote the orientation that coincides with the given orientation ν on Γ, and with $\nu_{\mathcal{F}\Omega}$ on $\mathcal{F}\Omega$. By our definition of $u^\pm$ (see Definition 3.9) and by proposition 2.20, (3.33) and (3.34), can be rewritten as:

$$\int_\Gamma|\underline{u}^\pm|^p\Theta^\pm\,d\mathcal{H}^{n-1}\le C(n,p)\bigg(\int_{\mathbb{R}^n}|\mathcal{E}\underline{u}|^p dx+\int_{\mathbb{R}^n}|\underline{u}|^p\,dx\bigg), \tag{3.35}$$

and

$$\int_\Gamma|\underline{u}^\pm|^{\frac{p(n-1)}{n-p}}\Theta^\pm\,d\mathcal{H}^{n-1}\le C'(n,p)\bigg(\int_{\mathbb{R}^n}|\mathcal{E}\underline{u}|^p\,dx+\int_{\mathbb{R}^n}|\underline{u}|^{p^*}\,dx\bigg). \tag{3.36}$$

We argue similarly to the proof of Theorem 3.11: consider $(\Gamma_i)_{i=1}^N$ the partition of Γ given in Theorem 3.11, and let Ξ_i be the orthonormal basis of $\mathbb{R}^n$ associated to Γ_i. Fix i and $\xi\in\Xi_i$.

From now on we will work on the points $y\in\pi^\xi(\Gamma_i)$ such that $\hat{u}^\xi_y\in SBV_{loc}(\mathbb{R})$ and $\mathcal{H}^0(\Gamma^\xi_y)<\infty$; from the Definition 2.9 of $GSBD$ and Remark 3.1 we already know that $\mathcal{H}^{n-1}$-almost all of y have these properties.

We call $(t_k)_{k=1}^{\mathcal{H}^0(\Gamma^\xi_y)}$ the points of the slicing Γ^ξ_y ordered such that $t_k<t_{k+1}$ for any k. Let $\theta^\xi:E\to\mathbb{R}^+$ be the one sectional distance introduced in Definition 3.2. For $x\in\Gamma$ let $\theta^{\xi^\pm}(x)$ to be the trace of θ^ξ according to ν.

Now we work on Γ_i. By Theorem 2.16

$$\underline{u}^+(y+t\xi)\cdot\xi=(\hat{\underline{u}}^\xi_y)^+(t)\text{ if }t\in(\Gamma_i)^\xi_y\text{ and }\nu(x)\cdot\xi>0, \tag{3.37}$$

and

$$\underline{u}^+(y+t\xi)\cdot\xi=(\hat{\underline{u}}^\xi_y)^-(t)\text{ if }t\in(\Gamma_i)^\xi_y\text{ and }\nu(x)\cdot\xi<0. \tag{3.38}$$

Since ξ has been fixed, in order to simplify the notation, we omit the dependence on ξ and write $\Gamma_i^+:=\Gamma_i\cap\{\nu\cdot\xi>0\}$ and $\Gamma_i^-:=\Gamma_i\cap\{\nu\cdot\xi<0\}$. Let's focus for example on the set Γ_i^+:

$$\hat{\underline{u}}^\xi_y(t)-(\hat{\underline{u}}^\xi_y)^+(t_k)=\int_{t_k}^t\nabla\hat{\underline{u}}^\xi_y(r)dr,\text{ for }t_i\in(\Gamma_i^+)^\xi_y\text{ and }t_k<t<t_{k+1}, \tag{3.39}$$

passing to the modulus and elevating to the power p:

$$\begin{aligned}|(\hat{\underline{u}}^\xi_y)^+(t_k)|^p\le 2^{p-1}(t_{k+1}-t_k)^{p-1}\int_{t_k}^{t_{k+1}}|\nabla\hat{\underline{u}}^\xi_y(r)|^p\,dr+2^{p-1}|\hat{\underline{u}}^\xi_y(t)|^p,\\ \text{for }t_k\in(\Gamma_i^+)^\xi_y\text{ and }t_k<t<t_{k+1}.\end{aligned} \tag{3.40}$$

The same holds true for $|(\hat{\underline{u}}^\xi_y)^-|$ on the set Γ_i^-. Notice that by Theorem 2.18 $\nabla\hat{u}^\xi_y(t)=\mathcal{E}u(y+t\xi)\xi\cdot\xi$ for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$ and $\mathcal{H}^1$-a.e. $t\in\Omega^\xi_y$. So exactly as in Theorem 3.11, at fixed y we can integrate on $t \in (t_k, t_{k+1})$ so that we don't touch points of the slicing $(\Gamma_i)^\xi_y$; then we integrate with respect to $y \in \pi^\xi(\Gamma_i)$ and we use Coarea formula with the fact that $|\nu\cdot\xi| > 1/\sqrt{n} - \delta$:

$$\begin{aligned} \frac{1-\sqrt{n}\delta}{\sqrt{n}} \int_{\Gamma_i} |\underline{u}^+(x)|^p \theta^{\xi^+}(x)\, d\mathcal{H}^{n-1} &\le \sum_{t_k \in (\Gamma_i)^\xi_y}^{\mathcal{H}^0((\Gamma_i)^\xi_y)} \int_{\pi^\xi(\Gamma_i)} |\underline{u}^+(y+t_k\xi)\cdot\xi|^p \theta^{\xi^+}(y,t_k)\, dy \\ &\le 2^p \int_{\pi^\xi(\Gamma_i)} \bigg(\int_{\mathbb{R}} |\mathcal{E}\underline{u}(y+t\xi)\xi\cdot\xi|^p\, dt \bigg)\, dy \\ &+ 2^p \int_{\mathbb{R}^n} |\underline{u}(x)|^p\, dx. \end{aligned} \tag{3.41}$$

Summing (3.41) for every $\xi_j \in \Xi_i$ we get:

$$\begin{aligned} \int_{\Gamma_i} \sum_{\xi_j\in\Xi_i} |\underline{u}^+\cdot\xi_j|^p \theta^{\xi_j^+}\, d\mathcal{H}^{n-1} &\le C(n,p) \sum_{\xi_j\in\Xi_i} \int_{\pi^{\xi_j}(\Gamma_i)} \bigg(\int_{\mathbb{R}} |\mathcal{E}\underline{u}(y+t\xi)\xi_j\cdot\xi_j|^p\, dt \bigg)\, dy \\ &+ C(n,p) \int_{\mathbb{R}^n} |\underline{u}(x)|^p\, dx \\ &\le C(n,p) \bigg(\int_{\mathbb{R}^n} |\mathcal{E}\underline{u}(x)|^p\, dx + \int_{\mathbb{R}^n} |\underline{u}(x)|\, dx \bigg). \end{aligned} \tag{3.42}$$

Now define for every $i = 1,\dots,N$ (where N is the dimensional constant introduced in Remark 3.6), exactly as in (3.29):

$$\Theta^+(x) := \min\{\theta^{\xi_j^+}(x) \mid \xi_j \in \Xi_i\} \text{ for } x \in \Gamma_i, \tag{3.43}$$

so that (3.35) holds for $\underline{u}^+$. Now (a) follows exactly as in Theorem 3.11. Analogously by defining

$$\Theta^-(x) := \min\{\theta^{\xi_j^-}(x) \mid \xi_j \in \Xi_i\} \text{ for } x \in \Gamma_i, \tag{3.44}$$

we can prove (a) for Θ^- and (3.35) for the trace from below $\underline{u}^-$.

To prove (3.36) fix i and $\xi \in \Xi_i$. Then we notice that for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have all the properties mentioned in the first lines of this proof and moreover that $\hat{\underline{u}}^\xi_y \in L^{p*}(\mathbb{R})$, $\nabla\hat{\underline{u}}^\xi_y \in L^p(\mathbb{R})$. Then we elevate the one dimensional sections $\hat{\underline{u}}^\xi_y$ to the power $p(n-1)/(n-p)$ and we notice that for $\mathcal{H}^{n-1}$-a.e. y we have $\underline{u}^\xi_y \in W^{1,p}\big((t_k,t_{k+1})\big)$ so by means of the *chain rule formula* we get

$$\begin{aligned} (t_{k+1}-t_k)|(\hat{\underline{u}}^\xi_y)^+(t_k)|^{\frac{p(n-1)}{n-p}} &\le (t_{k+1}-t_k)\frac{p(n-1)}{n-p} \int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(r)|^{\frac{p(n-1)}{n-p}-1} |\nabla\hat{\underline{u}}^\xi_y(r)|\, dr \\ &+ \int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(t)|^{\frac{p(n-1)}{n-p}}\, dt, \\ &\text{for } t_k \in \big(\Gamma_i^+\big)^\xi_y, \text{ and } t_{k+1}-t_k \le 1, \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} |(\hat{\underline{u}}^\xi_y)^+(t_k)|^{\frac{p(n-1)}{n-p}} &\le \frac{p(n-1)}{n-p} \int_{t_k}^{1+t_k} |\hat{\underline{u}}^\xi_y(r)|^{\frac{p(n-1)}{n-p}-1} |\nabla\hat{\underline{u}}^\xi_y(r)|\, dr \\ &+ \int_{t_k}^{1+t_k} |\hat{\underline{u}}^\xi_y(t)|^{\frac{p(n-1)}{n-p}}\, dt, \\ &\text{for } t_k \in \big(\Gamma_i^+\big)^\xi_y, \text{ and } t_{k+1}-t_k > 1, \end{aligned} \tag{3.46}$$

