EXTENDED AFFINE WEYL GROUPS OF BCD TYPE, FROBENIUS MANIFOLDS AND THEIR LANDAU-GINZBURG SUPERPOTENTIALS

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Abstract. For the root systems of type B_l, C_l and D_l, we generalize the result of [7] by showing the existence of Frobenius manifold structures on the orbit spaces of the extended affine Weyl groups that correspond to any vertex of the Dynkin diagram instead of a particular choice made in [7]. It also depends on certain additional data. We also construct LG superpotentials for these Frobenius manifold structures.

Contents

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1. Introduction

Let R be an irreducible reduced root system defined in an l-dimensional Euclidean space V with the Euclidean inner product $(\ ,\ )$. We fix a basis of simple roots $\alpha_1, \ldots, \alpha_l$ and denote by $\alpha_j^\vee,\ j = 1, 2, \cdots, l$ the corresponding coroots. The Weyl group W is generated by the reflections

$$\mathbf{x} \mapsto \mathbf{x} - (\alpha_j^\vee, \mathbf{x})\alpha_j, \quad \forall\, \mathbf{x} \in V,\ j = 1, \ldots, l. \tag{1.1}$$

Recall that the *Cartan matrix* of the root system has integer entries $A_{ij} = \left(\alpha_i, \alpha_j^\vee\right)$ satisfying $A_{ii} = 2,\ A_{ij} \leq 0$ for $i \neq j$. The semi-direct product of W by the lattice of coroots yields the affine Weyl group W_a that acts on V by the affine transformations

$$\mathbf{x} \mapsto w(\mathbf{x}) + \sum_{j=1}^{l} m_j \alpha_j^\vee, \quad w \in W,\ m_j \in \mathbb{Z}. \tag{1.2}$$

We denote by $\omega_1, \ldots, \omega_l$ the fundamental weights defined by the relations

$$(\omega_i, \alpha_j^\vee) = \delta_{ij}, \quad i, j = 1, \ldots, l. \tag{1.3}$$

Note that the root system R is one of the type $A_l, B_l, C_l, D_l, E_6, E_7, E_8,\ F_4, G_2$. In what follows the Euclidean space V and the basis $\alpha_1, \ldots, \alpha_l$ of the simple roots will be defined as in Plate I-IX of [2]. Let us fix a simple root α_k and define an extension of the affine Weyl group W_a in a similar way as it is done in [7].

Definition 1.1. *The extended affine Weyl group* $\widetilde{W} = \widetilde{W}^{(k)}(R)$ *acts on the extended space*

$$\widetilde{V} = V \oplus \mathbb{R}\,,$$

it is generated by the transformations

$$x = (\mathbf{x}, x_{l+1}) \mapsto (w(\mathbf{x}) + \sum_{j=1}^{l} m_j \alpha_j^\vee,\ x_{l+1}), \quad w \in W,\ m_j \in \mathbb{Z}, \tag{1.4}$$

and

$$x = (\mathbf{x}, x_{l+1}) \mapsto (\mathbf{x} + \gamma\, \omega_k,\ x_{l+1} - \gamma). \tag{1.5}$$

Here $1 \le k \le l$, $\gamma = 1$ *except for the cases when* $R = B_l, k = l$ *and* $R = F_4, k = 3$ *or* $k = 4$, *in these three cases* $\gamma = 2$.

The above definition of the extended affine Weyl group coincides with the one given in [7] for the particular choice of α_k that is made there. We note that in the cases for which $\gamma = 1$ the number $\frac{1}{2}(\alpha_k, \alpha_k)$ are integers, while for the three exceptional cases $\frac{1}{2}(\alpha_k, \alpha_k) = \frac{1}{2}$.

Let us introduce coordinates $x_1, \dots, x_l$ on the space V by

$$\mathbf{x} = x_1 \alpha_1^\vee + \cdots + x_l \alpha_l^\vee. \tag{1.6}$$

Denote by $f = \det(A_{ij})$ the determinant of the Cartan matrix of the root system R.

Definition 1.2 ([7])**.** $\mathcal{A} = \mathcal{A}^{(k)}(R)$ *is the ring of all* $\widetilde{W}$*-invariant Fourier polynomials of the form*

$$\sum_{m_1,\dots,m_{l+1} \in \mathbb{Z}} a_{m_1,\dots,m_{l+1}} e^{2\pi i (m_1 x_1 + \cdots + m_l x_l + \frac{1}{f} m_{l+1} x_{l+1})}$$

bounded in the limit

$$\mathbf{x} = \mathbf{x}^0 - i\omega_k \tau, \quad x_{l+1} = x_{l+1}^0 + i\tau, \quad \tau \to +\infty \tag{1.7}$$

for any $x^0 = (\mathbf{x}^0, x_{l+1}^0)$.

We introduce a set of numbers

$$d_j = (\omega_j, \omega_k), \quad j = 1, \dots, l \tag{1.8}$$

and define the following Fourier polynomials [7]

$$\tilde{y}_j(x) = e^{2\pi i d_j x_{l+1}} y_j(\mathbf{x}), \quad j = 1, \dots, l, \tag{1.9}$$

$$\tilde{y}_{l+1}(x) = e^{\frac{2\pi i}{\gamma} x_{l+1}}. \tag{1.10}$$

Here $y_1(\mathbf{x}), \dots, y_l(\mathbf{x})$ are the basic W_a-invariant Fourier polynomials defined by

$$y_j(\mathbf{x}) = \frac{1}{n_j} \sum_{w \in W} e^{2\pi i (\omega_j, w(\mathbf{x}))}, \quad n_j = \#\{w \in W | e^{2\pi i (\omega_j, w(\mathbf{x}))} = e^{2\pi i (\omega_j, \mathbf{x})}\}. \tag{1.11}$$

It was shown in [7] that for some particular choices of the simple root α_k, a Chevalley-type theorem holds true for the ring $\mathcal{A}$, i.e., it is isomorphic to the polynomial ring generated by $\tilde{y}_1, \ldots, \tilde{y}_{l+1}$, and thus the orbit space defined as $\mathcal{M} = \operatorname{Spec} \mathcal{A}$ of the extended affine Weyl group $\widetilde{W}$ is an affine algebraic variety of dimension $l+1$. In [7] it was further proved that on such an orbit space there exists a Frobenius manifold structure whose potential is a polynomial of $t^1, \ldots, t^{l+1}, e^{t^{l+1}}$. Here $t^1, \ldots, t^{l+1}$ are the flat coordinates of the Frobenius manifold. For the root system of type A_l, there is in fact no restrictions on the choice of α_k. However, for the root systems of type $B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2$ there is only one choice for each.

In [18] Slodowy pointed out that the Chevalley-type theorem of [7] is a consequence of the results of Looijenga and Wirthmüller [13, 14, 20], and in fact it holds true for any choice of the base element α_k, or equivalently, for any fixed vertex of the Dynkin diagram. So we have

Theorem 1.3 ([18, 20, 13, 14])**.** *The ring $\mathcal{A}$ is isomorphic to the ring of polynomials of $\tilde{y}_1(x), \cdots, \tilde{y}_{l+1}(x)$.*

A natural question, as it was pointed out in [7, 18], *is whether the geometric structures that were revealed in* [7] *also exist on the orbit spaces of the extended affine Weyl groups for an arbitrary choice of the root α_k?* The purpose of the present paper is to give an affirmative answer to this question for the root systems of type B_l, C_l and also for D_l. It will be organized as follows.

In Sec.2 we give an elementary proof of Theorem 1.3 that is based on the proof of the Chevalley type theorem given in [7].

Let $\mathcal{M}$ be the orbit space of the extended affine Weyl group $\widetilde{W}^{(k)}(C_l)$ and $\widetilde{\mathcal{M}}$ a covering of $\mathcal{M} \setminus \{\tilde{y}_{l+1} = 0\}$. In Sec.3, firstly we introduce an indefinite metric $(\ ,\ )^{\sim}$ on $\widetilde{V} = V \oplus \mathbb{R}$ given by

$$(dx_s, dx_n)^{\sim} = \frac{s}{4\pi^2}, \quad (dx_s, dx_{l+1})^{\sim} = 0, \quad (dx_{l+1}, dx_{l+1})^{\sim} = -\frac{1}{4k\pi^2} \tag{1.12}$$

for $1 \le s \le n \le l$. The projection

$$P : \widetilde{V} \to \widetilde{\mathcal{M}}$$

induces the following symmetric bilinear form on $T^*\widetilde{\mathcal{M}}$:

$$g^{ij}(y) := \sum_{a,b=1}^{l+1} \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b} (dx^a, dx^b)^{\sim}, \tag{1.13}$$

where $y^1 = \tilde{y}_1, \dots, y^l = \tilde{y}_l,\ y^{l+1} = \log \tilde{y}_{l+1} = 2\pi i x_{l+1}$. Afterwards, on certain open subset $\mathcal{U}$ of $\mathcal{M}$ we construct a flat pencil of metrics $g^{ij}(y)$ and $\eta^{ij}(y)$, where

$$\eta^{ij}(y) := \mathcal{L}_e g^{ij}(y) \tag{1.14}$$

and the vector field e has the form

$$e = \sum_{j=k}^{l} c_j \frac{\partial}{\partial y^j}. \tag{1.15}$$

It depends on the choice of an integer m in the range $0 \le m \le l-k$. Namely, for a given m the coefficients $c_k, \cdots, c_l$ are defined by the generating function $\sum_{j=k}^{l} c_j u^{l-j} = (u+2)^m (u-2)^{l-k-m}$. Furthermore, we show that

Main Theorem 1. (*Theorem 3.14*) *For any fixed integer $0 \le m \le l-k$, there exists a unique Frobenius manifold structure, denoted by $\mathcal{M}_{k,m}(C_l)$, of charge $d = 1$ living on the covering of the orbit space $\mathcal{M} \setminus \{t^{l-m} = 0\} \cup \{t^l = 0\}$ of $\widetilde{W}^{(k)}(C_l)$ polynomial in $t^1, \cdots, t^{l+1}, \frac{1}{t^{l-m}}, \frac{1}{t^l}, e^{t^{l+1}}$ for a suitable choice of flat coordinates $t^1, \cdots, t^{l+1}$ for the metric* (1.14) *(see Theorem 3.10 below) such that*

(1) *The unity vector field e coincides with $\sum_{j=k}^{l} c_j \frac{\partial}{\partial y^j} = \frac{\partial}{\partial t^k}$;*

(2) *The Euler vector field has the form*

$$E = \sum_{\alpha=1}^{l} \tilde{d}_\alpha t^\alpha \frac{\partial}{\partial t^\alpha} + \frac{1}{k} \frac{\partial}{\partial t^{l+1}}, \tag{1.16}$$

where $\tilde{d}_1, \dots, \tilde{d}_l$ are defined in (3.62)–(3.64).

(3) *The invariant flat metric and the intersection form of the Frobenius manifold structure coincide respectively with the metric $(\eta^{ij}(t))$ and $(g^{ij}(t))$ on the covering of $\mathcal{M} \setminus \{t^{l-m} = 0\} \cup \{t^l = 0\}$.*

In Sec.4 we further show that for the root systems of type B_l and D_l we can apply a similar construction as the one for the root system of type C_l. The resulting Frobenius manifolds are isomorphic to those obtained from the root system of type C_l.

Observe that in the case of the root system of type A_l, in [7] it is shown that the extended affine Weyl group $\widetilde{W}^{(k)}(A_l)$ describes monodromy of roots of trigonometric polynomials with a given bidegree being of the form

$$\lambda(\varphi) = e^{\mathbf{i}k\varphi} + a_1 e^{\mathbf{i}(k-1)\varphi} + \cdots + a_l e^{\mathbf{i}(k-l)\varphi}, \quad a_l \neq 0.$$

A natural question *is whether there exists a similar construction for the root systems of type B_l, C_l and D_l?* In Sec.5, let us denote by $\mathfrak{M}_{k,m,n}$ the space of a particular class of cosine Laurent series of one variable with a given tri-degree $(2k, 2m, 2n)$ being of the form

$$\lambda(\varphi) = \left(\cos^2(\varphi) - 1\right)^{-m} \sum_{j=0}^{k+m+n} a_j \cos^{2(k+m-j)}(\varphi), \quad a_0 a_{k+m+n} \neq 0,$$

where all $a_j \in \mathbb{C}$, $m, n \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{N}$. The space $\mathfrak{M}_{k,m,n}$ carries a natural structure of Frobenius manifold. Its invariant inner product η and the intersection form g of two vectors ∂', ∂'' tangent to $\mathfrak{M}_{k,m,n}$ at a point $\lambda(\varphi)$ can be defined by the following formulae

$$\eta(\partial', \partial'') = (-1)^{k+1} \sum_{|\lambda|<\infty} \operatorname*{res}_{d\lambda=0} \frac{\partial'(\lambda(\varphi)d\varphi)\partial''(\lambda(\varphi)d\varphi)}{d\lambda(\varphi)}, \tag{1.17}$$

and

$$g(\partial', \partial'') = -\sum_{|\lambda|<\infty} \operatorname*{res}_{d\lambda=0} \frac{\partial'(\log\lambda(\varphi)d\varphi)\partial''(\log\lambda(\varphi)d\varphi)}{d\log\lambda(\varphi)}. \tag{1.18}$$

Moreover, we will show that

Main Theorem 2. (*Theorem 5.6*) *There is an isomorphism of Frobenius manifolds between $\mathfrak{M}_{k,m,n}$ and $\mathcal{M}_{k,m}(C_{k+m+n})$.*

A function involved in the representation of the form (1.17), (1.18) of the flat pencil of metrics on the Frobenius manifold is called *Landau–Ginzburg* (LG) *superpotential* of the Frobenius manifold. Observe that the multiplication law on the tangent spaces to the Frobenius manifold can also be expressed in terms of the LG superpotential (see eq. (5.5) below).

Some concluding remarks are given in the last section.