Hölder's inequality with exponents $p/(p-1)$ and p, and then Young's inequality with the same exponents yelds to:

$$\begin{aligned} \int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(r)|^{\frac{p(n-1)}{n-p}-1} |\nabla\hat{\underline{u}}^\xi_y(r)|\, dr &\le \bigg(\int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(r)|^{p^*}\, dr \bigg)^{\frac{p}{p-1}} \bigg(\int_{t_k}^{t_{k+1}} |\nabla\hat{\underline{u}}^\xi_y(r)|^p\, dr \bigg)^{\frac{1}{p}} \\ &\le \frac{p}{p-1} \int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(r)|^{p^*}\, dr + \frac{1}{p} \int_{t_k}^{t_{k+1}} |\nabla\hat{\underline{u}}^\xi_y(r)|^p\, dr, \end{aligned} \tag{3.47}$$

Now we first integrate on the interval (t_k,t_{k+1}) both inequalities (3.45) and (3.46) using also (3.47), and then we integrate with respect to $y \in \Pi^\xi$. Finally we can conclude exactly as before, getting (3.36) for $\underline{u}^+$. The same argument works for $\underline{u}^-$ and we conclude. □

Definition 3.13. Given $\Gamma \subset \mathbb{R}^n$ a countably $(\mathcal{H}^{n-1}, n-1)$- rectifiable set oriented by ν, we say that Γ satisfies the *cone condition*, if there exist $r > 0$, $0 < L < 1$, and two $\mathcal{H}^{n-1}$-measurable maps $\eta^{\pm}\colon \Gamma \to \mathbb{S}^{n-1}$, such that for every $x \in \Gamma$ we have

$$\{x + C^{+}(\eta^{+}(x), L)\} \cap B_r(x) \cap \Gamma = \varnothing, \tag{3.48}$$

and

$$\{x + C^{-}(\eta^{-}(x), L)\} \cap B_r(x) \cap \Gamma = \varnothing. \tag{3.49}$$

Remark 3.14. For example if Γ is the boundary of some lipschitz-regular domain $\Omega \subset \mathbb{R}^n$ then it satisfies the cone condition.

Proposition 3.15. *(Trace inequality with no weights). Let Ω and Γ be as in Theorem 3.11. Suppose that $\Gamma \cup \mathcal{F}\Omega$ satisfies the cone condition with parameters r and L (see Definition 3.13), then we have:*

(a) If $u \in GBD(\Omega;\Gamma)$ then $u^{\pm} \in L^1(\Gamma \cup \mathcal{F}\Omega, \mathcal{H}^{n-1})$, and moreover there exists a constant $C(n,L,r) > 0$ such that:

$$\int_{\Gamma \cup \mathcal{F}\Omega} |u^{\pm}|\, d\mathcal{H}^{n-1} \le C(n,L,r)\left(\hat{\mu}_u(\Omega \setminus J_u) + \int_{\Omega} |u|\, dx\right) \tag{3.50}$$

(b) If $u \in GSBD^p_p(\Omega;\Gamma)$ $(p \ge 1)$ then $u^{\pm} \in L^p(\Gamma \cup \mathcal{F}\Omega, \mathcal{H}^{n-1})$, and moreover there exists a constant $C(n,L,r,p) > 0$ such that:

$$\int_{\Gamma \cup \mathcal{F}\Omega} |u^{\pm}|^p\, d\mathcal{H}^{n-1} \le C(n,L,r,p)\left(\int_{\Omega} |\mathcal{E}u|^p\, dx + \int_{\Omega} |u|^p\, dx\right) \tag{3.51}$$

Proof. We prove (a). The proof of (b) is similar.

Let $\underline{u} \in GBD(\mathbb{R}^n; \Gamma \cup \mathcal{F}\Omega)$ be the function extended to 0 outside of Ω as in Proposition 2.20. In order to simplify the notation, we write Γ to denote $\Gamma \cup \mathcal{F}\Omega$, and ν to denote the orientation that coincides with the given orientation ν on Γ, and with $\nu_{\mathcal{F}\Omega}$ on $\mathcal{F}\Omega$. By our definition of $u^{\pm}$ (see Definition 3.9) and by proposition 2.20, (3.50) and (3.51), can be rewritten as:

$$\int_{\Gamma} |\underline{u}^{\pm}|\, d\mathcal{H}^{n-1} \le C(n,L,r)\left(\hat{\mu}_{\underline{u}}(\mathbb{R}^n \setminus J_{\underline{u}}) + \int_{\mathbb{R}^n} |\underline{u}|\, dx\right), \tag{3.52}$$

and

$$\int_{\Gamma} |\underline{u}^{\pm}|^p\, d\mathcal{H}^{n-1} \le C'(n,L,r,p)\left(\int_{\mathbb{R}^n} |\mathcal{E}\underline{u}|^p\, dx + \int_{\mathbb{R}^n} |\underline{u}|^p\, dx\right). \tag{3.53}$$

We prove (3.52), the proof of (3.53) is similar. Let us focus on the trace from above $\underline{u}^{+}$: first notice that if $x \in \Gamma$ admits an approximate tangent space [1], say $\mathrm{Tan}(x,\Gamma)$, then it must lies on the set of points $y \in \mathbb{R}^n \setminus C^{+}(\eta^{+}(x), L)$: this is simply because by definition of approximate tangent space [2]:

$$\mathcal{H}^{n-1} \llcorner \left(\frac{\Gamma - x}{\lambda}\right) \rightharpoonup \mathcal{H}^{n-1} \llcorner \mathrm{Tan}(x,\Gamma) \text{ as } \lambda \to 0^{+},$$

weakly in the sense of measure, i.e. tested against every continuous functions with compact support in $\mathbb{R}^n$; by our hypothesis for every $\lambda > 0$, $\frac{\Gamma - x}{\lambda} \cap C^{+}(\eta^{+}(x), L) \cap B_{r/\lambda}(0) = \varnothing$, and this means that the limit measure $\mathcal{H}^{n-1} \llcorner \mathrm{Tan}(x,\Gamma)$ has support disjoint from the open set $C^{+}(\eta^{+}(x), L)$. Thus we have the uniform bound on the scalar product:

$$\nu(x) \cdot \eta^{+}(x) > \sqrt{1 - L^2}, \text{ for every } x \in \Gamma. \tag{3.54}$$

Now consider $\epsilon > 0$ small enough such that $(L+1)/2 - \epsilon > L$ and $\epsilon < \frac{\sqrt{1-L}(\sqrt{1+L}-1)}{2}$. By compactness we can find a finite covering of $\mathbb{S}^{n-1}$, made of closed balls of radius $\epsilon/2$, say $(B_i)_{i=1}^{N(\epsilon)}$. Define for each $i = 1, \dots, N(\epsilon)$, $\Gamma_i := {\eta^{+}}^{-1}(B_i)$, then $\Gamma \subset \bigcup_i \Gamma_i$. For every $\Gamma_i \ne \varnothing$ $(i = 1, \dots, N(\epsilon))$ choose $x_i \in \Gamma_i$ and define $\eta_i^{+} := \eta^{+}(x_i)$. We claim that:

$$\{x + C^{+}(\eta_i^{+}, (L+1)/2)\} \cap B_r(x) \cap \Gamma = \varnothing, \text{ for every } x \in \Gamma_i. \tag{3.55}$$

[1] By [11, Theorem 5.4.5] $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ admits an approximate tangent space.

[2] See [11, Definition 5.4.4] for the definition of approximate tangent space.

In order to show (3.55) it is enough to notice that if $y \in x + C^+(\eta_i^+,\ (L+1)/2)$ then by using $(L+1)/2 - \epsilon > L$, we have:

$$\begin{aligned} |\eta^+(x)\cdot(y-x)| = |\eta_i^+\cdot(y-x) + (\eta^+(x) - \eta_i^+)\cdot(y-x)| &\ge |\eta_i^+\cdot(y-x)| - \epsilon|y-x| \\ &\ge \left(\frac{L+1}{2} - \epsilon\right)|y-x| \\ &> L|y-x|, \end{aligned} \tag{3.56}$$

which implies $y \in x + C^+(\eta^+(x), L) \cap B_r(x)$ and proves the claim.
Now we work on Γ_i. Consider a basis of $\mathbb{R}^n$, say $\Xi_i := \{\xi_1, \ldots, \xi_n\}$, such that:

$$\xi_j \in C^+\big(\eta_i^+, (L+1)/2)\big), \text{ for every } j = 1, \ldots, n. \tag{3.57}$$

Notice that by the fact $\epsilon < \frac{\sqrt{1-L}(\sqrt{1+L}-1)}{2}$ we have:

$$\begin{aligned} \nu(x)\cdot\xi_j &= \nu(x)\cdot(\xi_j - \eta_i^+) + \nu(x)\cdot(\eta_i^+ - \eta_i^+(x)) + \nu(x)\cdot\eta_i^+(x) \\ &\ge -\sqrt{2(1-\xi_j\cdot\eta_i^+)} - \epsilon + \sqrt{1-L^2} \\ &\ge -\sqrt{1-L} - \epsilon + \sqrt{1-L^2} \\ &\ge \frac{\sqrt{1-L}(\sqrt{1+L}-1)}{2} \end{aligned}$$

Now proceeding exactly as in the proof of Theorem 3.11 we have for every $\xi_j \in \Xi_i$

$$\begin{aligned} \int_{\pi^{\xi_j}(\Gamma_i^\epsilon \cap \{\nu\cdot\xi_j > 0\})} \sum_{t_k} |(\underline{\hat{u}}_y^{\xi_j})^+(t_k)| \theta^{\xi_j^+}(y + t_k\xi_j)\ dy &\le \int_{\pi^{\xi_j}(\Gamma_i^\epsilon \cap \{\nu\cdot\xi_j > 0\})} |D\underline{\hat{u}}_y^{\xi_j}|(\mathbb{R} \setminus (J_{\underline{u}})_y^{\xi_j})\ dy \\ &\quad + \int_{\Gamma_i \times \mathbb{R}} |\underline{u}(x)|\ dx. \end{aligned} \tag{3.58}$$

Using (3.55) we have that $\theta^{\xi_j^+}(x) \ge r$ for every $\xi_j \in \Xi_i$ and every $x \in \Gamma_i$. So by means of Coarea Formula applied to π^{ξ_j} on the set Γ_i, we can write:

$$\begin{aligned} & r\frac{\sqrt{1-L}(\sqrt{1+L}-1)}{2} \int_{\Gamma_i} |\underline{u}^+(x)\cdot\xi_j|\ d\mathcal{H}^{n-1}(x) \le \\ & \int_{\pi^{\xi_j}(\Gamma_i \cap \{\nu\cdot\xi_j > 0\})} |D\underline{\hat{u}}_y^{\xi_j}|(\mathbb{R} \setminus (J_{\underline{u}})_y^{\xi_j})\ dy + \int_{\Gamma_i \times \mathbb{R}} |\underline{u}(x)|\ dx. \end{aligned} \tag{3.59}$$

Summing the inequalities (3.59) for every $\xi_j \in \Xi_i$ we get

$$\int_{\Gamma_i} \sum_{\xi_j \in \Xi_i} |\underline{u}^+(x)\cdot\xi_j|\ d\mathcal{H}^{n-1}(x) \le C'(n, L, r)\left(\hat{\mu}_{\underline{u}}(\mathbb{R}^n \setminus J_{\underline{u}}) + \int_{\Gamma_i \times \mathbb{R}} |\underline{u}(x)| dx\right). \tag{3.60}$$

Now call $A_i \in \mathbb{M}^{n\times n}(\mathbb{R}^n)$, the matrix whose j-th columns is composed by the vector $\xi_j \in \Xi_i$. Then we have:

$$\sum_{\xi_j \in \Xi_i} |\underline{u}^+(x)\cdot\xi_j| \ge \left(\sum_{\xi_j \in \Xi_i} |\underline{u}^+(x)\cdot\xi_j|^2\right)^{\frac{1}{2}} \ge \frac{|\underline{u}^+(x)|}{\|A_i^{-T}\|_{\mathbb{M}^{n\times n}}}. \tag{3.61}$$

So finally we can write:

$$\int_{\Gamma_i} |\underline{u}^+(x)|\ d\mathcal{H}^{n-1}(x) \le C''(n, L, r)\left(\hat{\mu}_{\underline{u}}((\Gamma_i \times \mathbb{R}) \setminus J_{\underline{u}}) + \int_{\mathbb{R}^n} |\underline{u}(x)|\ dx\right), \tag{3.62}$$

where $C''(n, L, r)$ is a constant which depends only on n, L, r. Analogously we have the same inequality for $|\underline{u}^-|$, so that by summing on $i = 1, \ldots, N(n)$ we obtain:

$$\int_{\Gamma} |\underline{u}^\pm(x)|\ d\mathcal{H}^{n-1}(x) \le C(n, L, r)\left(\hat{\mu}_{\underline{u}}(\mathbb{R}^n \setminus J_{\underline{u}}) + \int_{\mathbb{R}^n} |\underline{u}(x)| dx\right), \tag{3.63}$$

which concludes the proof. □

Remark 3.16. A particular case of Theorem 3.12 and of point (b) of Proposition 3.15 is when $u \in GSBV_p^p(\Omega;\Gamma)^n$ $(p \ge 1)$. By definition every $u \in GSBV_p^p(\Omega)^n$ is a vector field in $L^p(\Omega, \mathbb{R}^n)$ whose approximate gradient ∇u belongs to $\in L^p(\Omega, \mathbb{M}^{n\times n})$. Therefore $GSBV_p^p(\Omega;\Gamma)^n \subset GSBD_p^p(\Omega;\Gamma)$. In particular Theorem 3.12 and point (b) of Corollary 3.15 apply to $GSBV_p^p(\Omega;\Gamma)^n$ with $\mathcal{E}u$ replaced by ∇u.

An alternative way to obtain a trace estimate without weight on $\mathcal{F}\Omega \cup \Gamma$ is to consider a suitable weight Ψ defined on Ω as explained in the next theorem:

Theorem 3.17. *Let Ω and Γ be as in Theorem 3.11. Then there exists an $\mathcal{L}^{n-1}$-measurable function $\Psi : \Omega \to \mathbb{R}^+$ depending only on the geometry of Γ, its orientation ν, and on Ω, such that denoting with $u^\pm$ the traces of u according to Definition 3.9 ,we have:*

(a) The function $\Psi \in L^1_{loc}(\Omega)$. In particular:

$$\int_B \Psi \; dx \le C(n)\Big(\mathcal{H}^{n-1}(\Gamma \cap B) + \mathcal{H}^{n-1}(\partial B) + \mathcal{L}^n(B)\Big), \tag{3.64}$$

for every ball $B \subset \Omega$.

(b) The following inclusions hold true:

(i) $GBD(\Omega;\Gamma) \cap L^1(\Omega, \Psi\mathcal{L}^n) \subset BD(\Omega;\Gamma)$ and

$$\int_{\Gamma\cup\mathcal{F}\Omega} |u^\pm| \; d\mathcal{H}^{n-1} \le C(n)\Big(\hat{\mu}_u(\Omega \setminus J_u) + \int_\Omega |u| \, \Psi dx\Big); \tag{3.65}$$

(ii) $GSBD^p_p(\Omega;\Gamma) \cap L^p(\Omega, \Psi\mathcal{L}^n) \subset SBD^p_p(\Omega;\Gamma)$ $(p \ge 1)$ and

$$\int_{\Gamma\cup\mathcal{F}\Omega} |u^\pm|^p \; d\mathcal{H}^{n-1} \le C(n,p)\Big(\int_\Omega |\mathcal{E}u|^p \; dx + \int_\Omega |u|^p \, \Psi dx\Big). \tag{3.66}$$

(c) Given $p < n$ let $p^* = np/(n-p)$ be the usual critical Sobolev exponent, and consider $u \in GSBD^p_p(\Omega;\Gamma) \cap L^{p^*}(\Omega) \cap L^{\frac{p(n-1)}{n-p}}(\Omega, \Psi\mathcal{L}^n)$. Then:

$$\int_{\Gamma\cup\mathcal{F}\Omega} |u^\pm|^{\frac{p(n-1)}{n-p}} \; d\mathcal{H}^{n-1} \le C(n,p)\Big(\int_\Omega |\mathcal{E}u|^p \; dx + \int_\Omega |u|^{\frac{p(n-1)}{n-p}} \, \Psi dx + \int_\Omega |u|^{p^*} \; dx\Big). \tag{3.67}$$

(d) If $\Gamma \cup \mathcal{F}\Omega$ satisfies the cone condition (see Definition 3.13) , then Ψ can be chosen in such a way that:

$$\operatorname*{ess\,sup}_{x\in\Omega} \Psi < \infty. \tag{3.68}$$

(e) If Γ is such that:

$$\int_\Omega \Psi^\gamma dx < \infty, \; \textit{for some } \gamma > 1, \tag{3.69}$$

then we have $GSBD(\Omega;\Gamma) \cap L^{\frac{\gamma}{\gamma-1}}(\Omega) \subset SBD(\Omega;\Gamma)$.

Remark 3.18. If $\Omega \subset \mathbb{R}^n$ is a lipschitz-regular bounded domain, and $\Gamma = \varnothing$, then clearly $\mathcal{F}\Omega (= \partial\Omega)$ satisfies the cone condition. Thanks to point (d) of the previous theorem, $\operatorname{ess\,sup}_{x\in\Omega} \Psi < \infty$, therefore (3.67) becomes:

$$\int_{\partial\Omega} |Tr(u)|^{\frac{p(n-1)}{n-p}} \; d\mathcal{H}^{n-1} \le C(n,p,\Gamma,\Omega)\Big(\int_\Omega |\mathcal{E}u|^p \; dx + \int_\Omega |u|^{p^*} \; dx\Big). \tag{3.70}$$

Moreover one can prove that on the open set Ω holds true a Sobolev-like inequality of the form:

$$\|u\|_{p^*} \le C(n,p,\Gamma,\Omega)(\|\mathcal{E}u\|_p + \|u\|_p). \tag{3.71}$$

This last inequality, together with (3.70), proves the $L^{\frac{p(n-1)}{n-p}}(\partial\Omega, \mathcal{H}^{n-1})$- integrability of the trace of u, which is the usual critical exponent for the trace of Sobolev functions in $W^{1,p}(\Omega)$.

Proof. (Theorem 3.12) Let $\underline{u} \in GBD(\mathbb{R}^n; \Gamma \cup \mathcal{F}\Omega)$ be the function extended to 0 outside of Ω as in proposition 2.20. In order to simplify the notation, we write Γ to denote $\Gamma \cup \mathcal{F}\Omega$, and ν to denote the orientation that coincides with the given orientation ν on Γ, and with $\nu_{\mathcal{F}\Omega}$ on $\mathcal{F}\Omega$.

By following the proofs of Theorem 3.11 and 3.12, thanks to our definitions of $u^\pm$ and by Proposition 2.20, we can prove the analogous of inequalities (3.65), (3.66), and (3.67) for the function $\underline{u}$.