2. A proof of Theorem 1.3 related to the root systems of type B_l, C_l, D_l

In this section, we give an elementary proof of the Theorem 1.3 for the root systems of type B_l, C_l and D_l for any fixed vertex of the Dynkin diagram. To this end, we first write down the explicit expressions of the invariant Fourier polynomials $\tilde{y}_j(x)$ that are defined in (1.9), (1.10) for these root systems with the fixed simple root α_k, hereafter $\alpha_1, \dots, \alpha_l$ denote the standard base of simple roots as given in [2]. We then prove the theorem by using an approach that is similar to the one used in [7].

For the root system of type B_l, the numbers d_j defined in (1.8) have the values

$$d_i = i,\ 1 \le i \le k, \quad d_j = k,\ k+1 \le j \le l-1, \quad d_l = \frac{k}{2}, \tag{2.1}$$

for $k < l$ and

$$d_i = \frac{i}{2},\ 1 \le i \le l-1, \quad d_k = \frac{l}{4} \tag{2.2}$$

for $k = l$. The W_a-invariant Fourier polynomials $y_1(\mathbf{x}), \dots, y_l(\mathbf{x})$ defined in (1.11) have the expressions [12]

$$y_j(\mathbf{x}) = \sigma_j(\xi_1, \cdots, \xi_l), \quad j = 1, \dots, l-1, \tag{2.3}$$

$$y_l(\mathbf{x}) = \prod_{j=1}^{l} \left(e^{i\pi\, v_j} + e^{-i\pi\, v_j}\right), \tag{2.4}$$

where

$$\begin{aligned} &v_1 = x_1, \quad v_m = x_m - x_{m-1}, \quad 2 \le m \le l-1, \\ &v_l = 2x_l - x_{l-1}, \\ &\xi_j = e^{2\,i\pi\, v_j} + e^{-2\,i\pi\, v_j}, \quad 1 \le j \le l. \end{aligned} \tag{2.5}$$

Here and henceforth the functions $\sigma_j(\xi_1,\dots,\xi_l)$ denote the j-th elementary symmetric polynomial of $\xi_1,\cdots,\xi_l$ defined by

$$\prod_{j=1}^{l}(z+\xi_j)=\sum_{j=0}^{l}\sigma_j(\xi_1,\dots,\xi_l)z^{l-j}. \tag{2.6}$$

For the root system of type C_l, the numbers d_j are given by

$$d_1=1,\dots,d_{k-1}=k-1,\ d_j=k,\quad k\le j\le l. \tag{2.7}$$

The W_a-invariant Fourier polynomials $y_1(\mathbf{x}),\dots,y_l(\mathbf{x})$ defined in (1.11) have the expressions

$$y_j(\mathbf{x})=\sigma_j(\xi_1,\cdots,\xi_l). \tag{2.8}$$

Here ξ_j are defined by

$$\xi_j=e^{2\,i\pi\,(x_j-x_{j-1})}+e^{-2\,i\pi\,(x_j-x_{j-1})},\quad x_0=0,\ 1\le j\le l.$$

For the root system of type D_l, we have

i)

$$d_j=j,\ 1\le j\le k,\quad d_j=k,\ k+1\le j\le l-2, \tag{2.9}$$

$$d_j=\frac{k}{2},\ j=l-1,l \tag{2.10}$$

for $k\le l-2$; and

ii)

$$d_j=\frac{j}{2},\ 1\le j\le l-2,\quad d_{l-1}=\frac{l}{4},\quad d_l=\frac{l-2}{4} \tag{2.11}$$

for $k=l-1$; and

iii)

$$d_j=\frac{j}{2},\ 1\le j\le l-2,\quad d_{l-1}=\frac{l-2}{4},\quad d_l=\frac{l}{4} \tag{2.12}$$

for $k=l$. The basis of the W_a-invariant Fourier polynomials defined in (1.11) has the form

$$\begin{aligned} y_j(\mathbf{x}) &= \sigma_j(\xi_1,\cdots,\xi_l), \quad j=1,\dots,l-2,\\ y_{l-1}(\mathbf{x}) &= \frac{1}{2}\left(\prod_{j=1}^{l}\left(e^{i\pi v_j}+e^{-i\pi v_j}\right)+\prod_{j=1}^{l}\left(e^{i\pi v_j}-e^{-i\pi v_j}\right)\right), \\ y_l(\mathbf{x}) &= \frac{1}{2}\left(\prod_{j=1}^{l}\left(e^{i\pi v_j}+e^{-i\pi v_j}\right)-\prod_{j=1}^{l}\left(e^{i\pi v_j}-e^{-i\pi v_j}\right)\right), \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} &v_1 = x_1, \quad v_m = x_m - x_{m-1}, \quad 2\le m\le l-2,\\ &v_{l-1} = x_l + x_{l-1} - x_{l-2}, \quad v_l = x_{l-1}-x_l,\\ &\xi_j = e^{2\,i\pi\,v_j}+e^{-2\,i\pi\,v_j}, \quad 1\le j\le l. \end{aligned} \tag{2.14}$$

Proof of the Theorem 1.3 for the root system $R=B_l, C_l, D_l$. From the explicit expressions of the Fourier polynomials $\tilde{y}_1(x),\dots,\tilde{y}_{l+1}(x)$, it is not difficult to see that they are $\widetilde{W}^{(k)}(R)$-invariant. So in order to prove the theorem, we only need to show that any element $f(x)$ of the ring $\mathcal{A}$ can be expressed as a polynomial of $\tilde{y}_1(x),\dots,\tilde{y}_{l+1}(x)$. By using the fact that the ring of W_a-invariant Fourier polynomials is isomorphic to the polynomial ring generated by $y_1(\mathbf{x}),\dots,y_l(\mathbf{x})$ and by using the $\widetilde{W}$-invariance of the function $f(x)\in\mathcal{A}$, we can represent it as a polynomial of $\tilde{y}_1(x),\dots,\tilde{y}_l(x),\tilde{y}_{l+1}(x),\tilde{y}_{l+1}^{-1}$. Assume

$$f(x)=\sum_{n\ge -N}\tilde{y}_{l+1}^{n}\;P_n(\tilde{y}_1(x),\dots,\tilde{y}_l(x)),$$

and the polynomial $P_{-N}(\tilde{y}_1(x),\dots,\tilde{y}_l(x))$ does not vanish identically for certain positive integer N. From the definition of the functions $\tilde{y}_j(x)$ we know that in the limit (1.7)) we have

$$y_j(\mathbf{x}) = e^{2\pi d_j\tau}[y_j^0(\mathbf{x}^0)+\mathcal{O}(e^{-2\alpha\pi\tau})], \quad j=1,\dots,l, \tag{2.15}$$

where α is a certain positive integer and the expressions of the functions $y_j^0(\mathbf{x}^0)$ will be given below. So in the limit (1.7)) the function $f(x)$ behaves as

$$f(x) = e^{\frac{2\pi}{\gamma}N\tau - \frac{2\pi i}{\gamma}Nx_{l+1}^0}[P_{-N}(\tilde{y}_1^0(x^0), \ldots, \tilde{y}_l^0(x^0)) + \mathcal{O}(e^{-2\beta\pi\tau})]$$

for a certain positive integer β and

$$\tilde{y}_j^0(x^0) = e^{2\pi i\ d_j x_{l+1}^0}\ y_j^0(\mathbf{x}^0), \quad j = 1, \cdots, l.$$

Since the function $f(x)$ is bounded for $\tau \to +\infty$, we must have

$$P_{-N}(\tilde{y}_1^0(x^0), \ldots, \tilde{y}_l^0(x^0)) \equiv 0$$

for any $x^0 = (\mathbf{x}^0, x_{l+1}^0)$. This leads to a contradiction to the algebraic independence of the functions $\tilde{y}_1^0, \ldots, \tilde{y}_l^0$ that we will now prove case by case for the root systems of the type B_l, C_l and D_l.

i) For the root system of type B_l with $1 \le k \le l-1$,

$$\begin{aligned} y_j^0(\mathbf{x}^0) &= \rho_j, \quad j = 1, \cdots, k, \\ y_s^0(\mathbf{x}^0) &= \rho_k \rho_s, \quad s = k+1, \cdots, l-1, \\ y_l^0(\mathbf{x}^0) &= \sqrt{\rho_k \rho_l}\,, \end{aligned}$$

where the functions ρ_i are defined by

$$\begin{aligned} \rho_j &= \sigma_j(e^{2\pi i v_1^0}, \cdots, e^{2\pi i v_k^0}), \quad j = 1, \cdots, k, \\ \rho_s &= \sigma_{s-k}(\xi_{k+1}^0, \cdots, \xi_l^0), \quad s = k+1, \cdots, l \end{aligned}$$

with

$$\begin{aligned} &\xi_m^0 = e^{2\pi i v_m^0} + e^{-2\pi i v_m^0}, \quad m = 1, \cdots, l, \\ &v_1^0 = x_1^0, \quad v_j^0 = x_j^0 - x_{j-1}^0,\ 2 \le j \le l-1, \quad v_l^0 = 2\,x_l^0 - x_{l-1}^0. \end{aligned}$$

Thus we obtain

$$\det\left(\frac{\partial y_i^0(\mathbf{x}^0)}{\partial \rho_j}\right) = \frac{\rho_k^{l-k}}{2\sqrt{\rho_k \rho_l}}. \tag{2.16}$$

When $k=l$, we have

$$y^0_j(\mathbf{x}^0)=\rho_j=\sigma_j(e^{2\pi i v^0_1},\cdots,e^{2\pi i v^0_l}),\quad j=1,\cdots,l-1,$$
$$y^0_l(\mathbf{x}^0)=\sqrt{\rho_l},\quad \rho_l=\sigma_l(e^{2\pi i v^0_1},\cdots,e^{2\pi i v^0_l}),$$
$$\det(\frac{\partial y^0_i(\mathbf{x}^0)}{\partial \rho_j})=1. \tag{2.17}$$

ii) For the root system of type C_l,

$$y^0_j(\mathbf{x}^0)=\rho_j,\quad j=1,\cdots,k,$$
$$y^0_s(\mathbf{x}^0)=\rho_k\rho_s,\quad s=k+1,\cdots,l,$$

where the functions ρ_j are defined by

$$\rho_j=\sigma_j(e^{2\pi i v^0_1},\cdots,e^{2\pi i v^0_k}),\quad j=1,\cdots,k,$$
$$\rho_s=\sigma_{s-k}(\xi^0_{k+1},\cdots,\xi^0_l),\quad s=k+1,\cdots,l$$

with

$$\xi^0_m=e^{2\pi i v^0_m}+e^{-2\pi i v^0_m},$$
$$v^0_1=x^0_1,\ v^0_m=x^0_m-x^0_{m-1},\quad m=2,\cdots,l.$$

Thus we get

$$\det\left(\frac{\partial y^0_i(\mathbf{x}^0)}{\partial \rho_j}\right)=(\rho_k)^{l-k}. \tag{2.18}$$

iii) For the root system of type D_l with $k\le l-2$,

$$y^0_j(\mathbf{x}^0)=\rho_j,\quad j=1,\cdots,k,$$
$$y^0_s(\mathbf{x}^0)=\rho_k\rho_s,\quad s=k+1,\cdots,l-2,$$
$$y^0_{l-1}(\mathbf{x}^0)=\frac{1}{2}\sqrt{\rho_k}\left(\rho_l+\rho_{l-1}\right),\quad y^0_l(\mathbf{x}^0)=\frac{1}{2}\sqrt{\rho_k}\left(\rho_l-\rho_{l-1}\right)$$

where the functions ρ_j are given by

$$\rho_j=\sigma_j(e^{2\pi i v^0_1},\cdots,e^{2\pi i v^0_k}),\quad j=1,\cdots,k,$$
$$\rho_s=\sigma_{s-k}(\xi^0_{k+1},\cdots,\xi^0_l),\quad s=k+1,\cdots,l-2,$$
$$\rho_{l-1}=\prod_{s=k+1}^{l}\left(e^{i\pi v^0_j}+e^{-i\pi v^0_j}\right),\quad \rho_l=\prod_{s=k+1}^{l}\left(e^{i\pi v^0_j}-e^{-i\pi v^0_j}\right)$$

with

$$\xi_m^0 = e^{2\pi i v_m^0} + e^{-2\pi i v_m^0},$$

$$v_1^0 = x_1^0,\ v_m^0 = x_m^0 - x_{m-1}^0, \quad m = 2, \cdots, l-2,$$

$$v_{l-1}^0 = x_l^0 + x_{l-1}^0 - x_{l-2}^0, \quad v_l^0 = x_{l-1}^0 - x_l^0.$$

Thus we know

$$\det\left(\frac{\partial y_i^0(\mathbf{x}^0)}{\partial \rho_j}\right) = \frac{1}{2}(\rho_k)^{l-k-1}. \tag{2.19}$$

iv) For the case D_l with $k = l-1$ we have

$$y_m^0(\mathbf{x}^0) = \rho_m, \quad m = 1, \cdots, l-2,$$

$$y_{l-1}^0(\mathbf{x}^0) = \sqrt{\rho_l}, \quad y_l^0(\mathbf{x}^0) = \frac{\rho_{l-1}}{\sqrt{\rho_l}},$$

where the functions ρ_j are defined by

$$\rho_j = \sigma_j(e^{2\pi i v_1^0}, \cdots, e^{2\pi i v_l^0}), \quad j = 1, \cdots, l$$

with

$$v_1^0 = x_1^0,\ v_m^0 = x_m^0 - x_{m-1}^0, \quad 2 \le m \le l-2,$$

$$v_{l-1}^0 = x_l^0 + x_{l-1}^0 - x_{l-2}^0,\ v_l^0 = x_{l-1}^0 - x_l^0.$$

So we have

$$\det\left(\frac{\partial y_i^0(\mathbf{x}^0)}{\partial \rho_j}\right) = -\frac{1}{2\rho_l}. \tag{2.20}$$

v) For the case D_l with $k = l$ the functions $y_j^0(\mathbf{x}^0)$ and ρ_j are defined in the same way as we did in the above case iv) except

$$y_{l-1}^0(\mathbf{x}^0) = \frac{\rho_{l-1}}{\sqrt{\rho_l}}, \quad y_l^0(\mathbf{x}^0) = \sqrt{\rho_l}, \quad v_l^0 = x_l - x_{l-1}^0.$$

and we have

$$\det\left(\frac{\partial y_i^0(\mathbf{x}^0)}{\partial \rho_j}\right) = \frac{1}{2\rho_l}. \tag{2.21}$$

From the above calculation of the Jacobian $\det\left(\frac{\partial y_i^0(\mathbf{x}^0)}{\partial \rho_j}\right)$ and from the algebraic independence of the functions $\rho_1, \ldots, \rho_l$ we deduce the algebraic independence of the functions $y_1^0(\mathbf{x}^0), \cdots, y_l^0(\mathbf{x}^0)$. This completes the proof of the theorem. $\square$

3. Frobenius manifold structures on the orbit space of $\widetilde{W}^{(k)}(C_l)$

3.1. **Flat pencils of metrics on the orbit space of $\widetilde{W}^{(k)}(C_l)$.** Let $\mathcal{M}$ be the orbit space defined as Spec$\mathcal{A}$ of the extended affine Weyl group $\widetilde{W}^{(k)}(C_l)$ for any fixed $1 \le k \le l$. As in [7] we define an indefinite metric $(\ ,\ )^{\sim}$ on $\widetilde{V} = V \oplus \mathbb{R}$ such that $\widetilde{V}$ is the orthogonal direct sum of V and $\mathbb{R}$. Here V is endowed with the W-invariant Euclidean metric

$$(dx_s, dx_n)^{\sim} = \frac{s}{4\pi^2}, \quad 1 \le s \le n \le l \tag{3.1}$$

and $\mathbb{R}$ is endowed with the metric

$$(dx_{l+1}, dx_{l+1})^{\sim} = -\frac{1}{4k\,\pi^2}. \tag{3.2}$$

The set of generators for the ring $\mathcal{A} = \mathcal{A}^{(k)}(C_l)$ are defined by (1.9), (1.10), (2.8) with $\gamma = 1$. They form a system of global coordinates on $\mathcal{M}$. We now introduce a system of local coordinates on $\mathcal{M}$ as follows

$$y^1 = \tilde{y}_1, \dots, y^l = \tilde{y}_l,\ y^{l+1} = \log \tilde{y}_{l+1} = 2\pi i\, x_{l+1}. \tag{3.3}$$

They live on a covering $\widetilde{\mathcal{M}}$ of $\mathcal{M} \setminus \{\tilde{y}_{l+1} = 0\}$. The projection

$$P : \widetilde{V} \to \widetilde{\mathcal{M}} \tag{3.4}$$

induces a symmetric bilinear form on $T^*\widetilde{\mathcal{M}}$

$$(dy^i, dy^j)^{\sim} \equiv g^{ij}(y) := \sum_{a,b=1}^{l+1} \frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^b}(dx^a, dx^b)^{\sim}. \tag{3.5}$$

Denote

$$\Sigma = \{y |\, \det(g^{ij}(y)) = 0\}, \tag{3.6}$$

then it was shown in [7] that Σ is the P-image of the hyperplanes

$$\{(\mathbf{x}, x_{l+1}) | (\beta, \mathbf{x}) = r \in \mathbb{Z},\ x_{l+1} = \text{arbitrary}\}, \quad \beta \in \Phi^+, \tag{2.3}$$

where Φ^+ is the set of all positive roots.

Proposition 3.1. *The functions $g^{ij}(y)$ and the contravariant components of its Levi-Civita connection*

$$\Gamma_n^{ij}(y) = -\sum_{s=1}^{l+1} g^{is}(y)\Gamma_{sn}^j(y), \quad 1 \le i, j, n \le l+1 \tag{3.7}$$

are weighted homogeneous polynomials in $y^1, \cdots, y^l$, $e^{y^{l+1}}$ of the degree

$$\deg g^{ij}(y) = \deg y^i + \deg y^j, \tag{3.8}$$

$$\deg \Gamma_n^{ij}(y) = \deg y^i + \deg y^j - \deg y^n \tag{3.9}$$

where $\deg y^j = d_j$ *and* $\deg y^{l+1} = d_{l+1} = 0$.

Proof. The proposition follows from $\Gamma_n^{ij}(y)dy^n = \dfrac{\partial y^i}{\partial x^p}\dfrac{\partial^2 y^j}{\partial x^q \partial x^r}(dx^p, dx^q)^{\sim} dx^r$ and Theorem 1.3. □

From this Proposition we see that Σ is an algebraic subvariety in $\mathcal{M}$ and the matrix (g^{ij}) is invertible on $\mathcal{M} \setminus \Sigma$, the inverse matrix $(g^{ij})^{-1}$ defines a flat metric on $\mathcal{M} \setminus \Sigma$. We now proceed to look for other flat metrics on a certain subvariety in $\mathcal{M}$ that are compatible with the metric $(g^{ij})^{-1}$. To this end, let us introduce the following new coordinates on $\mathcal{M}$:

$$\theta^j = \begin{cases} e^{k\,y^{l+1}}, & j = 0, \\ y^j e^{(k-j)y^{l+1}}, & j = 1, \cdots, k-1, \\ y^j, & j = k, \cdots, l \end{cases} \tag{3.10}$$

and denote

$$\mu_j = 2\pi i(x_j - x_{j-1}), \quad \mu_{l+1} = y^{l+1} = 2\pi i x_{l+1}, \quad j = 1, \cdots, l. \tag{3.11}$$

In the coordinates $\mu_1, \ldots, \mu_{l+1}$ the indefinite metric on $\widetilde{V}$ has the form

$$\big((d\mu_i, d\mu_j)^{\sim}\big) = \mathrm{diag}(-1, \ldots, -1, \frac{1}{k}). \tag{3.12}$$

Define

$$P(u) := \sum_{j=0}^{l} u^{l-j}\theta^j = e^{k\mu_{l+1}} \prod_{j=1}^{l}(u + \xi_j). \tag{3.13}$$

We can easily verify that the function $P(u)$ satisfies

$$\frac{\partial P(u)}{\partial \mu_a} = \frac{1}{u+\xi_a} P(u)(e^{\mu_a} - e^{-\mu_a}), \quad 1 \le a \le l; \tag{3.14}$$

$$\frac{\partial P(u)}{\partial \mu_{l+1}} = kP(u), \quad P'(u) := \frac{\partial P(u)}{\partial u} = P(u) \sum_{a=1}^{l} \frac{1}{u+\xi_a}. \tag{3.15}$$

Lemma 3.2. *The following formulae hold true for the generating functions of the metric* (g^{ij}) *and the contravariant components of its Levi-Civita connection* Γ_k^{ij} *in the coordinates* $\theta^0, \ldots, \theta^l$*:*

$$\begin{aligned}
&\sum_{i,j=0}^{l} (d\theta^i, d\theta^j)^{\sim} u^{l-i} v^{l-j} = (dP(u), dP(v))^{\sim} \\
&= (k-l)P(u)P(v) + \frac{u^2-4}{u-v} P'(u)P(v) - \frac{v^2-4}{u-v} P(u)P'(v),
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\sum_{i,j,r=0}^{l} \Gamma_r^{ij}(\theta) d\theta^r u^{l-i} v^{l-j} = \sum_{a,b,r=1}^{l+1} \frac{\partial P(u)}{\partial \mu_a} \frac{\partial^2 P(v)}{\partial \mu_b \partial \mu_r} d\mu_r (d\mu_a, d\mu_b) \\
&= (k-l)P(u)dP(v) + \frac{u^2-4}{u-v} P'(u)dP(v) - \frac{v^2-4}{u-v} P(u)dP'(v) \\
&\quad + \frac{uv-4}{(u-v)^2} P(v)dP(u) - \frac{uv-4}{(u-v)^2} P(u)dP(v).
\end{aligned} \tag{3.17}$$

Here $\Gamma_r^{ij}(\theta) = -\sum_{s=1}^{l+1} g^{is}(\theta) \Gamma_{sr}^{j}(\theta)$.

Proof. By using (3.14) and (3.15), we have

$$\begin{aligned}
(dP(u), dP(v))^{\sim} &= \frac{1}{k} \frac{\partial P(u)}{\partial \mu_{l+1}} \frac{\partial P(v)}{\partial \mu_{l+1}} - \sum_{a=1}^{l} \frac{\partial P(u)}{\partial \mu_a} \frac{\partial P(v)}{\partial \mu_a} \\
&= kP(u)P(v) - \sum_{a=1}^{l} P(u)P(v) \frac{\xi_a^2 - 4}{(u+\xi_a)(v+\xi_a)} \\
&= kP(u)P(v) - \sum_{s=1}^{l} P(u)P(v) \left(1 - \frac{u^2-4}{u-v} \frac{1}{u+\xi_a} + \frac{v^2-4}{u-v} \frac{1}{v+\xi_a}\right) \\
&= (k-l)P(u)P(v) + \frac{u^2-4}{u-v} P'(u)P(v) - \frac{v^2-4}{u-v} P(u)P'(v).
\end{aligned}$$

So we proved the first formula, the second formula can be proved in the same way. The lemma is proved. ∎

The above lemma shows that in the coordinates $\theta^0, \dots, \theta^l$ the functions $g^{ij}(\theta)$ are quadratic polynomials, and the contravariant components Γ^{ij}_s are homogeneous linear functions[1]. To find flat metrics that are compatible with this quadratic metric $g^{ij}(\theta)$, we need the following lemma.

Lemma 3.3. *If there is a set of constants* $\{c_0, \dots, c_l\}$ *such that*

(i) the functions

$$g^{ij}(\theta^0 + c_0\lambda, \theta^1 + c_1\lambda, \dots, \theta^l + c_l\lambda),$$
$$\Gamma^{ij}_s(\theta^0 + c_0\lambda, \theta^1 + c_1\lambda, \dots, \theta^l + c_l\lambda)$$

are linear in the parameter λ *for* $1 \le i, j, s \le l+1$, *and*

(ii) the matrix (η^{ij}) *with*

$$\eta^{ij} = \mathcal{L}_e g^{ij}, \quad e = \sum_{j=0}^{l} c_j \frac{\partial}{\partial \theta^j} \tag{3.18}$$

is nondegenerate on certain open subset $\mathcal{U}$ *of* $\mathcal{M}$.

Then the metrics $(g^{ij}), (\eta^{ij})$ *form a flat pencil, i.e., the linear combination* $(g^{ij} + \lambda\eta^{ij})$ *yields a flat metric on* $\mathcal{U}$ *for any* λ *satisfying* $\det(g^{ij} + \lambda\eta^{ij}) \neq 0$, *and the contravariant components of the Levi-Civita connection for this metric equal*

$$\Gamma^{ij}_s + \lambda\, \gamma^{ij}_s. \tag{3.19}$$

Here γ^{ij}_s *are the contravariant components of the Levi-Civita connection for the metric* (η^{ij}) *which can be evaluated by* $\gamma^{ij}_s = \mathcal{L}_e \Gamma^{ij}_s$.

Proof. For the proof of this lemma, see Appendix D of [6]. ∎

[1] These metrics give rise to a quadratic Poisson structure on the space of "loops" $\{S^1 \to M\}$ (see [5] for the details):

$$\{\theta^i(a), \theta^j(b)\} = g^{ij}(\theta(a))\delta'(a-b) + \Gamma^{ij}_s(\theta(a))\theta^s_a \delta(a-b).$$

We plan to study such important class of quadratic metrics and Poisson structures in a separate publication.

Theorem 3.4. *For any fixed integer $0 \le m \le l-k$ there is a flat pencil of metrics $(g^{ij}), (\eta^{ij})$ on a certain open subset $\mathcal{U}$ of $\mathcal{M}$ with (g^{ij}) given by (3.5) and $\eta^{ij} = \mathcal{L}_e g^{ij}$. Here the vector field e has the form*

$$e := \sum_{j=k}^{l} c_j \frac{\partial}{\partial \theta^j} = \sum_{j=k}^{l} c_j \frac{\partial}{\partial y^j}, \tag{3.20}$$

where the constants $c_k, \cdots, c_l$ are defined by the generating function

$$P_0(u) = \sum_{j=k}^{l} c_j u^{l-j} = (u+2)^m (u-2)^{l-k-m}. \tag{3.21}$$

Explicitly, $c_j = (-2)^{j-k} \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \binom{l-k-m}{l-j-s}$ *for* $j = k, \cdots, l$.

Proof. Firstly we want to find the constants $c_0, \ldots, c_l$ satisfying the condition (i) in Lemma 3.3. It suffices to find a polynomial $P_0(u) = \sum_{j=0}^{l} c_j u^{l-j}$ such that after the shift

$$P(u) \mapsto P(u) + \lambda P_0(u), \quad P(v) \mapsto P(v) + \lambda P_0(v),$$

the right hand side of (3.16) and (3.17) are linear in λ. This yields that $P_0(u)$ and $P_0(v)$ must satisfy

$$(k-l)P_0(u)P_0(v) + \frac{u^2-4}{u-v} P_0'(u)P_0(v) - \frac{v^2-4}{u-v} P_0(u)P_0'(v) = 0. \tag{3.22}$$

Separating the variables and integrating one obtains

$$P_0(u) = a \left(\frac{u-2}{u+2}\right)^b [(u-2)(u+2)]^{\frac{l-k}{2}} = (u-2)^{\frac{l-k}{2}+b} (u+2)^{\frac{l-k}{2}-b}$$

for some constants a, b. This is a polynomial *iff* $m := \frac{l-k}{2} - b$ is a non-negative integer. Hence any polynomial solution to eq. (3.22) must have the form $P_0(u) = a(u+2)^m (u-2)^{l-k-m}$ for an integer where $0 \le m \le l-k$. Thus, up to a common factor the constants $c_0, \ldots, c_l$ are determined by

$$\sum_{j=0}^{l} c_j u^{l-j} = (u+2)^m (u-2)^{l-k-m}.$$

Actually, by comparing the degrees of u, we know $c_j = 0$ for $j = 0, \cdots, k-1$.