We first prove (a) and (b): consider Λ the covering of $\mathbb{S}^{n-1}$ as in Remark 3.6 and by compactness we consider a subcovering $(C(\Xi_i,\delta))_{i=1}^N$. If we define for any $i = 1,\dots,N$, $\Gamma_i := \nu^{-1}\big(C(\Xi_i,\delta)\big)$ then $(\Gamma_i)_{i=1}^N$ is a finite measurable cover of Γ. Note that by definition of the covering Λ, for any $\xi \in \Xi_i$ we have $|\xi \cdot \nu(x)| > 1/\sqrt{n} - \delta$ for every $x \in \Gamma_i$.

Now we fix i and $\xi \in \Xi_i$. We write the generic point $x \in \mathbb{R}^n$ as $y + t\xi$ where $(y,t) \in \Pi^\xi \times \mathbb{R}$, and from now on we will work on the set of points $y \in \pi^\xi(\Gamma_i)$ such that $\hat{\underline{u}}^\xi_y \in BV_{loc}(\mathbb{R})$ and

$\mathcal{H}^0(\Gamma^\xi_y) < \infty$; from the Definition 2.7 of GBD and Remark 3.1 we already know that $\mathcal{H}^{n-1}$ almost all of y have these properties.

We call $(t_k)_{k=1}^{\mathcal{H}^0(\Gamma^\xi_y)}$ the point of the slicing Γ^ξ_y ordered such that $t_k < t_{k+1}$ for any k.

Since ξ has been fixed, in order to simplify the notation, we omit the dependence on ξ and write $\Gamma_i^+ := \Gamma_i \cap \{\nu \cdot \xi > 0\}$ and $\Gamma_i^- := \Gamma_i \cap \{\nu \cdot \xi < 0\}$. Let's focus for example on the set Γ_i^+. Proceeding exactly as in Theorem 3.11, we have for $\mathcal{H}^{n-1}$-a.e. $y \in \pi^\xi(\Gamma_i^+)$

$$\begin{gathered}(t_{k+1}-t_k)|(\underline{\hat{u}}^\xi_y)^+(t_k)| \le (t_{k+1}-t_k)\int_{t_k}^{t_{k+1}} d|D\underline{\hat{u}}^\xi_y| + \int_{t_k}^{t_{k+1}} |\underline{\hat{u}}^\xi_y(t)|\,dt,\\ \text{for } t_k \in \left(\Gamma_i^+\right)^\xi_y, \text{ and } t_{k+1}-t_k \le 1,\end{gathered} \tag{3.72}$$

and

$$\begin{gathered}|(\underline{\hat{u}}^\xi_y)^+(t_k)| \le \int_{t_k}^{1+t_k} d|D\underline{\hat{u}}^\xi_y| + \int_{t_k}^{1+t_k} |\underline{\hat{u}}^\xi_y(t)|\,dt,\\ \text{for } t_k \in (\Gamma_i^+)^\xi_y, \text{ and } t_{k+1}-t_k > 1.\end{gathered} \tag{3.73}$$

Since θ^ξ coincides with $t_{k+1}-t_k$ or 1 on the set $\{y+t\xi \mid t_k < t < t_{k+1}\}$, we can divide both sides of the previous inequality by θ^ξ and then we sum on $t_k \in \left(\Gamma_i^+\right)^\xi_y$ to get

$$\begin{aligned}\sum_{t_k}|(\underline{\hat{u}}^\xi_y)^+(t_k)| &\le \sum_{t_k}\left(\int_{t_k}^{t_k+\theta^{\xi^+}(y+t_k\xi)} d|D\underline{\hat{u}}^\xi_y| + \int_{t_k}^{t_k+\theta^{\xi^+}(y+t_k\xi)} \frac{|\underline{\hat{u}}^\xi_y(t)|}{\theta^\xi(y+t\xi)}dt\right)\\ &\le |D\underline{\hat{u}}^\xi_y|(\mathbb{R}\setminus J_{\underline{\hat{u}}^\xi_y}) + \sum_{t_k}\int_{t_k}^{t_k+\theta^{\xi^+}(y+t_k\xi)} \frac{|\underline{\hat{u}}^\xi_y(t)|}{\theta^\xi(y+t\xi)}dt.\end{aligned} \tag{3.74}$$

The term in the left hand-side, and the last two addends on the right hand-side of (3.74) are measurable functions of y (as explained in the proof of Theorem 3.11). By Theorem 2.16 $J_{\underline{\hat{u}}^\xi_y} = (J_{\underline{u}})^\xi_y$ for a.e. y, so by integrating over $\pi^\xi(\Gamma_i^+)$:

$$\begin{aligned}\int_{\pi^\xi(\Gamma_i^+)}\sum_{t_k}|(\underline{\hat{u}}^\xi_y)^+(t_k)|\,dy &\le \int_{\pi^\xi(\Gamma_i^+)}|D\underline{\hat{u}}^\xi_y|(\mathbb{R}\setminus(J_{\underline{u}})^\xi_y)\,dy\\ &+ \int_{\pi^\xi(\Gamma_i^+)}\left(\sum_{t_k\in(\Gamma_i^+)^\xi_y}\int_{t_k}^{t_k+\theta^{\xi^+}(y+t_k\xi)} \frac{|\underline{u}(y+t\xi)|}{\theta^\xi(y+t\xi)}\,dt\right)dy.\end{aligned} \tag{3.75}$$

Again by arguing as in the proof of Theorem 3.11, we find the same inequality on the set Γ_i^-, then by means of the Coarea formula on the rectifiable set Γ_i applied to the projection π^ξ, and by summing on every directions in Ξ_i, we get:

$$\int_{\Gamma_i}|\underline{u}^+(x)|\,d\mathcal{H}^{n-1}(x) \le C(n)\left(\hat{\mu}_{\underline{u}}(\mathbb{R}^n\setminus J_{\underline{u}}) + \int_{\mathbb{R}^n}\sum_{\xi_j\in\Xi_i}\frac{|\underline{u}(x)|}{\theta^{\xi_j}}(x)\,dx\right). \tag{3.76}$$

Now define $\Psi^i, \Psi : \mathbb{R}^n \to \mathbb{R}^+$ as:

$$\Psi^i(x) := \sum_{\xi_j\in\Xi_i}\frac{1}{\theta^{\xi_j}(x)} \text{ and } \Psi(x) := \sum_{i=1}^{N(n)}\Psi^i(x). \tag{3.77}$$

To prove (a) it is enough to notice that for each $\xi \in \mathbb{S}^{n-1}$ and for each ball $B \subset \mathbb{R}^n$ we have:

$$\begin{aligned}\int_B \frac{1}{\theta^\xi}\,dx &= \int_{\pi^\xi(B)}\left(\int_{B^\xi_y}\frac{1}{\theta^\xi(y+t\xi)}\,dt\right)dy\\ &= \int_{\pi^\xi(B)}\left(\sum_{t_k\in((\Gamma\cap B)\cup\partial B)^\xi_y}\frac{t_{k+1}-t_k}{\theta^\xi(y+t\xi)}\right)dy\\ &\le \int_{\pi^\xi(B)}\left(\mathcal{H}^0((\Gamma\cap B)^\xi_y)+1+\mathcal{H}^1(B^\xi_y)\right)dy\\ &\le \mathcal{H}^{n-1}(\Gamma\cap B)+\mathcal{H}^{n-1}(\partial B)+\mathcal{L}^n(B),\end{aligned} \tag{3.78}$$

Hence $\Psi \in L^1_{loc}(\Omega)$. By summing on $i = 1, \ldots, N(n)$ inequality (3.76) becomes:

$$\int_\Gamma |\underline{u}^+(x)|\,d\mathcal{H}^{n-1}(x) \le C'(n)\left(\hat{\mu}_{\underline{u}}(\mathbb{R}^n\setminus J_{\underline{u}}) + \int_{\mathbb{R}^n}|\underline{u}(x)|\,\Psi(x)dx\right). \tag{3.79}$$

Analogously we can prove the same inequality for the trace from below:

$$\int_\Gamma |\underline{u}^-(x)|\, d\mathcal{H}^{n-1}(x) \le C'(n)\bigg(\hat{\mu}_{\underline{u}}(\mathbb{R}^n \smallsetminus J_{\underline{u}}) + \int_{\mathbb{R}^n} |\underline{u}(x)|\,\Psi(x)dx\bigg). \tag{3.80}$$

Thanks to Proposition 2.20, (3.79) and (3.80) are exactly (3.65). In particular this means that the jump function $[u](x) = u^+(x) - u^-(x)$ belongs to $L^1(J_u, \mathcal{H}^{n-1})$, and as a consequence that $u \in BD(\Omega)$ (see Remark 2.17).

In order to pass to the L^p-norm in (3.66), we can proceed as in the proof of Theorem 3.12. Then by arguing as in the previous proof of inequality (3.65), we get also (ii) of point (b).