Next we want to check the condition (ii) in Lemma 3.3. In order to do this, taking any fixed integer $0 \le m \le l-k$ we consider the following linear change of coordinates

$$(y^1, \dots, y^{l+1}) \mapsto (\tau^1, \dots, \tau^{l+1})$$

defined by the relations $\tau^{l+1} = y^{l+1}$ and

$$\sum_{j=0}^{l} \theta_j u^{l-j} = \sum_{j=0}^{l-m} \varpi^j \,(u+2)^m \,(u-2)^{l-m-j} \\ - \sum_{j=l-m+1}^{l} \varpi^j \,(u+2)^{l-j} \,(u-2)^{j-k-1}\,, \tag{3.23}$$

where

$$\varpi^j = \begin{cases} e^{k\,\tau^{l+1}}, & j=0, \\ \tau^j e^{(k-j)\tau^{l+1}}, & j=1,\cdots,k-1, \\ \tau^j, & j=k,\cdots,l. \end{cases} \tag{3.24}$$

Then,

$$\sum_{j=0}^{l} \frac{\partial \theta^j}{\partial \tau^k} u^{l-j} = (u+2)^m (u-2)^{l-k-m} = \sum_{j=0}^{l} c_j u^{l-j}.$$

This means that in terms of the new coordinates τ^i the vector field e defined in (3.20) has the expression

$$e = \sum_{j=0}^{l} \frac{\partial \theta^j}{\partial \tau^k} \frac{\partial}{\partial \theta^j} = \frac{\partial}{\partial \tau^k}.$$

Furthermore, observe that the left hand side of (3.23) coincides with the polynomial $P(u)$, by substituting the expressions of $P(u), P(v)$ given by the right hand side of (3.23) and a careful analysis, the matrix $(\eta^{ij}(\tau))$ with entries

$$\eta^{ij}(\tau) = \mathcal{L}_e g^{ij}(\tau) \tag{3.25}$$

has the block form

$$\begin{pmatrix} & W_1 & \begin{matrix} P_1 \\ \vdots \\ P_{k-1} \end{matrix} & & & \\ P_1 & \cdots P_{k-1} & P_k & 0 & 0 & 1 \\ & & 0 & W_2 & 0 & 0 \\ & & 0 & 0 & W_3 & 0 \\ & & 1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.26}$$

where W_i are triangular blocks

$$W_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & k \\ 0 & 0 & 0 & \cdots & k & R_1 \\ 0 & 0 & 0 & \cdots & R_1 & R_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ k & R_1 & R_2 & \cdots & & R_{k-2} \end{pmatrix}, \tag{3.27}$$

$$W_2 = \begin{pmatrix} Q_1 & Q_2 & \cdots & Q_{l-k-m} \\ Q_2 & Q_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ Q_{l-k-m} & 0 & \cdots & 0 \end{pmatrix}, \quad W_3 = \begin{pmatrix} S_1 & S_2 & \cdots & S_m \\ S_2 & S_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ S_m & 0 & \cdots & 0 \end{pmatrix} \tag{3.28}$$

with entries

$$\begin{aligned} R_j &= 4(k-j+1)\tau^{j-1}e^{\tau^{l+1}} + (k-j)\tau^j, \\ P_j &= 4(k-j+1)\tau^{j-1}e^{\tau^{l+1}}, \\ Q_s &= 4s\tau^{k+s} + (1-\delta_{s,l-k-m})(s+1)\tau^{k+s+1}, \\ S_r &= 4r\tau^{l-m+r} - 4(1-\delta_{r,m})r\tau^{l-m+r+1}, \\ 1 &\le j \le k, \quad 1 \le r \le m, \quad 1 \le s \le l-k-m. \end{aligned} \tag{3.29}$$

A simple computation gives

$$\det(\eta^{ij}) = (-1)^l k^{k-1} 4^{l-k} m^m (l-k-m)^{l-k-m} (\tau^{l-m})^{l-k-m} (\tau^l)^m. \tag{3.30}$$

So the metric $(\eta^{ij}(\tau))$ does not degenerate on $\mathcal{M}\setminus\{\tau\in\mathcal{M}|\tau^l=0,\tau^{l-m}=0\}$. We complete the proof of this theorem. $\square$

Remark 3.5. *(1). The block W_2 or W_3 does not appears in the matrix (3.26) when $m=l-k$ or $m=0$. (2). The flat pencil of metrics that corresponds to a fixed integer m is equivalent to the one that corresponds to the integer $l-k-m$, this is due to the fact that under replacement $u\mapsto -u$ the polynomial $P_0(u)=(u+2)^m(u-2)^{l-k-m}$ is transformed to the polynomial $(-1)^{l-k}(u+2)^{l-k-m}(u-2)^m$.*

Corollary 3.6. *In the coordinates $\tau^1,\dots,\tau^{l+1}$ the components $g^{ij}(\tau)$, $\Gamma^{ij}_m(\tau)$ of the metric (3.5) and its Levi-Civita connection are weighted homogeneous polynomials of the degrees*

$$\deg g^{ij}=d_i+d_j,\quad \deg\Gamma^{ij}_s(\tau)=d_i+d_j-d_s. \tag{3.31}$$

They are at most linear in τ^k.

3.2. **Flat coordinates of the metric (η^{ij}).** In this subsection, we will show that the flat coordinates of the metric (η^{ij}) defined in the last subsection are algebraic functions of $\tau^1,\dots,\tau^{l+1},e^{\tau^{l+1}}$. To this end, we first perform changes of coordinates to simplify the matrix $(\eta^{ij}(\tau))$.

Lemma 3.7. *There exists a system of coordinates $z^1,\dots,z^{l+1}$ of the form*

$$z^j=\tau^j+p_j(\tau^1,\dots,\tau^{j-1},e^{\tau^{l+1}}),\ 1\le j\le k, \tag{3.32}$$

$$z^j=\tau^j+\sum_{s=j+1}^{l-m}c^j_s\tau^s,\quad k+1\le j\le l-k-m, \tag{3.33}$$

$$z^j=\tau^j+\sum_{s=j+1}^{l}h^j_s\tau^s,\quad l-k-m+1\le j\le l, \tag{3.34}$$

$$z^{l+1}=\tau^{l+1},$$

where c^j_s and h^j_s are some constants and p_j are homogeneous polynomials of degree d_j such that in the new coordinates z^i the components of the metric (η^{ij}) can *still been encoded into a block diagonal matrix of the form (3.26)–(3.28) with the entries replaced by*

$$R_j = 0, \quad P_j = 0, \quad Q_s = 4sz^{k+s}, S_r = 4rz^{l-m+r}, \tag{3.35}$$
$$1 \le j \le k, \quad 1 \le s \le l-k-m, \quad 1 \le r \le m.$$

Proof. Let us first note that the $(k+1)\times(k+1)$ matrix $(\tilde\eta^{ij})$ which has entries

$$\tilde\eta^{ij} = \eta^{ij}(\tau),\ \tilde\eta^{k+1,m} = \tilde\eta^{m,k+1} = \delta_{m,k}, \quad 1 \le i,j \le k,\ 1 \le m \le k+1 \tag{3.36}$$

coincides, under renaming of the label of coordinate $\tau^{l+1} \mapsto \tau^{k+1}$, with the matrix $(\eta^{ij}(\tau))_{(k+1)\times(k+1)}$ that is constructed as in the last subsection with respect to the extended affine Weyl group $\widetilde{W}^{(k)}(C_k)$. Thus by using the results of [7] we can find homogeneous polynomials $p_j, 1 \le j \le k$ such that under the change of coordinates (3.32) and $z^j = \tau^j$, $k+1 \le j \le l+1$ the matrix $(\eta^{ij}(z))$ has the form (3.26)–(3.28) with entries

$$R_j = 0, \quad P_j = 0, \quad Q_s = 4sz^{k+s} + (1-\delta_{s,l-k-s})(s+1)z^{k+s+1},$$
$$S_r = 4rz^{l-m+r} - 4(1-\delta_{m,r})rz^{l-m+r+1},$$
$$1 \le j \le k, \quad 1 \le r \le m, \quad 1 \le s \le l-k-m.$$

To finish the proof of the lemma, we need to perform a second change of coordinates. To this end, denote by Ψ an $n\times n$ matrix with entries as linear functions of $a^1, \dots, a^n$

$$\psi^{ij}(a) = 4(i+j-1)a^{i+j-1} + \kappa(i,j)a^{i+j}, \quad i,j \ge 1, \tag{3.37}$$
$$\kappa(i,j) = i+j, \quad or \quad -4(i+j-1). \tag{3.38}$$

Here $a^s = 0$ for $s \ge n+1$. We are to find a linear transformation of the triangular form

$$a^j = \sum_{\alpha=j}^{n} B^j_\alpha b^\alpha, \quad B^j_j = 1,\ j \ge 1 \tag{3.39}$$

such that

$$\sum_{r,s=1}^{n} 4(r+s-1)b^{r+s-1}\frac{\partial a^i}{\partial b^r}\frac{\partial a^j}{\partial b^s} = \psi^{ij}(a). \tag{3.40}$$

Equivalently, the constants B^i_j must satisfy the relations

$$\begin{aligned} &4(i+j-1)B^{i+j-1}_\gamma + \kappa(i,j)B^{i+j}_\gamma = 4\,\gamma \sum_{\alpha+\beta=\gamma+1} B^i_\alpha B^j_\beta,\\ &i+j\le\gamma\le n. \end{aligned} \tag{3.41}$$

Introduce the generating functions

$$f^i(t) = \sum_{\alpha\ge 0} B^i_{i+\alpha}\, t^\alpha, \quad i=1,2,\dots. \tag{3.42}$$

Then the relations in (3.41) can be encoded into the following equations:

$$4(i+j-1)t^{i+j-2}f^{i+j-1} + \kappa(i,j)t^{i+j-1}f^{i+j} = 4\frac{d}{dt}\left(t^{i+j-1}f^i f^j\right). \tag{3.43}$$

When $\kappa(i,j)=i+j$ and $\kappa(i,j)=-4(i+j-1)$, this system of equations has the following solution respectively

$$f^i(t) = \cosh\left(\frac{\sqrt{t}}{2}\right)\left(\frac{2\sinh\left(\frac{\sqrt{t}}{2}\right)}{\sqrt{t}}\right)^{2i-1}, \tag{3.44}$$

and

$$f^i(t) = \left(\frac{\tanh(\sqrt{t})}{\sqrt{t}}\right)^{2i-1}. \tag{3.45}$$

From the above result we know the existence of constants c^j_s and h^j_s such that under the change of coordinates

$$\begin{aligned} &z^i\mapsto z^i,\ i=1,\dots,k,l+1,\\ &z^j\mapsto z^j+\sum_{s=j+1}^{l-m} c^j_s\, z^s, \quad k+1\le j\le l-m,\\ &z^j\mapsto z^j+\sum_{s=j+1}^{l} h^j_s\, z^s, \quad l-m+1\le j\le l, \end{aligned}$$

the matrix $(\eta^{ij}(z))$ has the form (3.26)–(3.28) and with entries given by (3.35). The lemma is proved. $\square$

Lemma 3.8. *Under the change of coordinates*

$$w^i = z^i, \quad i = 1, \ldots, k,\ l+1, \tag{3.46}$$

$$w^{k+1} = z^{k+1}(z^{l-m})^{-\frac{1}{2(l-m-k)}}, \tag{3.47}$$

$$w^s = z^s(z^{l-m})^{-\frac{s-k}{l-m-k}},\ s = k+2, \cdots, l-m-1, \tag{3.48}$$

$$w^{l-m} = (z^{l-m})^{\frac{1}{2(l-m-k)}}, \tag{3.49}$$

$$w^{l-m+1} = z^{l-m+1}(z^l)^{-\frac{1}{2m}}, \tag{3.50}$$

$$w^r = z^r(z^l)^{-\frac{r+m-l}{m}},\ r = l-m+2, \cdots, l-1, \tag{3.51}$$

$$w^l = (z^l)^{\frac{1}{2m}}, \tag{3.52}$$

the components of the metric $(\eta^{ij}(z))$ are transformed to the form

$$\begin{pmatrix} A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & B_1 & 0 & 0 \\ 0 & 0 & 0 & B_2 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.53}$$

where the matrix $A = A_{(k-1)\times(k-1)}$ has entries $A^{ij} = \delta_{i,k-j}k$ and the upper triangular matrices B_1 and B_2 have the form

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 2 \\ 0 & H_{k+3} & H_{k+4} & \cdots & H_{l-m-1} & H_{l-m} & \\ 0 & H_{k+4} & H_{k+5} & \cdots & H_{l-m} & & \\ \vdots & \vdots & \vdots & & & & \\ 0 & H_{l-m} & & & & & \\ 2 & & & & & & \end{pmatrix} \tag{3.54}$$

and

$$B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 2 \\ 0 & H_{l-m+3} & H_{l-m+4} & \cdots & H_{l-1} & H_l & \\ 0 & H_{l-m+4} & H_{l-m+5} & \cdots & H_l & & \\ \vdots & \vdots & \vdots & & & & \\ 0 & H_l & & & & & \\ 2 & & & & & & \end{pmatrix} \tag{3.55}$$

with

$$\begin{aligned} & H_{k+s} = 4s(w^{l-m})^{-2}w^{k+s},\ H_{l-m} = 4(l-m-k)\left(w^{l-m}\right)^{-2}, \\ & H_{l-m+j} = 4j(w^{l})^{-2}w^{l-m+j},\ H_l = 4m\left(w^{l}\right)^{-2}, \\ & 3 \le s \le l-m-k-1, \quad 3 \le j \le m-1. \end{aligned} \tag{3.56}$$

Proof. By a straightforward calculation. □

Remark 3.9. *When $m = l-k$, the matrix B_1 does not appear in (3.53), i.e., the matrix given in (3.53) has the form*

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & B_2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

In this case we use the formulae (3.46), (3.50)–(3.52) for the change of coordinates. When $m = l-k-1$, we have $B_1 = 1$, and we use the formulae (3.46), (3.49)–(3.52) to define the new coordinates. When $m = l-k-2$, the matrix B_1 has the form $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$. We understand the above lemma in a similar way as we did for the cases when $m = 0, 1, 2$.

Theorem 3.10. *We can choose the flat coordinates of the metric* $(\eta^{ij}(w))$ *in the form*

$$
\begin{aligned}
&t^1 = w^1, \dots, t^k = w^k,\ t^{l+1} = w^{l+1},\\
&t^{k+1} = w^{k+1} + w^{l-m}\, h_{k+1}(w^{k+2}, \dots, w^{l-m-1}),\\
&t^j = w^{l-m}(w^j + h_j(w^{j+1}, \dots, w^{l-m-1})),\ k+2 \le j \le l-m-1,\\
&t^{l-m} = w^{l-m},\\
&t^{l-m+1} = w^{l-m+1} + w^l\, h_{l-m+1}(w^{l-m+2}, \dots, w^{l-1}),\\
&t^s = w^l(w^s + h_s(w^{s+1}, \dots, w^{l-1})),\ l-m+2 \le s \le l-1,\\
&t^l = w^l.
\end{aligned}
$$

Here $h_{l-m-1} = h_{l-1} = 0$, h_j *are weighted homogeneous polynomials of degree* $\frac{k\,(l-m-j)}{l-m-k}$ *for* $j = k+1, \dots, l-m-2$ *and* h_s *are weighted homogeneous polynomials of degree* $\frac{k\,(l-s)}{m}$ *for* $s = l-m+2, \dots, l-1$. *The degrees of the coordinates* w^i *are defined in a natural way through the degrees of* y^i *given in (2.7).*

Proof. From the block diagonal form (3.53) of the matrix $(\eta^{ij}(w))$ and the definition (3.54)–(3.56) of its entries, we know that the flat coordinates can be chosen to have the form

$$t^i = w^i,\ 1 \le i \le k,\ i = l+1, \tag{3.57}$$

$$t^j = t^j(w^{k+1}, \dots, w^{l-m}), \quad k+1 \le j \le l-m \tag{3.58}$$

$$t^s = t^s(w^{l-m+1}, \dots, w^l), \quad l-m+1 \le s \le l. \tag{3.59}$$

Since the matrices B_1 and B_2 have the same form, and B_1 becomes constant when $m = l-k$ or $m = l-k-1$, we only need to consider the flat coordinates (3.58) for the metric that corresponds to the matrix B_1 defined in (3.54) with $m \le l-k-3$.

The functions $t^j = t^j(w^{k+1}, \dots, w^{l-m})$ must satisfy the following system of PDEs

$$\frac{\partial^2 t}{\partial w^a \partial w^b} - \sum_{c=1}^{l+1} \gamma^c_{ab} \frac{\partial t}{\partial w^c} = 0, \quad a, b = k+1, \dots, l-m. \tag{3.60}$$

Introduce the $(l-m-k)\times(l-m-k)$ matrix

$$\Phi=(\phi^i_j),\quad \phi^i_j=\frac{\partial t^{k+i}}{\partial w^{k+j}},\quad 1\le i,j\le l-m-k,$$

Then the system (3.60) can be written in the form

$$\partial_s\Phi=\Phi A_s,\quad \partial_s=\frac{\partial}{\partial w^s},\quad s=k+1,\dots,l-m, \tag{3.61}$$

where the entries of the coefficient matrices A_s are rational functions of $w^{k+1},\dots,w^{l-m}$. It follows from the simple expressions of the entries of the matrix B_1 that the systems (3.61) are regular at $\mathbf{w}=(w^{k+1},\ \dots,w^{l-m})=0$ except for case when $s=l-m$, in this case the coefficient matrix has the form

$$A_{l-m}=\mathrm{diag}(0,\frac{1}{w^{l-m}},\dots,\frac{1}{w^{l-m}},0).$$

Note for all the cases with $m=k+1,\dots,l-m-1$ the entries of the matrices A_s are weighted homogeneous polynomials of $w^{k+1},\dots,w^{l-m}$.

Now we put Φ in the form

$$\Phi=\Psi\,\mathrm{diag}(1,w^{l-m},\dots,w^{l-m},1),$$

then the systems in (3.61) are converted to

$$\partial_s\Psi=\Psi B_s,\quad \partial_l\Psi=0,\quad s=k+1,\dots,l-m-1.$$

The entries of the coefficient matrices B_s are now weighted homogeneous polynomials of $w^{k+1},\dots,w^{l-m}$, thus we can find a unique solution Ψ of the above systems such that it is analytic at $\mathbf{w}=0$ and

$$\Psi|_{\mathbf{w}=0}=\mathrm{diag}(1,\dots,1).$$

From the weighted homogeneity of the coefficient matrices B_s it follows that the elements of Ψ are also weighted homogeneous. Since $\deg w^j>0$ for $j=k+1,\dots,l-m$ we know that they are in fact polynomials of $w^{k+1},\dots,w^{l-m}$, and thus the results of the theorem follow. The theorem is proved. □

Due to the above construction, we can associate the following natural degrees to the flat coordinates

$$\tilde{d}_j = \deg t^j := \frac{j}{k}, \quad 1 \le j \le k, \tag{3.62}$$

$$\tilde{d}_s = \deg t^s := \frac{2l-2m-2s+1}{2(l-m-k)}, \quad k+1 \le s \le l-m, \tag{3.63}$$

$$\tilde{d}_\alpha = \deg t^\alpha := \frac{2l-2\alpha+1}{2m}, \quad l-m+1 \le \alpha \le l, \tag{3.64}$$

$$\tilde{d}_{l+1} = \deg t^{l+1} := 0, \quad \deg e^{t^{l+1}} := \frac{1}{k}, \tag{3.65}$$

and we readily have the following corollary.

Corollary 3.11. *In the flat coordinates $t^1, \dots, t^{l+1}$, the nonzero entries of the matrix $(\eta^{ij}(t))$ are given by*

$$\eta^{ij} = \begin{cases} k, & j = k-i, \qquad 1 \le i \le k-1, \\ 1, & i = l+1, j = k \quad \text{or } i = k,\ j = l+1, \\ 4(l-m-k), & j = l-m+k-i+1, \quad k+2 \le i \le l-m-1, \\ 2, & i = l-m, j = k+1 \quad \text{or } i = k+1,\ j = l-m, \\ 4m, & j = 2l-m-i+1, \quad l-m+2 \le i \le l-1, \\ 2, & i = l, j = l-m+1 \quad \text{or } i = l-m+1,\ j = l. \end{cases} \tag{3.66}$$

The entries of the matrix $(g^{ij}(t))$ and the Christoffel symbols $\Gamma^{ij}_m(t)$ are weighted homogeneous polynomials of $t^1, \dots, t^l, \frac{1}{t^{l-m}}, \frac{1}{t^l}, e^{t^{l+1}}$ of degrees $\tilde{d}_i + \tilde{d}_j$ and $\tilde{d}_i + \tilde{d}_j - \tilde{d}_m$ respectively. In particular,

$$\begin{aligned} g^{s,l+1} &= \tilde{d}_s t^s, \qquad 1 \le s \le l, \quad g^{l+1,l+1} = \frac{1}{k}, \\ \Gamma^{l+1,i}_j &= \tilde{d}_j\, \delta_{i,j}, \qquad 1 \le i, j \le l+1. \end{aligned} \tag{3.67}$$

The numbers $\tilde{d}_1, \dots, \tilde{d}_{l+1}$ satisfy a duality relation that is similar to that of [7]. To describe this duality relation, let us delete the k-th vertex of the Dynkin diagram $\mathcal{R}$. We then obtain two components $\mathcal{R} \setminus \alpha_k = \mathcal{R}_1 \cup \mathcal{R}_2$. For any given integer $0 \le m \le l-k$, we denote $\mathcal{R}_2 = \mathcal{R}_{21} \cup \mathcal{R}_{22}$, where $\mathcal{R}_{21} = \{\alpha_{k+1}, \cdots, \alpha_{l-m}\}$ and $\mathcal{R}_{22} = \{\alpha_{l-m+1}, \cdots, \alpha_l\}$. On each component we have an involution $i \mapsto i^*$

given by the reflection with respect to the center of the component. Define

$$k^* = l+1, \quad (l+1)^* = k, \tag{3.68}$$

then we have

$$\tilde{d}_i + \tilde{d}_{i^*} = 1, \quad i = 1, \dots, l+1, \tag{3.69}$$

and from the above corollary we see that η^{ij} is a nonzero constant iff $j = i^*$.

3.3. **Frobenius manifold structures on the orbit space of $\widetilde{W}^{(k)}(C_l)$.** Now we are ready to describe the Frobenius manifold structures on the orbit space of the extended affine Weyl group $\widetilde{W}^{(k)}(C_l)$. Let us first recall the definition of Frobenius manifold, see [6] for details.

Definition 3.12. *A Frobenius algebra is a pair $(A, < \, , \, >)$ where A is a commutative associative algebra with a unity e over a field $\mathcal{K}$ (in our case $\mathcal{K} = \mathbb{C}$) and $< \, , \, >$ is a $\mathcal{K}$-bilinear symmetric nondegenerate invariant form on A, i.e.,*

$$< x \cdot y, z > = < x, y \cdot z >, \quad \forall\, x, y, z \in A.$$

Definition 3.13. *A Frobenius structure of charge d on an n-dimensional manifold M is a structure of Frobenius algebra on the tangent spaces $T_tM = (A_t, < \, , \, >_t)$ depending (smoothly, analytically etc.) on the point t. This structure satisfies the following axioms:*

- FM1. *The metric $< \, , \, >_t$ on M is flat, and the unity vector field e is covariantly constant, i.e., $\nabla e = 0$. Here we denote ∇ the Levi-Civita connection for this flat metric.*
- FM2. *Let c be the 3-tensor $c(x, y, z) := < x \cdot y, z >$, x, y, $z \in T_tM$. Then the 4-tensor $(\nabla_w c)(x, y, z)$ is symmetric in x, y, z, $w \in T_tM$.*
- FM3. *The existence on M of a vector field E, called the Euler vector field, which satisfies the conditions $\nabla\nabla E = 0$ and*

$$[E, x \cdot y] - [E, x] \cdot y - x \cdot [E, y] = x \cdot y,$$

$$E < x, y > - < [E, x], y > - < x, [E, y] > = (2 - d) < x, y >$$

for any vector fields x, y on M.

A manifold M equipped with a Frobenius structure on it is called a Frobenius manifold.

Let us choose local flat coordinates $t^1, \cdots t^n$ for the invariant flat metric, then locally there exists a function $F(t^1, \cdots, t^n)$, called the *potential* of the Frobenius manifold, such that

$$< u \cdot v, w >= u^i v^j w^s \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^s} \tag{3.70}$$

for any three vector fields $u = u^i \frac{\partial}{\partial t^i}$, $v = v^j \frac{\partial}{\partial t^j}$, $w = w^s \frac{\partial}{\partial t^s}$. Here and in what follows summations over repeated indices are assumed. By definition, we can also choose the coordinates t^1 such that $e = \frac{\partial}{\partial t^1}$. Then in the flat coordinates the components of of the flat metric $< \frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j} >$ can be expressed in the form

$$\frac{\partial^3 F}{\partial t^1 \partial t^i \partial t^j} = \eta_{ij}, \quad i, j = 1, \ldots, n. \tag{3.71}$$

The associativity of the Frobenius algebras is equivalent to the following overdetermined system of equations for the function F

$$\frac{\partial^3 F}{\partial t^i \partial t^j \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^k \partial t^m} = \frac{\partial^3 F}{\partial t^k \partial t^j \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^i \partial t^m} \tag{3.72}$$

for arbitrary indices i, j, k, m from 1 to n.

In the flat coordinates the Euler vector field E has the form

$$E = \sum_{i=1}^{n} (\hat{d}_i t^i + r_i) \frac{\partial}{\partial t^i} \tag{3.73}$$

for some constants $\hat{d}_i, r_i$, $i = 1, \ldots, n$ which satisfy $\hat{d}_1 = 1, r_1 = 0$. From the axiom FM3, it follows that the potential F satisfies the quasi-homogeneity condition

$$\mathcal{L}_E F = (3 - d)F + \text{quadratic polynomial in t.} \tag{3.74}$$

The system (3.71)–(3.74) is called the $WDVV$ *equations of associativity* which is equivalent to the above definition of Frobenius manifold in the chosen system of local coordinates.

Let us also recall an important geometrical structure on a Frobenius manifold M, the *intersection form* of M. This is a symmetric bilinear form $(\ , \ )^*$ on T^*M

defined by the formula

$$(w_1, w_2)^* = i_E(w_1 \cdot w_2), \tag{3.75}$$

here the product of two 1-forms w_1, w_2 at a point $t \in M$ is defined by using the algebra structure on T_tM and the isomorphism

$$T_tM \to T_t^*M \tag{3.76}$$

established by the invariant flat metric $<\ ,\ >$. In the flat coordinates $t^1, \cdots, t^n$ of the invariant metric, the intersection form can be represented by

$$(dt^i, dt^j)^* = \mathcal{L}_E F^{ij} = (d - 1 + \hat{d}_i + \hat{d}_j)F^{ij}, \tag{3.77}$$

where

$$F^{ij} = \eta^{ii'}\eta^{jj'}\frac{\partial^2 F}{\partial t^{i'}\partial t^{j'}} \tag{3.78}$$

and $F(t)$ is the potential of the Frobenius manifold. Denote by $\Sigma \subset M$ the *discriminant* of M on which the intersection form degenerates, then an important property of the intersection form is that on $M \setminus \Sigma$ its inverse defines a new flat metric.