To prove (c) fix i and $\xi \in \Xi_i$. Notice that for $\mathcal{H}^{n-1}$-a.e. $y \in \pi^\xi(\Gamma_i)$ we have all the properties mentioned in the first lines of this proof and moreover $\hat{\underline{u}}^\xi_y \in L^{p^*}(\mathbb{R})$, $\nabla \hat{\underline{u}}^\xi_y \in L^p(\mathbb{R})$. So we elevate the one dimensional sections $\hat{\underline{u}}^\xi_y$ to the power $p(n-1)/(n-p)$ and we notice that for $\mathcal{H}^{n-1}$-a.e. y we have $\hat{\underline{u}}^\xi_y \in W^{1,p}\big((t_k, t_{k+1})\big)$. Thus by means of the *chain rule formula* we get:

$$\begin{aligned}(t_{k+1} - t_k)|(\hat{\underline{u}}^\xi_y)^+(t_k)|^{\frac{p(n-1)}{n-p}} &\le (t_{k+1} - t_k)\frac{p(n-1)}{n-p}\int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(r)|^{\frac{p(n-1)}{n-p}-1}|\nabla \hat{\underline{u}}^\xi_y(r)|\, dr \\ &+ \int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(t)|^{\frac{p(n-1)}{n-p}}\, dt, \\ &\text{for } t_k \in \big(\Gamma_i^+\big)^\xi_y, \text{ and } t_{k+1} - t_k \le 1,\end{aligned} \tag{3.81}$$

and:

$$\begin{aligned}|(\hat{\underline{u}}^\xi_y)^+(t_k)|^{\frac{p(n-1)}{n-p}} &\le \frac{p(n-1)}{n-p}\int_{t_k}^{1+t_k} |\hat{\underline{u}}^\xi_y(r)|^{\frac{p(n-1)}{n-p}-1}|\nabla \hat{\underline{u}}^\xi_y(r)|\, dr \\ &+ \int_{t_k}^{1+t_k} |\hat{\underline{u}}^\xi_y(t)|^{\frac{p(n-1)}{n-p}}\, dt, \\ &\text{for } t_k \in \big(\Gamma_i^+\big)^\xi_y, \text{ and } t_{k+1} - t_k > 1,\end{aligned} \tag{3.82}$$

Hölder's inequality with exponents $p/(p-1)$ and p, and then Young's inequality with the same exponents yelds to:

$$\begin{aligned}\int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(r)|^{\frac{p(n-1)}{n-p}-1}|\nabla \hat{\underline{u}}^\xi_y(r)|\, dr &\le \bigg(\int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(r)|^{p^*}\, dr\bigg)^{\frac{p}{p-1}}\bigg(\int_{t_k}^{t_{k+1}} |\nabla \hat{\underline{u}}^\xi_y(r)|^p\, dr\bigg)^{\frac{1}{p}} \\ &\le \frac{p}{p-1}\int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(r)|^{p^*}\, dr + \frac{1}{p}\int_{t_k}^{t_{k+1}} |\nabla \hat{\underline{u}}^\xi_y(r)|^p\, dr,\end{aligned} \tag{3.83}$$

First we can use inequality (3.83) to estimate the first term in the right hand side of (3.81) and of (3.82), then we can argue in the same way as in the proof of (b) in order to get (c).

The proof of (d) is similar to the one of Theorem 3.15 always starting from inequalities (3.72) and (3.73).

Finally we prove (e). It is enough to apply Hölder inequality with the two conjugate exponents γ and $\gamma/(\gamma-1)$ to the integral on the right hand side of (3.65):

$$\int_\Omega |u(x)|\Psi(x)\, dx \le \bigg(\int_\Omega \Psi(x)^\gamma\, dx\bigg)^{\frac{1}{\gamma}}\bigg(\int_\Omega |u(x)|^{\frac{\gamma}{\gamma-1}}\, dx\bigg)^{\frac{\gamma-1}{\gamma}}. \tag{3.84}$$

□

Remark 3.19. Under hypothesis of Theorem (3.17) inequalities (3.66), (3.67), statements (d) and (e), hold true in $GSBV^p_p(\Omega;\Gamma)^n$ with the full approximate gradient instead of the symmetric one as specified in Remark 3.16.

4. Convergence of trace in measure

This section is devoted to prove a fundamental result about the continuity of the trace operator. We will show that the trace operator acting on the space $GSBD^p_p(\Omega;\Gamma)$, is continuous in measure with respect to the notion of convergence (4.15). This result, together with our previous trace inequalities, allow us to deduce the continuity properties of the trace cited so far in the introduction.

For convenience of the reader, we remind the notion of convergence in measure:

Definition 4.1. (Convergence in measure). Let $\mu \in \mathcal{M}_b^+(\mathbb{R}^n)$ a bounded positive Radon measure. Consider $(v_i)_{i\in\mathbb{N}}$ $v_i:\mathbb{R}^n \to \mathbb{R}$ a sequence of μ-measurable functions and let $v:\mathbb{R}^n \to \mathbb{R}$ be a μ-measurable function. Then the v_i converge to v in μ-measure, if for any $\epsilon > 0$ and $\delta > 0$ there exists an index $\bar{i} \in \mathbb{N}$ such that:

$$\mu\Big(\{x \in \mathbb{R}^n \mid |v_i(x) - v(x)| > \epsilon\}\Big) \le \delta, \quad \forall i \ge \bar{i}. \tag{4.1}$$

Remark 4.2. If v_i converge to v in measure, then there exists a subsequence v_{i_j} that converges to v pointwise μ-a.e. Moreover if $\mu \in \mathcal{M}_b^+(\mathbb{R}^n)$ is concentrated on A, and $(A_j)_{j\in\mathbb{N}}$ is a μ-measurable covering of A, in order to check the convergence in measure, it is enough to check the convergence in measure of the $v_i \llcorner A_j$ to $v \llcorner A_j$, for each $j = 1, 2, \ldots$.

Now we introduce the notation for the truncation functions:

Definition 4.3. (Truncated function). Let a, b be two real numbers. Define $\sigma_a, \sigma_b:\mathbb{R} \to \mathbb{R}$ to be the truncation function from below at level a, respectively at level b, as:

$$\sigma_a(t) := \begin{cases} a \text{ if } & t < a \\ t \text{ if } & t \ge a, \end{cases} \quad \sigma^b(t) := \begin{cases} t \text{ if } & t < b \\ b \text{ if } & t \ge b. \end{cases} \tag{4.2}$$

Define $\sigma_a^b:\mathbb{R} \to \mathbb{R}$ to be the truncation function from below and above at level a and b $(a < b)$, as:

$$\sigma_a^b(t) := \begin{cases} a \text{ if } & t < a \\ t \text{ if } & a \le t < b, \\ b \text{ if } & t \ge b. \end{cases} \tag{4.3}$$

Proposition 4.4. *Let $\mu \in \mathcal{M}_b^+(\mathbb{R}^n)$ be a bounded positive Radon measure. Let $(v_j)_{j\in\mathbb{N}}$, $v_j:\mathbb{R}^n \to \mathbb{R}$, be a sequence of μ-measurable functions and let $v:\mathbb{R}^n \to \mathbb{R}$ be a μ-measurable function. Suppose that for any $a < b$ holds:*

$$\sigma_a^b(v_j) \rightharpoonup \sigma_a^b(v) \text{ for } j \to \infty, \text{ weakly* in } L^\infty(\mathbb{R}^n, \mu). \tag{4.4}$$

Then the v_j converge to v in measure.

Proof. First of all fix two positive parameters ϵ and δ as in Definition 4.1. Then find $M > 0$ big enough such that $\mu\big(\mathbb{R}^n \smallsetminus \{-M \le v < M\}\big) \le \delta/2$ (this is possible because μ is a finite measure). To simplify the notation we write $V_M := \{-M \le v < M\}$.

Let $\gamma := \min\{\frac{c\epsilon\delta}{2}, \frac{\epsilon}{2}\}$, where $c = \frac{1}{4\mu(\mathbb{R}^n)}$, and consider a partition of $[-M, M)$ made of interval of the form $[t_i, t_{i+1})$, such that $t_{i+1} - t_i = \gamma$, for any $i = 1, \ldots, 2M/\gamma$ (we may suppose that $M = \gamma \cdot N$ where N is a sufficiently large natural number).

Define for any i the set $A_i := v^{-1}([t_i, t_{i+1}))$ and $\bar{t}_i := (t_{i+1} + t_i)/2$ the middle point between t_i and t_{i+1}. Notice that by triangular inequality and by recalling that $\gamma \le \epsilon$:

$$\{|v_j - v| > \epsilon\} \cap A_i \subseteq \{|v_j - \bar{t}_i| > \frac{\epsilon}{2}\} \cap A_i \text{ for } i = 1, \ldots, \frac{2M}{\gamma}, \text{ and } j \in \mathbb{N}. \tag{4.5}$$

This means that:

$$\begin{aligned} \mu\Big(\{|v_j - v| > \epsilon\} \cap V_M\Big) &\le \sum_{i=1}^{2M/\gamma} \mu\big(\{|v_j - \bar{t}_i| > \frac{\epsilon}{2}\} \cap A_i\big) \\ = \sum_{i=1}^{2M/\gamma} &\Big[\mu\Big(\{v_j - \bar{t}_i > \frac{\epsilon}{2}\} \cap A_i\Big) + \mu\Big(\{v_j - \bar{t}_i < -\frac{\epsilon}{2}\} \cap A_i\Big)\Big]. \end{aligned} \tag{4.6}$$

For every i, let us introduce the function:

$$\frac{2[\sigma_{\bar{t}_i}^{\bar{t}_i+\epsilon/2}(v_j) - \bar{t}_i]}{\epsilon} = \begin{cases} 0 & \text{if } v_j - \bar{t}_i < 0 \\ 1 & \text{if } v_j - \bar{t}_i \ge \frac{\epsilon}{2} \\ \frac{2}{\epsilon}(v_j - \bar{t}_i) & \text{if } 0 \le v_j - \bar{t}_i < \frac{\epsilon}{2}. \end{cases} \tag{4.7}$$

For every i and j we have:

$$\mathbb{1}_{\{v_j - \bar{t}_i > \epsilon/2\}}(x) \le \frac{2[\sigma_{\bar{t}_i}^{\bar{t}_i+\epsilon/2}(v_j(x)) - \bar{t}_i]}{\epsilon}, \; \forall x \in \mathbb{R}^n. \tag{4.8}$$