Theorem 3.14. *For any fixed integer $0 \le m \le l-k$, there exists a unique Frobenius manifold structure of charge $d = 1$ living on the covering of the orbit space $\mathcal{M} \setminus \{t^{l-m} = 0\} \cup \{t^l = 0\}$ of $\widetilde{W}^{(k)}(C_l)$ polynomial in $t^1, \cdots, t^{l+1}, \frac{1}{t^{l-m}}, \frac{1}{t^l}, e^{t^{l+1}}$ such that*

(1) *The unity vector field e coincides with $\sum_{j=k}^{l} c_j \frac{\partial}{\partial y^j} = \frac{\partial}{\partial t^k}$;*

(2) *The Euler vector field has the form*

$$E = \sum_{\alpha=1}^{l} \tilde{d}_\alpha t^\alpha \frac{\partial}{\partial t^\alpha} + \frac{1}{k}\frac{\partial}{\partial t^{l+1}} \tag{3.79}$$

where $\tilde{d}_1, \ldots, \tilde{d}_l$ are defined in (3.62)–(3.64).

(3) *The invariant flat metric and the intersection form of the Frobenius manifold structure coincide respectively with the metric $(\eta^{ij}(t))$ and $(g^{ij}(t))$ on the covering of $\mathcal{M} \setminus \{t^{l-m} = 0\} \cup \{t^l = 0\}$.*

Proof. By following the lines of the proof of Lemma 2.6 given in [7] we can show the existence of a unique weighted homogeneous polynomial

$$G := G(t^1, \ldots, t^{k-1}, t^{k+1}, \ldots, t^l, \frac{1}{t^{l-m}}, \frac{1}{t^l}, e^{t^{l+1}})$$

of degree 2 such that the function

$$F = \frac{1}{2}(t^k)^2 t^{l+1} + \frac{1}{2} t^k \sum_{i,j \neq k} \eta_{ij}\, t^i t^j + G \tag{3.80}$$

satisfies the equations

$$g^{ij} = \mathcal{L}_E F^{ij}, \quad \Gamma^{ij}_m = \tilde{d}_j\, c^{ij}_m, \quad i, j, m = 1, \ldots, l+1, \tag{3.81}$$

where $c^{ij}_m = \frac{\partial F^{ij}}{\partial t^m}$. Obviously, the function F satisfies the equations

$$\frac{\partial^3 F}{\partial t^k \partial t^i \partial t^j} = \eta_{ij}, \quad i, j = 1, \ldots, l+1 \tag{3.82}$$

and the quasi-homogeneity condition

$$\mathcal{L}_E F = 2F. \tag{3.83}$$

From the properties of a flat pencil of metrics [6] it follows that F also satisfies the associativity equations

$$c^{ij}_m\, c^{mp}_q = c^{ip}_m\, c^{mj}_q \tag{3.84}$$

for any set of fixed indices i, j, p, q. Now the theorem follows from above properties of the function F and the simple identity $\mathcal{L}_E e = -e$. The theorem is proved. □

Remark 3.15. *It follows from Remark 3.5 that the Frobenius manifold structures which correspond to the integers m and $l-k-m$ are equivalent. From the above construction we see that the potential F is in general a polynomial of $t^1, \ldots, t^{l+1}, \frac{1}{t^{l-m}}, \frac{1}{t^l}, e^{t^{l+1}}$, in the particular cases when $m = 1$ and $m = l-k-1$ it does not depend on $\frac{1}{t^l}$ and $\frac{1}{t^{l-1}}$ respectively. When $k = l$ the Frobenius manifold structure coincides with the one that is constructed in* [7].

3.4. **Examples.** To end up this section we give some examples to illustrate the above construction of Frobenius manifold structures. For brevity, instead of $t^1, \dots, t^{l+1}$ we will denote the flat coordinates of the metric η^{ij} by $t_1, \dots, t_{l+1}$, and we will also denote $\partial_i = \frac{\partial}{\partial t_i}$ in the the following examples.

Example 3.16. $[C_3, k=1]$ *Let R be the root system of type C_3, take $k=1$, then $d_1 = d_2 = d_3 = 1$, and*

$$
\begin{aligned}
y^1 &= e^{2\,i\pi\,x_4}\left(\xi_1+\xi_2+\xi_3\right),\\
y^2 &= e^{2\,i\pi\,x_4}\left(\xi_1\xi_2+\xi_1\xi_3+\xi_2\xi_3\right),\\
y^3 &= e^{2\,i\pi\,x_4}\xi_1\xi_2\xi_3,\\
y^4 &= 2\,i\pi\,x_4,
\end{aligned}
$$

where $\xi_j = e^{2\,i\pi\,(x_j-x_{j-1})} + e^{-2\,i\pi\,(x_j-x_{j-1})}$ and $x_0 = 0$, $j = 1,2,3$. The metric $(\ ,\ )^{\sim}$ has the form

$$
((dx_i, dx_j)^{\sim}) = \frac{1}{4\pi^2}\begin{pmatrix} 1 & 1 & 1 & 0\\ 1 & 2 & 2 & 0\\ 1 & 2 & 3 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}.
$$

Case I. *$m = 0$, i.e., $e = \frac{\partial}{\partial y^1} - 4\frac{\partial}{\partial y^2} + 4\frac{\partial}{\partial y^3}$.*
We first introduce the variables

$$
\begin{aligned}
&z^1 = y^1 + 6\,e^{y^4},\ \ z^2 = y^2 + 4\,y^1 + 12\,e^{y^4},\\
&z^3 = y^3 + 2\,y^2 + 4\,y^1 + 8\,e^{y^4},\ \ z^4 = y^4.
\end{aligned}
$$

Then the flat coordinates are given by

$$
t_1 = z^1 - 2\,e^{z_4},\ \ t_2 = (z^2 - \frac{1}{6}\,z^3)(z^3)^{-\frac{1}{4}},\ \ t_3 = (z^3)^{\frac{1}{4}},\ \ t_4 = z^4
$$

and the intersection form has the expression

$$
\begin{aligned}
g^{11} &= 2\,t_2t_3\,e^{t_4} + \frac{1}{3}\,{t_3}^4e^{t_4} + 4\,e^{2t_4},\\
g^{12} &= \frac{7}{3}\,{t_3}^3e^{t_4} + \frac{7}{2}\,t_2e^{t_4},\ g^{13} = \frac{5}{2}\,t_3e^{t_4},\ g^{14} = t_1,\\
g^{22} &= 12\,{t_3}^2e^{t_4} - \frac{1}{4}\,{t_2}^2 + \frac{1}{12}\,{t_3}^3t_2 - \frac{1}{108}\,{t_3}^6 + \frac{1}{4}\,\frac{{t_2}^3}{{t_3}^3},\\
g^{23} &= 2\,t_1 + 4\,e^{t_4} - \frac{1}{3}\,t_2t_3 + \frac{1}{72}\,{t_3}^4 - \frac{1}{4}\,\frac{{t_2}^2}{{t_3}^2},\\
g^{24} &= \frac{3}{4}\,t_2,\ g^{33} = \frac{1}{4}\,\frac{t_2}{t_3} - \frac{1}{12}\,{t_3}^2,\ g^{34} = \frac{1}{4}\,t_3,\ g^{44} = 1.
\end{aligned}
$$

The potential has the form

$$
\begin{aligned}
F = {}& \frac{1}{2}\,{t_1}^2t_4 + \frac{1}{2}\,t_1t_2t_3 - \frac{1}{48}\,{t_2}^2{t_3}^2 + \frac{1}{1440}\,t_2{t_3}^5 - \frac{1}{36288}\,{t_3}^8\\
&+t_2t_3e^{t_4} + \frac{1}{6}\,{t_3}^4e^{t_4} + \frac{1}{2}\,e^{2t_4} + \frac{1}{48}\,\frac{{t_2}^3}{t_3}
\end{aligned}
$$

and the Euler vector field is given by

$$
E = t_1\partial_1 + \frac{3}{4}t_2\partial_2 + \frac{1}{4}t_3\partial_3 + \partial_4.
$$

Case II. $m = 1$, *i.e.*, $e = \frac{\partial}{\partial y^1} - 4\frac{\partial}{\partial y^3}$.
Define

$$
\begin{aligned}
z^1 &= y^1 + 2\,e^{y^4},\ z^2 = \frac{1}{2}\,y^2 + \frac{1}{4}\,y^3 + y^1 + 2\,e^{y^4},\\
z^3 &= \frac{1}{4}\,y^3 - \frac{1}{2}\,y^2 + y^1 - 2\,e^{y^4},\ z^4 = y^4.
\end{aligned}
$$

Then the flat coordinates are

$$
t_1 = z^1 - 2e^{z^4},\ t_2 = \sqrt{z^2},\ t_3 = \sqrt{z^3},\ t_4 = z^4
$$

and the intersection form is given by

$$
\begin{aligned}
&g^{11} = 2\,{t_2}^2 e^{t_4} - 2\,{t_3}^2 e^{t_4} + 4\,e^{2t_4},\\
&g^{12} = 3\,t_2 e^{t_4},\;\; g^{13} = -3\,t_3 e^{t_4},\;\; g^{14} = t_1,\\
&g^{22} = 2\,e^{t_4} + t_1 - \frac{1}{4}\,{t_3}^2 - \frac{1}{4}\,{t_2}^2,\;\; g^{23} = -\frac{1}{2}\,t_2 t_3,\\
&g^{33} = -2\,e^{t_4} + t_1 - \frac{1}{4}\,{t_2}^2 - \frac{1}{4}\,{t_3}^2,\\
&g^{24} = \frac{1}{2}\,t_2,\;\; g^{34} = \frac{1}{2}\,t_3,\;\; g^{44} = 1.
\end{aligned}
$$

The potential has the expression

$$
\begin{aligned}
F = \quad & \frac{1}{2}\,t_1 {t_2}^2 + \frac{1}{2}\,t_1 {t_3}^2 + \frac{1}{2} {t_1}^2\,t_4 - \frac{1}{48}\,{t_2}^4\\
& - \frac{1}{48}\,{t_3}^4 - \frac{1}{8}\,{t_2}^2 {t_3}^2 + {t_2}^2 e^{t_4} - {t_3}^2 e^{t_4} + \frac{1}{2}\,e^{2t_4}
\end{aligned}
$$

and the Euler vector field is given by

$$
E = t_1\partial_1 + \frac{1}{2} t_2\partial_2 + \frac{1}{2} t_3\partial_3 + \partial_4.
$$

The Frobenius manifold structure that we obtain for this case is isomorphic to the one given in Example 2.6 $[A_3, k = 2]$ of [7].

Example 3.17. $[C_3, k = 2]$ *Let R be the root system of type C_3, take $k = 2$, then $d_1 = 1, d_2 = d_3 = 2$, and*

$$
\begin{aligned}
&y^1 = e^{2\,i\pi\,x_4}\left(\xi_1 + \xi_2 + \xi_3\right),\\
&y^2 = e^{2\,i\pi\,x_4}\left(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3\right),\\
&y^3 = e^{2\,i\pi\,x_4}\xi_1\xi_2\xi_3,\\
&y^4 = 2\,i\pi\,x_4,
\end{aligned}
$$

where $\xi_j = e^{2\,i\pi\,(x_j - x_{j-1})} + e^{-2\,i\pi\,(x_j - x_{j-1})}$ *and* $x_0 = 0$, $j = 1, 2, 3$. *The metric* $(\ ,\ )^{\sim}$ *has the form*

$$((dx_i, dx_j)^{\sim}) = \frac{1}{4\pi^2}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

Case I. $m = 0$, *i.e.,* $e = \frac{\partial}{\partial y^2} - 2\frac{\partial}{\partial y^3}$. *The Frobenius manifold structure that we obtain for this case is isomorphic to the one given in Example 2.7 [*$B_3, k = 2$*] of* [7].

Case II. $m = 1$, *i.e.,* $e = \frac{\partial}{\partial y^2} + 2\frac{\partial}{\partial y^3}$.

We first introduce the following variables

$$z^1 = y^1 + 2\,e^{y^4},\ z^2 = y^2 + 4e^{2y^4},$$
$$z^3 = 2\,y^2 - 4\,y^1 e^{y_4} - y^3 + 8\,e^{2y^4},\ z^4 = y^4.$$

Then the flat coordinates given by

$$t_1 = z^1 - 4e^{z^4},\ t_2 = z^2 - 2\,z^1 e^{z^4} + 6\,e^{2\,z^4},\ t_3 = \sqrt{z^3},\ t_4 = z^4.$$

The potential has the expression

$$\begin{aligned} F = \quad & \frac{1}{2}\,t_2 {t_3}^2 + \frac{1}{4}\,{t_1}^2 t_2 + \frac{1}{2}\,{t_2}^2 t_4 - \frac{1}{48}\,{t_3}^4 \\ & -\frac{1}{96}\,{t_1}^4 + {t_3}^2 e^{2t_4} - {t_3}^2 t_1 e^{t_4} + \frac{1}{2}\,{t_1}^2 e^{2t_4} + \frac{1}{4}\,e^{4t_4} \end{aligned}$$

and the Euler vector field is given by

$$E = \frac{1}{2} t_1 \partial_1 + t_2 \partial_2 + \frac{1}{2} t_3 \partial_3 + \frac{1}{2}\partial_4.$$

This Frobenius manifold structure is exactly the one given in Example 2.7 [$B_3, k = 2$*] of* [7].

Example 3.18. $[C_4, k=1, m=0]$ *Let R be the root system of type C_4, take $k=1$, then $d_1=d_2=d_3=d_4=1$, and*

$$\begin{aligned}
y^1 &= e^{2\,i\pi\,x_5}\left(\xi_1+\xi_2+\xi_3+\xi_4\right),\\
y^2 &= e^{2\,i\pi\,x_5}\sum_{1\le a<b\le 4}\xi_a\xi_b,\\
y^3 &= e^{2\,i\pi\,x_5}\sum_{1\le a<b<c\le 4}\xi_a\xi_b\xi_c,\\
y^4 &= e^{2\,i\pi\,x_5}\xi_1\xi_2\xi_3\xi_4,\\
y^5 &= 2\,i\pi\,x_5,
\end{aligned}$$

where $\xi_j = e^{2\,i\pi\,(x_j-x_{j-1})}+e^{-2\,i\pi\,(x_j-x_{j-1})}$ and $x_0=0$, $j=1,2,3,4$. The metric $(\ ,\ )^{\sim}$ has the form

$$((dx_i,dx_j)^{\sim}) = \frac{1}{4\pi^2}\begin{pmatrix} 1 & 1 & 1 & 1 & 0\\ 1 & 2 & 2 & 2 & 0\\ 1 & 2 & 3 & 3 & 0\\ 1 & 2 & 3 & 4 & 0\\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Introduce the variables

$$\begin{aligned}
&z^1 = y^1+8\,e^{y^5},\ \ z^2 = y^2+6\,y^1+24\,e^{y^5},\\
&z^3 = y^3+4\,y^2+12\,y^1+32\,e^{y^5},\ \ z^5=y^5,\\
&z^4 = y^4+2\,y^3+8\,y^1+4\,y^2+16\,e^{y^5},
\end{aligned}$$

and

$$\begin{aligned}
&w_1 = z^1-2e^{z^5},\ \ w_2 = (z^2-\frac{1}{6}\,z^3+\frac{1}{30}\,z^4)(z^4)^{-\frac{1}{6}},\\
&w_3 = (z^3-\frac{1}{4}\,z^4)(z^4)^{-\frac{2}{3}},\ \ w_4 = (z^4)^{\frac{1}{6}},\ \ w_5 = z^5.
\end{aligned}$$

Then we have the expression of the flat coordinates

$$t_1 = w_1,\ \ t_2 = w_2-\frac{1}{12}w_3^2\,w_4,\ \ t_3 = w_3w_4,\ \ t_4=w_4,\ \ t_5=w_5.$$

The potential F is given by

$$\begin{aligned}F = &\frac{1}{2}\,{t_1}^2t_5 + \frac{1}{2}\,t_1t_2t_4 - \frac{1}{6912}\,{t_3}^4 + \frac{1}{17280}\,{t_3}^3{t_4}^3\\ &-\frac{1}{288}\,t_2t_4{t_3}^2 - \frac{1}{34560}\,{t_4}^6{t_3}^2 + \frac{1}{24}\,t_1{t_3}^2 + \frac{1}{1440}\,t_3{t_4}^4t_2\\ &-\frac{1}{48}\,{t_2}^2{t_4}^2 - \frac{1}{60480}\,{t_4}^7t_2 + \frac{1}{345600}\,{t_4}^9t_3 - \frac{1}{7603200}\,{t_4}^{12}\\ &+\frac{1}{12}\,e^{t_5}{t_3}^2 + \frac{1}{6}\,e^{t_5}t_3{t_4}^3 + \frac{1}{120}\,e^{t_5}{t_4}^6 + t_2t_4e^{t_5} + \frac{1}{2}\,e^{2t_5}\\ &+\frac{1}{24}\,\frac{t_3{t_2}^2}{t_4} - \frac{1}{216}\,\frac{t_2{t_3}^3}{{t_4}^2} + \frac{1}{4320}\,\frac{{t_3}^5}{{t_4}^3}\end{aligned}$$

with the Euler vector field

$$E = t_1\partial_1 + \frac{5}{6}t_2\partial_2 + \frac{1}{2}t_3\partial_3 + \frac{1}{6}t_4\partial_4 + \partial_5.$$

Example 3.19. $[C_4, k = 2, m = 0]$ *Let R be the root system of type C_4, take $k = 2$, then $d_1 = 1, d_2 = d_3 = d_4 = 2$, and*

$$\begin{aligned}y^1 &= e^{2\,i\pi\,x_5}\,(\xi_1 + \xi_2 + \xi_3 + \xi_4)\,,\\ y^2 &= e^{4\,i\pi\,x_5}\sum_{1\le a<b\le 4}\xi_a\xi_b,\\ y^3 &= e^{4\,i\pi\,x_5}\sum_{1\le a<b<c\le 4}\xi_a\xi_b\xi_c,\\ y^4 &= e^{4\,i\pi\,x_5}\xi_1\xi_2\xi_3\xi_4,\\ y^5 &= 2\,i\pi\,x_5,\end{aligned}$$

where ξ_j are defined as in the last example. The metric $(\ ,\ )^{\sim}$ has the form

$$((dx_i, dx_j)^{\sim}) = \frac{1}{4\pi^2}\begin{pmatrix} 1 & 1 & 1 & 1 & 0\\ 1 & 2 & 2 & 2 & 0\\ 1 & 2 & 3 & 3 & 0\\ 1 & 2 & 3 & 4 & 0\\ 0 & 0 & 0 & 0 & -\frac{1}{2}\end{pmatrix}.$$

Introduce the following variables

$$
\begin{aligned}
z^1 &= y^1 + 8\,e^{y^5},\ \ z^5 = y^5,\\
z^2 &= y^2 + 6\,y^1 e^{y^5} + 24\,e^{2y^5},\\
z^3 &= y^3 + 4\,y^2 + 12\,y^1 e^{y^5} + 32\,e^{2y^5},\\
z^4 &= y^4 + 2\,y^3 + 4\,y^2 + 8\,y^1 e^{y^5} + 16\,e^{2y_5}.
\end{aligned}
$$

Then the flat coordinates are given by

$$
\begin{aligned}
&t_1 = z^1 - 4e^{z^5},\ \ t_2 = z^2 - 2z^1 e^{z^5} + 6e^{2z^5},\\
&t_3 = (z^3 - \frac{1}{6}\,z^4)(z^4)^{-\frac{1}{4}},\ \ t_4 = (z^4)^{\frac{1}{4}},\ \ t_5 = z^5.
\end{aligned}
$$

The Euler vector field and the potential are given respectively by

$$
\begin{aligned}
E =\ & \frac{1}{2}t_1\partial_1 + t_2\partial_2 + \frac{3}{4}t_3\partial_3 + \frac{1}{4}t_4\partial_4 + \frac{1}{2}\partial_5.\\
F =\ & \frac{1}{2}\,{t_2}^2 t_5 + \frac{1}{4}\,{t_1}^2 t_2 + \frac{1}{2}\,t_4 t_3 t_2 + \frac{1}{1440}\,{t_4}^5 t_3 - \frac{1}{48}\,{t_4}^2 {t_3}^2\\
& - \frac{1}{36288}\,{t_4}^8 - \frac{1}{96}\,{t_1}^4 + \frac{1}{2}\,e^{2\,t_5}{t_1}^2 + \frac{1}{6}\,e^{t_5} t_1 {t_4}^4 + \frac{2}{3}\,{t_4}^4 e^{2\,t_5}\\
& + e^{t_5} t_1 t_3 t_4 + t_3 t_4 e^{2\,t_5} + \frac{1}{4}\,e^{4\,t_5} + \frac{1}{48}\,\frac{{t_3}^3}{t_4}.
\end{aligned}
$$

In the following, we present two more examples and omit all computations and only list the potentials and the Euler vector fields.

Example 3.20. $[C_5, k = 1, m = 2]$ *Let* R *be the root system of type* C_5*, take* $k = 1, m = 2$*, then*

$$
\begin{aligned}
F =\ \ & \frac{1}{2}\,t_6 {t_1}^2 + \frac{1}{2}\,t_1 t_2 t_3 + \frac{1}{2}\,t_1 t_4 t_5 - \frac{1}{72}\,{t_3}^4 {t_5}^4 - \frac{1}{8}\,t_2 t_3 t_4 t_5\\
& - \frac{1}{2268}\,{t_5}^8 - \frac{1}{36288}\,{t_3}^8 - \frac{1}{48}\,{t_3}^2 {t_2}^2 - \frac{1}{48}\,{t_4}^2 {t_5}^2 + \frac{1}{24}\,{t_5}^4 t_2 t_3\\
& + \frac{1}{96}\,{t_3}^4 t_4 t_5 + \frac{1}{1440}\,{t_3}^5 t_2 + \frac{1}{360}\,t_4 {t_5}^5 + t_2 t_3 e^{t_6} - t_4 t_5 e^{t_6}\\
& - \frac{2}{3}\,{t_5}^4 e^{t_6} + \frac{1}{6}\,{t_3}^4 e^{t_6} + \frac{1}{2}\,e^{2t_6} + \frac{1}{48}\,\frac{{t_2}^3}{t_3} + \frac{1}{192}\,\frac{{t_4}^3}{t_5}.
\end{aligned}
$$

The Euler vector field is given by

$$E = t_1\partial_1 + \frac{3}{4}t_2\partial_2 + \frac{1}{4}t_3\partial_3 + \frac{3}{4}t_4\partial_4 + \frac{1}{4}t_5\partial_5 + \partial_6.$$

Example 3.21. $[C_6, k = 1, m = 2]$ *Let R be the root system of type C_6, take $k = 1$, then*

$$\begin{aligned} F = \quad & \frac{1}{2}\,t_1{}^2t_7 + \frac{1}{24}\,t_1t_3{}^2 + \frac{1}{2}\,t_1t_2t_4 + \frac{1}{2}\,t_1t_5t_6 - \frac{1}{48}\,t_2{}^2t_4{}^2 \\ & + \frac{1}{17280}\,t_4{}^3t_3{}^3 - \frac{1}{48}\,t_5{}^2t_6{}^2 + \frac{1}{360}\,t_5t_6{}^5 + \frac{1}{288}\,t_3{}^2t_6{}^4 \\ & + \frac{17}{5760}\,t_6{}^4t_4{}^6 - \frac{1}{60480}\,t_4{}^7t_2 - \frac{1}{72}\,t_6{}^4t_4{}^3t_3 - \frac{1}{288}\,t_2t_3{}^2t_4 \\ & + \frac{1}{1440}\,t_2t_3t_4{}^4 - \frac{1}{96}\,t_3{}^2t_5t_6 - \frac{1}{2268}\,t_6{}^8 - \frac{1}{34560}\,t_4{}^6t_3{}^2 \\ & - \frac{1}{6912}\,t_3{}^4 - \frac{1}{7603200}\,t_4{}^{12} + \frac{1}{24}\,t_6{}^4t_2t_4 - \frac{1}{960}\,t_6t_4{}^6t_5 \\ & + \frac{1}{345600}\,t_4{}^9t_3 - \frac{1}{8}\,t_6t_2t_4t_5 + \frac{1}{96}\,t_6t_4{}^3t_3t_5 + \frac{1}{6}\,t_4{}^3t_3e^{t_7} \\ & + \frac{1}{120}\,t_4{}^6e^{t_7} + t_2t_4e^{t_7} - t_5t_6e^{t_7} + \frac{1}{12}\,t_3{}^2e^{t_7} - \frac{2}{3}\,t_6{}^4e^{t_7} \\ & + \frac{1}{2}\,e^{2\,t_7} + \frac{1}{24}\,\frac{t_2{}^2t_3}{t_4} - \frac{1}{216}\,\frac{t_2t_3{}^3}{t_4{}^2} + \frac{1}{4320}\,\frac{t_3{}^5}{t_4{}^3} + \frac{1}{192}\,\frac{t_5{}^3}{t_6}, \end{aligned}$$

and the Euler vector field is given by

$$E = t_1\partial_1 + \frac{5}{6}t_2\partial_2 + \frac{1}{2}t_3\partial_3 + \frac{1}{6}t_4\partial_4 + \frac{3}{4}t_5\partial_5 + \frac{1}{4}t_6\partial_6 + \partial_7.$$

4. On the Frobenius manifold structures related to the root system of type B_l and D_l

For the root system R of type B_l, we also define an indefinite metric $(\ ,\ )^{\sim}$ on $\widetilde{V} = V \oplus \mathbb{R}$ such that $\widetilde{V}$ is the orthogonal direct sum of V and $\mathbb{R}$. V is endowed with the W-invariant Euclidean metric

$$(dx_s, dx_n)^{\sim} = \frac{1}{4\pi^2}[(1 - \frac{1}{2}\delta_{n,l})s - \frac{l}{4}\,\delta_{n,l}\delta_{s,l}], \quad 1 \le s \le n \le l \tag{4.1}$$

and $\mathbb{R}$ is endowed with the metric

$$(dx_{l+1}, dx_{l+1})^{\sim} = -\frac{1}{4\pi^2 d_k}. \tag{4.2}$$

Here the numbers d_k are defined in (2.1) and (2.2). The basis of the W_a-invariant Fourier polynomials $y_1(\mathbf{x}),\ldots,y_{l-1}(\mathbf{x})$, $y_l(\mathbf{x})$ are defined in (2.3)–(2.5). The generators of the ring $\widetilde{W}^{(k)}(B_l)$ have the same form as that of (1.9) and (1.10). It is easy to see that the components of the resulting metric $(g^{ij}(y))$ coincide with those corresponding to the root system of type C_l if we perform the change of coordinates

$$y^j\mapsto\bar{y}^j=y^j,\ y^l\mapsto\bar{y}^l=(y^l)^2,\ y^{l+1}\mapsto\bar{y}^{l+1}=y^{l+1},\ j=1,\ldots,l-1 \tag{4.3}$$

for $1\le k\le l-1$ and

$$y^j\mapsto\bar{y}^j=y^j,\ y^l\mapsto\bar{y}^l=(y^l)^2,\ y^{l+1}\mapsto\bar{y}^{l+1}=\frac{1}{2}y^{l+1},\ j=1,\ldots,l-1 \tag{4.4}$$

for the case when $k=l$. Thus, the Frobenius manifold structure that we obtain in this way from B_l, by fixing the k-th vertex of the corresponding Dynkin diagram, is isomorphic to the one that we obtain from C_l by choosing the k-th vertex of the Dynkin diagram of C_l.