We claim that for every $i = 1, \dots, 2M/\gamma$, there exists a $j(i) \in \mathbb{N}$ (depending on i) such that for any $j > j(i)$:

$$\int_{A_i} \frac{2[\sigma_{\bar{t}_i}^{\bar{t}_i+\epsilon/2}(v_j(x)) - \bar{t}_i]}{\epsilon}\, d\mu(x) \le c\delta\mu(A_i), \tag{4.9}$$

where $c = \frac{1}{4\mu(A)}$. In fact using the hypothesis of weak convergence at any level of truncation we can write:

$$\begin{aligned}
&\limsup_{j\to\infty} \int_{A_i} \frac{2[\sigma_{\bar{t}_i}^{\bar{t}_i+\epsilon/2}(v_j(x)) - \bar{t}_i]}{\epsilon}\, d\mu(x) = \\
&= \limsup_{j\to\infty} \Bigg(\int_{A_i} \frac{2[\sigma_{\bar{t}_i}^{\bar{t}_i+\epsilon/2}(v_j(x)) - \sigma_{\bar{t}_i}^{\bar{t}_i+\epsilon/2}(v(x))]}{\epsilon}\, d\mu(x) \\
&\qquad + \int_{A_i} \frac{2[\sigma_{\bar{t}_i}^{\bar{t}_i+\epsilon/2}(v(x)) - \bar{t}_i]}{\epsilon}\, d\mu(x) \Bigg) \\
&\le \int_{A_i} \frac{2[\sigma_{\bar{t}_i}^{\bar{t}_i+\epsilon/2}(v(x)) - \bar{t}_i]}{\epsilon}\, d\mu(x) \le \frac{2\gamma}{\epsilon}\mu(A_i) < c\delta\mu(A_i).
\end{aligned} \tag{4.10}$$

Now define:

$$\tilde{j} := \max\{j(i) \mid i = 1, \dots, 2M/\gamma\}, \tag{4.11}$$

hence the following estimate holds true:

$$\sum_{i=1}^{2M/\gamma} \mu(\{v_j - \bar{t}_i > \frac{\epsilon}{2}\} \cap A_i) < \sum_{i=1}^{2M/\gamma} c\delta\mu(A_i) \le c\delta\mu(\mathbb{R}^n) = \frac{\delta}{4}, \quad \forall j \ge \tilde{j}. \tag{4.12}$$

Analogously we repeat the same argument for the other addends of (4.6). So finally we have:

$$\mu(\{|v_j - v| > \epsilon\} \cap V_M) \le \delta/2. \tag{4.13}$$

By using (4.13) and by the definition of M, for any $j > \tilde{j}$ we get:

$$\begin{aligned}
\mu(\{|v_j - v| > \epsilon\}) &= \mu\big(\{|v_j - v| > \epsilon\} \cap V_M\big) + \mu\big(\{|v_j - v| > \epsilon\} \setminus V_M\big) \\
&\le \frac{\delta}{2} + \mu(\mathbb{R}^n \setminus V_M) \\
&\le \frac{\delta}{2} + \frac{\delta}{2} = \delta,
\end{aligned} \tag{4.14}$$

obtaining the desired estimate. □

In the case of vector fields having bounded deformation there is a notion of convergence analogous to the one given in the introduction in the SBV context [3]. As we mentioned so far, this notion of convergence is useful in order to ensure the compactness for suitable minimizing sequences in several minimization problems that come out in the framework of variational models for fracture mechanics.

When we will speak about continuity of the traces, we will always refer to the following notion of convergence:

Definition 4.5. Let $(u_i)_{i\in\mathbb{N}}$ be a sequence in $GSBD^p_p(\Omega)$, and let $u \in GSBD^p_p(\Omega)$. We say that the sequance $(u_i)_i$ converges to u if and only if there exists a constant $C > 0$ such that the following three conditions hold true:

$$\begin{cases} \sup_i \big(\|u_i\|_p + \|\mathcal{E}u_i\|_p + \mathcal{H}^{n-1}(J_{u_i})\big) \le C \\ u_i \to u \text{ in } L^1(\Omega), \text{ as } i \to \infty \\ \mathcal{E}u_i \rightharpoonup \mathcal{E}u \text{ weakly in } L^1(\Omega), \text{ as } i \to \infty. \end{cases} \tag{4.15}$$

Definition 4.6. If Γ is a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with orientation ν, for every $\xi \in \mathbb{S}^{n-1}$ we define the set $\Gamma^\xi := \{x \in \Gamma \mid \nu(x) \cdot \xi \neq 0\}$.

We are now in position to prove our result about the convergence of traces in measure.

[3] See the introduction.

Theorem 4.7. *(Convergence in measure). Let $\Omega \subset \mathbb{R}^n$ be an open set of finite perimeter, and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with finite measure oriented by ν. Let $(u_i)_{i\in\mathbb{N}}$ be a sequence converging to $u \in GSBD^p_p(\Omega;\Gamma)$ $(p \geq 1)$ with respect to the convergence (4.15), then $(u_i)^{\pm}_{\Gamma\cup\mathcal{F}\Omega}$ converge in measure (with respect to $\mathcal{H}^{n-1} \llcorner \Gamma \cup \mathcal{F}\Omega$) to $u^{\pm}_{\Gamma\cup\mathcal{F}\Omega}$.*

Remark 4.8. Just to simplify the notation we prefer to give the proof of the previous theorem when Ω is the entire space $\mathbb{R}^n$. Using the extension argument given by proposition 2.20 the same argument works for the general case.

Proof. Thanks to Proposition 2.14 there exists a countable family of bounded open sets of finite perimeter, say $\{U_j\}_{j=1}^{\infty}$, such that

$$\Gamma \subset \bigcup_{j=1}^{\infty} \mathcal{F}U_j \tag{4.16}$$

up to a $\mathcal{H}^{n-1}$-negligible set. Hence, by Remark 4.2, in order to prove our statement we can reduce ourselves to prove that for every $j \in \mathbb{N}$, $(u_i)^{\pm}_{\Gamma}$ converges in $\mathcal{H}^{n-1} \llcorner (\Gamma \cap \mathcal{F}U_j)$-measure to $u^{\pm}_{\Gamma}$. Because of the fact that:

$$(u_i)^{+}_{\Gamma}(x) = \begin{cases} (u_i)^{+}_{\mathcal{F}U_j}(x) & \text{if } \nu_\Gamma(x) = \nu_{\mathcal{F}U_j}(x) \\ (u_i)^{-}_{\mathcal{F}U_j}(x) & \text{if } \nu_\Gamma(x) = -\nu_{\mathcal{F}U_j}(x), \end{cases}$$

up to a measurable change of sign of ν_Γ, it is equivalent to prove that $(u_i)^{\pm}_{\mathcal{F}U_j}$ converges to $u^{\pm}_{\mathcal{F}U_j}$ in $\mathcal{H}^{n-1} \llcorner \mathcal{F}U_j$-measure.
Now we fix $j \in \mathbb{N}$ and we prove that for any $\xi \in \mathbb{S}^{n-1}$, $(u_i)^{+}_{\mathcal{F}U_j} \cdot \xi$ converges to $u^{+}_{\mathcal{F}U_j} \cdot \xi$ in $\mathcal{H}^{n-1} \llcorner \mathcal{F}U^{\xi}_j$-measure, and to simplify the notation, we denote $u^+_i := (u_i)^+_{\mathcal{F}U_j}$ and $u^+ := u^+_{\mathcal{F}U_j}$.

By Proposition 4.4 it is enough to show that given any pair $a, b \in \mathbb{R}$ with $a < b$ we have:

$$\sigma^b_a(u^+_i \cdot \xi) \rightharpoonup \sigma^b_a(u^+ \cdot \xi), \text{ weakly* in } L^\infty(\mathcal{F}U^{\xi}_j, \mathcal{H}^{n-1}). \tag{4.17}$$

For each $i \in \mathbb{N}$ let $\underline{u}_i$ and $\underline{u}$ be the functions extended to zero outside of U_j (see Proposition 2.20). We know that $\underline{u}^+_i = u^+_i$ and $\underline{u}^+ = u^+$ on $\mathcal{F}U_j$, so we can prove our assertion for the sequence $(\underline{u}_i)_{i\in\mathbb{N}}$.

By hypothesis we have that $\underline{u}_i \to \underline{u}$ strongly in $L^1(\mathbb{R}^n, \mathbb{R}^n)$. As a consequence also $\sigma^b_a(\underline{u}_i \cdot \xi) \to \sigma^b_a(\underline{u} \cdot \xi)$ strongly in $L^1(\mathbb{R}^n)$ and in particular this means that:

$$D_\xi \sigma^b_a(\underline{u}_i \cdot \xi) \rightharpoonup D_\xi \sigma^b_a(\underline{u} \cdot \xi) \text{ in } \mathcal{D}'(\mathbb{R}^n), \tag{4.18}$$

in the sense of distributions. Moreover we have the bound on the total variations along the direction ξ:

$$\begin{aligned} \sup_{i\in\mathbb{N}} |D_\xi \sigma^b_a(\underline{u}_i \cdot \xi)|(\mathbb{R}^n) &\leq \sup_{i\in\mathbb{N}} \int_{\mathbb{R}^n} |(\mathcal{E}(\underline{u}_i)\xi,\ \xi)|\ dx + |b-a| \mathcal{H}^{n-1}\big((\Gamma^\xi \cap U_j) \cup \mathcal{F}U^{\xi}_j\big) \\ &= \int_{U_j} |(\mathcal{E}(u_i)\xi,\ \xi)|\ dx + |b-a| \mathcal{H}^{n-1}\big((\Gamma^\xi \cap U_j) \cup \mathcal{F}U^{\xi}_j\big) \\ &\leq \sup_{i\in\mathbb{N}} \mathcal{L}^n(U_j)^{1-\frac{1}{p}} \left(\int_{U_j} |\mathcal{E}u_i|^p\ dx \right)^{\frac{1}{p}} + |b-a| \mathcal{H}^{n-1}\big((\Gamma^\xi \cap U_j) \cup \mathcal{F}U^{\xi}_j\big) \\ &< +\infty. \end{aligned} \tag{4.19}$$

Hence the convergence in (4.18) still holds true in the weak sense of bounded Radon measure. Since by hypothesis $\mathcal{E}(\underline{u}_i) \rightharpoonup \mathcal{E}(\underline{u})$ weakly in $L^1(\mathbb{R}^n; \mathbb{M}^{n\times n}_{sym})$, we can write:

$$D_\xi \sigma^b_a(\underline{u}_i \cdot \xi) - \mathcal{E}(\underline{u}_i)\xi \cdot \xi \mathcal{L}^n \rightharpoonup D_\xi \sigma^b_a(\underline{u} \cdot \xi) - \mathcal{E}(\underline{u})\xi \cdot \xi \mathcal{L}^n \text{ weakly in } \mathcal{M}_b(\mathbb{R}^n), \tag{4.20}$$

and it follows:

$$[\sigma^b_a(\underline{u}_i \cdot \xi)]\xi \cdot \nu \mathcal{H}^{n-1} \rightharpoonup [\sigma^b_a(\underline{u} \cdot \xi)]\xi \cdot \nu \mathcal{H}^{n-1} \text{ weakly in } \mathcal{M}_b(\mathbb{R}^n). \tag{4.21}$$

On the other hand, thanks to the truncation between a and b, the sequence $([\sigma^b_a(\underline{u}_i \cdot \xi)])_{i\in\mathbb{N}}$ is relatively sequentially compact in the weak* topology of L^∞, and call for example α one of its limits. Given any $\phi \in L^1\big(\mathbb{R}^n, \mathcal{H}^{n-1} \llcorner [(\Gamma \cap U_j) \cup \mathcal{F}U_j]\big)$ we can use $\phi\xi \cdot \nu$ as test function in the weak* convergence:

$$\lim_{i_k\to\infty} \int_{(\Gamma\cap U_j)\cup\mathcal{F}U_j} [\sigma^b_a(\underline{u}_{i_k} \cdot \xi)]\phi\xi \cdot \nu\ d\mathcal{H}^{n-1}(x) = \int_{(\Gamma\cap U_j)\cup\mathcal{F}U_j} \alpha\phi\xi \cdot \nu\ d\mathcal{H}^{n-1}(x), \tag{4.22}$$

this together with (4.21) means that every weak* limits α is equal to $[\sigma_a^b(\underline{u}\cdot\xi)]$ on the set $(\Gamma^\xi \cap U_j)\cup \mathcal{F}U_j^\xi$.

Recall that by Remark 2.21 $\underline{u}_i^- = 0$ a.e. on $\mathcal{F}U_j$, and by Proposition 2.20 $\underline{u}_i^+ = u_i^+$ a.e. on $\mathcal{F}U_j$, hence for every $i \in \mathbb{N}$:

$$[\sigma_a^b(\underline{u}_i\cdot\xi)] = \sigma_a^b(u_i^+\cdot\xi),\ \mathcal{H}^{n-1}\text{-a.e. on } \mathcal{F}U_j,$$

and also:

$$[\sigma_a^b(\underline{u}\cdot\xi)] = \sigma_a^b(u^+\cdot\xi),\ \mathcal{H}^{n-1}\text{-a.e. on } \mathcal{F}U_j.$$

Therefore:

$$\sigma_a^b(u_i^+\cdot\xi) \rightharpoonup \sigma_a^b(u^+\cdot\xi) \text{ weakly* in } L^\infty(\mathcal{F}U_j^\xi,\mathcal{H}^{n-1}). \tag{4.23}$$

Using $\mathbb{R}^n \setminus U_j$ instead of U_j we can prove in the very same way that:

$$\sigma_a^b(u_i^-\cdot\xi) \rightharpoonup \sigma_a^b(u^-\cdot\xi) \text{ weakly-* in } L^\infty(\mathcal{F}U_j^\xi,\mathcal{H}^{n-1}).$$

Thanks to the arbitrariness of $\xi \in \mathbb{S}^{n-1}$, we can use the argument of Remark 3.6 to deduce:

$$\sigma_a^b(u_i^\pm) \rightharpoonup \sigma_a^b(u^\pm) \text{ weakly-* in } L^\infty(\mathcal{F}U_j,\mathcal{H}^{n-1}),$$

and thanks to the arbitrariness of $a,b \in \mathbb{R}$, by Proposition 4.4 we have:

$$u_i^\pm \to u^\pm \text{ in } \mathcal{H}^{n-1} \llcorner \mathcal{F}U_j\text{-measure},$$

which is our desired result. □

Remark 4.9. Let Ω and Γ be as in Theorem 4.7. As explained in Remark 3.16 we have the following inclusion $GSBV_p^p(\Omega;\Gamma)^n \subset GSBD_p^p(\Omega;\Gamma)$, hence thanks to Theorem 4.7, if $(u_i)_{i\in\mathbb{N}} \subset GSBV_p^p(\Omega;\Gamma)$ converges to $u \in GSBV_p^p(\Omega;\Gamma)$ with respect to the following notion of convergence:

$$\begin{cases} \sup_i \left(\|u_i\|_p + \|\nabla u_i\|_p + \mathcal{H}^{n-1}(J_{u_i})\right) \le C \\ u_i \to u, \text{ in } L^1(\Omega) \\ \nabla u_i \rightharpoonup \nabla u, \text{ weakly in } L^1(\Omega), \end{cases} \tag{4.24}$$

then $(u_i)^\pm_{\Gamma\cup\mathcal{F}\Omega}$ converges in $\mathcal{H}^{n-1} \llcorner (\Gamma\cup\mathcal{F}\Omega)$-measure with respect to $u^\pm_{\Gamma\cup\mathcal{F}\Omega}$.

5. Continuity of the trace and an application

Now we summarize our previous results, Theorems 3.12, 3.17, and 4.7, into the following theorem:

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set of finite perimeter, and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set, with $\mathcal{H}^{n-1}(\Gamma) < \infty$ and oriented by ν. Consider the space $GSBD_p^p(\Omega;\Gamma)$ $(p > 1)$ endowed with the notion of convergence (4.15), the functions $\Theta^\pm$ defined in Theorem 3.12 and Ψ given in Theorem 4.7. Then:*

(a) the trace operators from above and from below:

$$(\cdot)^\pm_{\Gamma\cup\mathcal{F}\Omega}: GSBD_p^p(\Omega;\Gamma) \to L^q(\Gamma\cup\mathcal{F}\Omega, \Theta^\pm\mathcal{H}^{n-1})\ (p>1), \tag{5.1}$$

are strongly continuous for every $q \in [1,p)$ and weakly continuous for every $q \in [1,p]$ if $p > 1$.

(b) if we add the uniform bound on the $\|\cdot\|_{L^p(\Omega,\Psi\mathcal{L}^n)}$-norm along the sequence in the notion of convergence (4.15), the trace operators from above and from below:

$$(\cdot)^\pm_{\Gamma\cup\mathcal{F}\Omega}: GSBD^p(\Omega;\Gamma) \cap L^p(\Omega,\Psi\mathcal{L}^n) \to L^q(\Gamma\cup\mathcal{F}\Omega, \mathcal{H}^{n-1})\ (p>1), \tag{5.2}$$

are strongly continuous for every $q \in [1,p)$ and weakly continuous for every $q \in [1,p]$.

Remark 5.2. By Remarks 3.16, 3.19 and 4.9, the previous theorem applies also to the space $GSBV_p^p(\Omega;\Gamma)$. Moreover the continuity properties of the trace operators mentioned so far in the introduction, are simply a consequence of this previous theorem, when we restrict our attention on $\partial\Omega$. In fact when Ω is lipschitz regular the reduced boundary $\mathcal{F}\Omega$ coincides with the topological boundary $\partial\Omega$, and our notion of trace operator coincides with the usual one.

Proof of Theorem 5.1. It is a consequence of the convergence in measure of the traces given in Theorem 4.7 plus estimate (3.33) to prove (a), and estimate (3.66) to prove (b). □

Now we give a counterexample to the strong continuity of the trace operator in (5.1) when $q = p$:

Example 5.3. *Consider in* $\mathbb{R}^2$ *the set*

$$E := \bigcup_{n=1}^{\infty}\left(\left[-\frac{1}{2n^2},\frac{1}{2n^2}\right]^2 + (n,0)\right),$$

made of infinitely many square E_n *of length* $1/n^2$ *and centered at* $(n,0)\in\mathbb{R}^2$*. Clearly* E *is a set of finite perimeter so we can choose as* Γ *its reduced boundary* $\mathcal{F}E$ *oriented with respect to its inner theoretical unit normal* ν_E*. Define the sequence of functions* $(u_n)_{n=1}^{\infty}\subset GSBD^2_2(\Omega;\Gamma)$ *as*

$$u_n(x) := \frac{1}{\sqrt{\mathcal{L}^2(E_n)}}\mathbb{1}_{E_n}(x) \text{ for every } n,$$

and notice that $\|u_n\|_2 = 1$ *for any* n*.*
Clearly the trace functions u_n^+ *converges pointwise* $(\Theta^+\mathcal{H}^1 \llcorner \Gamma)$*-a.e. to* 0 *for any choice of* Θ^+ *i.e. for any choice of an orthogonal basis* $\{\xi_1,\xi_2\}$ *of* $\mathbb{R}^2$ *as in* (3.43)*. This means that any strong* $L^2(\Gamma,\Theta^+\mathcal{H}^1)$ *limit of* u_n^+ *must be the zero function. But we claim that for each choice of* Θ^+ *as in* (3.43) *we have that*

$$\int_\Gamma |u_n^+|^2\Theta^+\, d\mathcal{H}^1 \geq C(\Theta^+)\|u_n\|_2^2 > 0,$$

where $C(\Theta^+)$ *is a strictly positive constant which depends only on* Θ^+*, which is a contradiction. Remember that in order to construct* Θ^+*, we divide* Γ *in finitely many parts* $(\Gamma_i)_{i=1}^N$*, and we associate to each* Γ_i *an orthonormal basis* $\{\xi_1^i,\xi_2^i\}$ *such that* $|\xi_1^i\cdot\nu_E(x)|, |\xi_2^i\cdot\nu_E(x)| > \frac{1}{\sqrt{2}}$ *for* $x\in\Gamma_i$*. This means that for each* n *there exists* $i(n)\in\{1,\dots,N\}$*, such that:*

$$\mathcal{H}^1(\mathcal{F}E_n\cap\Gamma_{i(n)}) \geq \frac{\mathcal{H}^1(\mathcal{F}E_n)}{N}. \tag{5.3}$$

Eventually passing through a sub-sequence we may suppose for example that for every n*:*

$$\mathcal{H}^1\left(\left\{x\in\mathcal{F}E_n\cap\Gamma_{i(n)} \mid \theta^{\xi_1^{i(n)}} \leq \theta^{\xi_2^{i(n)}}\right\}\right) \geq \frac{1}{2}\mathcal{H}^1(\mathcal{F}E_n\cap\Gamma_{i(n)}). \tag{5.4}$$

To simplify the notation we omit the dependence on n *and we write* Γ_i*,* ξ_1^i*,* ξ_2^i *to denote respectively* $\Gamma_{i(n)}$*,* $\xi_1^{i(n)}$*,* $\xi_2^{i(n)}$*; so we have:*

$$\begin{aligned}\int_\Gamma |u_n^+|^2\Theta^+\, d\mathcal{H}^1 &\geq \int_{\mathcal{F}E_n\cap\Gamma_i}|u_n^+(x)|^2\Theta^+(x)\, d\mathcal{H}^1(x)\\ &\geq \frac{1}{\mathcal{L}^2(E_n)}\int_{\mathcal{F}E_n\cap\Gamma_i\cap\{\theta^{\xi_1^i}\leq\theta^{\xi_2^i}\}}\theta^{\xi_1^i}(x)\, d\mathcal{H}^1(x)\\ &\geq \frac{1}{\mathcal{L}^2(E_n)}\int_{\pi^{\xi_1^i}(E_n^{\xi_1^i})}\frac{1}{\sqrt{2}}\mathcal{H}^1\left((E_n)_y^{\xi_1^i}\right)\, d\mathcal{L}^1(y)\\ &= \frac{\mathcal{L}^2(E_n^{\xi_1^i})}{\mathcal{L}^2(E_n)},\end{aligned}$$

where for each n*,* $E_n^{\xi_1^i} = \left\{(y,t)\in E_n \mid y\in\pi^{\xi_1^i}\left(\mathcal{F}E_n\cap\Gamma_i\cap\{\theta^{\xi_1^i}\leq\theta^{\xi_2^i}\}\right)\right\}$*. Finally from* (5.3)*,* (5.4)*, and* $|\xi_1^i\cdot\nu_E(x)| > \frac{1}{\sqrt{2}}$ *(for every* i*), it is a geometric fact that for every* n *the quotient* $\frac{\mathcal{L}^2(E_n^{\xi_1^i})}{\mathcal{L}^2(E_n)}$ *is greater then a strictly positive real number which depends only on the* $\mathcal{H}^1$*-measure of the projection* $\pi^{\xi_1}(E_n^{\xi_1^i})$ *and on the scalar product* $|\xi_1^i\cdot\nu_E|$*. Hence we have showed the claim.*

Finally, we show an application of our results in the theory of elasticity with cracks:

Example 5.4. *Let* $\Omega\subset\mathbb{R}^n$ *be a regular domain which represents the reference configuration of an elastic body, and let* $\Gamma\subset\mathbb{R}^n$ *be a crack described by a countably* $(\mathcal{H}^{n-1},n-1)$*-rectifiable set with finite* $\mathcal{H}^{n-1}$*-measure; we consider two disjoint measurable subsets of* $\partial\Omega$*, respectively* $\partial_D\Omega$ *and* $\partial_N\Omega$ *which are respectively the Dirichlet part and the Neumann part of the boundary. On the set* $\partial\Omega\setminus(\partial_D\Omega\cup\partial_N\Omega)$ *and on the crack we impose the homogeneous Neumann condition.*

We consider the following minimization problem:

$$\min_{\substack{u\in GSBD^2_2(\Omega;\Gamma)\\ u=w \text{ on } \partial_D\Omega}} E(u) := \int_\Omega |\mathcal{E}u|^2\, dx + \int_\Omega |u-g|^2\, dx - \int_{\partial_N\Omega} F\cdot Tr(u)\, d\mathcal{H}^{n-1}, \tag{5.5}$$

where w *is some function in* $GSBD^2_2(\Omega;\Gamma)$*,* F *is a vector field representing the traction force, and* g *is some square integrable vector field. Usually, in the variational model for quasistatic* *growth of brittle fracture, for example in [6], the functional takes into account also the $\mathcal{H}^{n-1}$-measure of the jump sets of all possible displacements u; in fact, in this case the jump sets are free to move inside an open set $\Omega_B \subset \Omega$ such that $\overline{\Omega}_B \cap \partial_D\Omega = \varnothing$ and $\overline{\Omega}_B \cap \partial_N\Omega = \varnothing$. In our case the finitness of the jump sets is ensured by requiring the stronger condition $J_u \subset \Gamma$, but on the other hand J_u might have possible interaction with both Dirichlet and Neumann part. Minimization problems like (5.5), arise for example in the minimizing movements technique, for example in [5] or in [13], in order to solve respectively the wave equation or the equations of elastodynamics, in a prescribed arbitrary growing cracks domain.*

In order to prove the existence of a minimum, we need to specify the space of all admissible Neumann terms: let Θ^+ be the weight function given in Theorem 5.1, then we consider all the measurable vector fields F such that $\int_{\partial_N\Omega} \frac{F^2}{\Theta^+}\, d\mathcal{H}^{n-1} < \infty$, or equivalently such that $F = G\sqrt{\Theta^+}$ for some vector field $G \in L^2(\Omega)$.

Roughly speaking the function Θ^+ measures, somehow, how much Γ is close to the boundary. From a physical point of view, this might be interpreted as the fact that, when the elastic material between the Neumann boundary and the crack is infinitesimally small, then the elastic reaction to the traction force will be infinitesimally too; hence, in order to reach the equilibrium, the traction forces need to decrease their intensity (proportionally to Θ^+).

First of all we show the coercivity of $E(\cdot)$. By Theorem (5.1) we can bound the Neumann term from above as:

$$\begin{aligned}\int_{\partial_N\Omega} F\cdot Tr(u)\, d\mathcal{H}^{n-1} &\le \bigg(\int_{\partial_N\Omega}|G|^2\, d\mathcal{H}^{n-1}\bigg)^{1/2}\bigg(\int_{\partial_N\Omega}|Tr(u)|^2\Theta^+\, d\mathcal{H}^{n-1}\bigg)^{1/2}\\ &\le C\bigg(\|\mathcal{E}u\|_2 + \|u\|_2\bigg),\end{aligned}$$

where $C > 0$ is a constant which depends only on the dimension n and on F. As a consequence we immediately deduce the coercivity:

$$E(u) \ge \|\mathcal{E}u\|_2^2 + 2\|u\|_2^2 - 2\|g\|_2^2 - C(\|\mathcal{E}u\|_2 + \|u\|_2).$$

Hence every minimizing sequence satisfies the uniform bound

$$\sup_k(\|\mathcal{E}u_k\|_2 + \|u_k\|_2 + \mathcal{H}^{n-1}(J_{u_k})) < \infty,$$

and we are in position to use the compactness result in [4, Theorem 11.3] to deduce that there exists $u \in GSBD^2_2(\Omega;\Gamma)$ such that (up to subsequences):

$$\begin{cases} u_k \to u, \text{ in } L^1(\Omega)\\ \mathcal{E}u_k \rightharpoonup \mathcal{E}u, \text{ weakly in } L^1(\Omega).\end{cases} \tag{5.6}$$

This means that u_k converges to u with respect to the notion of convergence (4.15), and by Theorem 4.7 u still satisfies $Tr(u) = Tr(w)$ on $\partial_D\Omega$.

The first two terms of $E(\cdot)$ are clearly lower semi-continuous with respect to the convergence (4.15), while the Neumann term is even continuous: this is a simple consequence of the fact that by Theorem 5.1 the trace operator is weakly continuous in $L^2(\Omega, \Theta^+\mathcal{H}^{n-1})$, thus we can write:

$$\begin{aligned}\lim_k \int_{\partial_N\Omega} F\cdot Tr(u_k)\, d\mathcal{H}^{n-1} &= \lim_k \int_{\partial_N\Omega}\frac{G}{\sqrt{\Theta^+}}\cdot Tr(u_k)\ \Theta^+ d\mathcal{H}^{n-1}\\ &= \int_{\partial_N\Omega}\frac{G}{\sqrt{\Theta^+}}\cdot Tr(u)\ \Theta^+ d\mathcal{H}^{n-1}\\ &= \int_{\partial_N\Omega} F\cdot Tr(u)\, d\mathcal{H}^{n-1}.\end{aligned}$$

Hence our functional is coercive and lower semi-continuous, so we are in position to apply the standard direct method in the calculus of variation to deduce the existence of a minimum.

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