For the root system R of type D_l, the indefinite metric $(\ ,\ )^{\sim}$ on $\widetilde{V}=V\oplus\mathbb{R}$ is defined through the W-invariant Euclidean metric

$$\begin{aligned}
&(dx_s,dx_n)^{\sim}_1=\frac{s}{4\pi^2},\quad 1\le s\le n\le l-2,\\
&(dx_s,dx_n)^{\sim}=\frac{s}{8\pi^2},\quad 1\le s\le l-2, n=l-1,l-2,\\
&(dx_{l-1},dx_{l-1})^{\sim}=(dx_l,dx_l)^{\sim}=\frac{l}{16\pi^2},\quad (dx_{l-1},dx_l)^{\sim}=\frac{l-2}{16\pi^2},
\end{aligned}$$

and

$$(dx_{l+1},dx_{l+1})^{\sim}=-\frac{1}{4\pi^2 d_k}.$$

Here the numbers d_k are defined in (2.9). The set of generators for the ring $\mathcal{A}=\mathcal{A}^{(k)}(D_l)$ have the same form as that of (1.9) and (1.10), where $y_j(\mathbf{x})$ are defined in (2.13) and (2.14). It can be verified that the components of the resulting metric $(g^{ij}(y))$ coincide with those corresponding to the root system of type C_l if we perform the change of coordinates

$$
\begin{aligned}
y^j \mapsto \bar{y}^j &= y^j,\ j = 1, \cdots, l-2,\ l+1, \\
y^{l-1} \mapsto \bar{y}^{l-1} &= y^{l-1} y^l - \sum_{s=2}^{l-k} [1 - (-1)^s]\ 2^{s-2} y^{l-s} \\
&\quad - \sum_{j=0}^{l-k} [1 - (-1)^j]\ 2^{l-j-2} y^j e^{(k-j) y^{l+1}}, \\
y^l \mapsto \bar{y}^l &= (y^l)^2 + (y^{l-1})^2 - \sum_{s=2}^{l-k} [1 + (-1)^s]\ 2^{s-1} y^{l-s} \\
&\quad - \sum_{j=0}^{l-k} [1 + (-1)^j]\ 2^{l-j-1} y^j e^{(k-j) y^{l+1}}.
\end{aligned}
$$

Thus, the Frobenius manifold structure that we obtain in this way from D_l, by fixing the k-th vertex of the corresponding Dynkin diagram, is isomorphic to the one that we obtain from C_l by choosing the k-th vertex of the Dynkin diagram of C_l.

5. LG superpotentials for the Frobenius manifolds of $\mathcal{M}_{k,m}(C_l)$-type

We consider a particular class of cosine-Laurent series of one variable with a given tri-degree $(2k, 2m, 2n)$, which is a function of the form[2]

$$
\lambda(\varphi) = \left(\cos^2(\varphi) - 1\right)^{-m} \sum_{j=0}^{k+m+n} a_j \cos^{2(k+m-j)}(\varphi), \quad a_0 a_{k+m+n} \neq 0, \tag{5.1}
$$

where all $a_j \in \mathbb{C}$, $m, n \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{N}$. The cosine is considered as an analytic function on the cylinder $\varphi \simeq \varphi + 2\pi$, so $\cos^2(\varphi)$ has four critical points $\varphi = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. We denote by $\mathfrak{M}_{k,m,n}$ the space of this kind of cosine Laurent

[2]When $k = 1$ and $m = n = 0$, this reduces to $\lambda(\varphi) = a_1 + a_0 \cos^2(\varphi)$. If we set

$$
\cos^2(\varphi) = \frac{1 + \cos(2\varphi)}{2}, \quad a_0 = -4e^{\frac{t_2}{2}}, \quad a_1 = t_1 + 2e^{\frac{t_2}{2}}, \quad p = 2\varphi,
$$

then the LG superpotential is rewritten as

$$
\lambda(p) = t_1 - 2e^{\frac{t_2}{2}} \cos(p),
$$

which is exactly the LG superpotential of the $\mathbb{CP}^1$-model obtained in Example I.1 [6].

series. By analogy with the construction in [6, 1, 21], the space $\mathfrak{M}_{k,m,n}$ carries a natural structure of Frobenius manifold. The invariant inner product η and the intersection form g of two vectors ∂', ∂'' tangent to $\mathfrak{M}_{k,m,n}$ at a point $\lambda(\varphi)$ can be defined by the following formulae

$$\eta(\partial',\partial'') = (-1)^{k+1}\sum_{|\lambda|<\infty}\operatorname*{res}_{d\lambda=0}\frac{\partial'(\lambda(\varphi)d\varphi)\partial''(\lambda(\varphi)d\varphi)}{d\lambda(\varphi)}, \tag{5.2}$$

and

$$g(\partial',\partial'') = -\sum_{|\lambda|<\infty}\operatorname*{res}_{d\lambda=0}\frac{\partial'(\log\lambda(\varphi)d\varphi)\partial''(\log\lambda(\varphi)d\varphi)}{d\log\lambda(\varphi)}. \tag{5.3}$$

In these formulae, the derivatives $\partial'(\lambda(\varphi)d\varphi)$ *etc.* are to be calculated keeping φ fixed. The formulae (5.2) and (5.3) uniquely determine multiplication of tangent vectors on $\mathfrak{M}_{k,m,n}$ assuming that the Euler vector field E has the form

$$E = \sum_{j=0}^{k+m+n} a_j\frac{\partial}{\partial a_j}. \tag{5.4}$$

For tangent vectors ∂', ∂'' and ∂''' to $\mathfrak{M}_{k,m,n}$, one has

$$c(\partial',\partial'',\partial''') = -\sum_{|\lambda|<\infty}\operatorname*{res}_{d\lambda=0}\frac{\partial'(\lambda(\varphi)d\varphi)\partial''(\lambda(\varphi)d\varphi)\partial'''(\lambda(\varphi)d\varphi)}{d\lambda(\varphi)d\varphi}. \tag{5.5}$$

The canonical coordinates $u_1,\cdots,u_{k+m+n+1}$ for this multiplication are the critical values of $\lambda(\varphi)$ and

$$\partial_{u_\alpha}\cdot\partial_{u_\beta} = \delta_{\alpha\beta}\partial_{u_\alpha},\quad \text{where}\quad \partial_{u_\alpha} = \frac{\partial}{\partial u_\alpha}. \tag{5.6}$$

For the clarity, we use the notations

$$\begin{aligned} \lambda(P) &= (P^2-1)^{-m}\sum_{j=0}^{l} a_j P^{2(k+m-j)},\quad l = k+m+n,\\ \dot\lambda(P) &= \frac{d\lambda(P)}{dP},\quad P=\cos(\varphi),\quad P'(\varphi) = \frac{dP}{d\varphi} = -\sin(\varphi) \end{aligned} \tag{5.7}$$

and

$$\lambda(\varphi) = a_0\,(P^2-1)^{-m}\,P^{-2n}\prod_{j=1}^{l}(P^2-p_j^2),\quad a_0 = e^{2k\mathbf{i}\,\varphi_{l+1}},\quad p_j = P(\varphi_j). \tag{5.8}$$

Without confusion, we always use $\lambda(P)$ instead of $\lambda(\varphi)$. Before preceding the main result, we give some useful identities.

Lemma 5.1.

$$\lambda'(\varphi_j) = \frac{2PP'(\varphi)\,\lambda(\varphi)}{P^2 - p_j^2}|_{\varphi=\varphi_j}, \quad j = 1, \cdots, l. \tag{5.9}$$

Proof. This follows from

$$\lambda'(\varphi) = 2PP'(\varphi)\lambda(\varphi)\left(\sum_{j=1}^{l} \frac{1}{P^2 - p_j^2} - \frac{m}{P^2-1} - \frac{n}{P^2}\right)$$

and the definition of $\lambda(\varphi)$ in (5.8). □

Let us factorize

$$\lambda'(\varphi) = 2k\,a_0\,(P^2-1)^{-m-1}P^{-2n-1}\prod_{\alpha=1}^{l+1}(P^2 - q_\alpha^2)P'(\varphi), \quad q_\alpha = P(\psi_\alpha), \tag{5.10}$$

where all q_α^2 are distinct. When $m = 0$, we choose $P'(\psi_{l+1}) = 0$, that is to say,

$$\psi_{l+1} = 0, \pi, \quad \text{i.e.,} \quad q_{l+1} = P(\psi_{l+1}) = 1.$$

Lemma 5.2. *For $1 \le \alpha \le l+1$, we have*

$$\lambda''(\psi_\alpha) = \frac{c_{\alpha,m}PP'(\varphi)\,\lambda'(\varphi)}{P^2 - q_\alpha^2}|_{\varphi=\psi_\alpha}, \quad c_{\alpha,m} = 2 - \delta_{\alpha,l+1}\delta_{m,0}. \tag{5.11}$$

Proof. By definition, we have

$$\begin{aligned}
\lambda''(\varphi) &= 2k\,a_0\,\frac{d}{d\varphi}\left((P^2-1)^{-m-1}P^{-2n-1}\right)\prod_{\alpha=1}^{l+1}(P^2-q_\alpha^2)P'(\varphi)\\
&+ 2k\,a_0\,(P^2-1)^{-m-1}P^{-2n-1}\frac{d}{d\varphi}\left(\prod_{\alpha=1}^{l+1}(P^2-q_\alpha^2)\right)P'(\varphi)\\
&+ 2k\,a_0\,(P^2-1)^{-m-1}P^{-2n-1}\prod_{\alpha=1}^{l+1}(P^2-q_\alpha^2)\,\frac{d^2P}{d^2\varphi}\\
&= \sum_{\alpha=1}^{l+1}\frac{2PP'(\varphi)\lambda'(\varphi)}{P^2-p_\alpha^2} - \frac{(2n+1)P'(\varphi)\lambda'(\varphi)}{P} + \frac{(2m+1)P\lambda'(\varphi)}{P'(\varphi)}.
\end{aligned}$$

So, with the use of (5.10), we get

$$\begin{aligned}\lambda''(\psi_\alpha) &= \left(\sum_{\alpha=1}^{l+1} \frac{2PP'(\varphi)\lambda'(\varphi)}{P^2-p_\alpha^2} + \frac{(2m+1)P\lambda'(\varphi)}{P'(\varphi)}\right)|_{\varphi=\psi_\alpha} \\ &= \begin{cases} \dfrac{2PP'(\varphi)\,\lambda'(\varphi)}{P^2-q_\alpha^2}|_{\varphi=\psi_\alpha}, & \alpha=1,\cdots,l, \\ -\dfrac{P\,\lambda'(\varphi)}{P_\varphi}|_{\varphi=\psi_{l+1}}, & \alpha=l+1, \quad m=0, \\ \dfrac{2PP'(\varphi)\,\lambda'(\varphi)}{P^2-q_\alpha^2}|_{\varphi=\psi_{l+1}}, & \alpha=l+1, \quad m\neq 0 \end{cases} \\ &= \frac{c_{\alpha,m}PP'(\varphi)\,\lambda'(\varphi)}{P^2-q_\alpha^2}|_{\varphi=\psi_\alpha}.\end{aligned}$$

Thus the lemma is proved. □

We define

$$u_\alpha = \lambda(\psi_\alpha), \quad \alpha=1,\cdots,l+1,$$

then

$$\partial_{u_\alpha}\lambda(\varphi)|_{\varphi=\psi_\beta} = \delta_{\alpha\beta}. \tag{5.12}$$

Observe that

$$(P^2-1)^{m+1}P^{2n-1}\partial_{u_\alpha}\lambda(P) = (\partial_{u_\alpha}a_0)P^{2l} + \cdots + (\partial_{u_\alpha}a_l)$$

is a polynomial of P and

$$(P^2-1)^{m+1}P^{2n-1}\partial_{u_\alpha}\lambda(P)|_{P=q_\beta} = (q_\beta^2-1)^{m+1}q_\beta^{2n-1}\delta_{\alpha\beta},$$

we thus obtain, using the Lagrange interpolation formula,

$$\partial_{u_\alpha}\lambda(\varphi) = \frac{c_{\alpha,m}PP'(\varphi)}{P^2-q_\alpha^2}\frac{\lambda'(\varphi)}{\lambda''(\psi_\alpha)}, \quad \alpha=1,\cdots,l+1. \tag{5.13}$$

Lemma 5.3.

$$\partial_{u_\alpha}\varphi_\beta = \begin{cases} -\dfrac{c_{\alpha,m}\,p_\beta\,P'(\varphi_\beta)}{\lambda''(\psi_\alpha)\,(p_\beta^2-q_\alpha^2)}, & \beta=1,\cdots,l, \\ \dfrac{1}{2k\mathbf{i}}\left(\dfrac{\delta_{\alpha,l+1}}{\lambda(\psi_\alpha)} + \dfrac{2\,c_{\alpha,m}}{\lambda''(\psi_\alpha)}\displaystyle\sum_{s=1}^{l}\frac{p_s^2\,P'(\varphi_s)^2}{(q_{l+1}^2-p_s^2)(q_\alpha^2-p_s^2)}\right), & \beta=l+1. \end{cases} \tag{5.14}$$

Proof. By the definition of $\lambda(\varphi)$ in (5.8) and using (5.13), we get

$$\frac{c_{\alpha,m} P P_\varphi}{P^2 - q_\alpha^2} \frac{\lambda'(\varphi)}{\lambda''(\psi_\alpha)} = \partial_{u_\alpha}\lambda(\varphi) = 2k\mathbf{i}\lambda(\varphi)\partial_{u_\alpha}\varphi_{l+1} - \sum_{s=1}^{l} \frac{2p_s\, P'(\varphi_s)\lambda(\varphi)}{P^2 - p_s^2}\partial_{u_\alpha}\varphi_s. \quad (5.15)$$

Putting $\varphi = \varphi_\beta$ for $\beta = 1, \cdots, l$ into (5.15) and using (5.9), we obtain

$$\partial_{u_\alpha}\varphi_\beta = -\frac{c_{\alpha,m}\, p_\beta\, P'(\varphi_\beta)}{\lambda''(\psi_\alpha)\,(p_\beta^2 - q_\alpha^2)}, \quad \beta = 1, \cdots, l$$

and furthermore,

$$\begin{aligned} \frac{\partial_{u_\alpha}\lambda(\varphi)}{\lambda(\varphi)} &= 2k\mathbf{i}\partial_{u_\alpha}\varphi_{l+1} - \sum_{s=1}^{l} \frac{2p_s\, P'(\varphi_s)}{P^2 - p_s^2}\partial_{u_\alpha}\varphi_s \\ &= 2k\mathbf{i}\partial_{u_\alpha}\varphi_{l+1} - \frac{2\, c_{\alpha,m}}{\lambda''(\psi_\alpha)} \sum_{s=1}^{l} \frac{p_s^2\, P'(\varphi_s)^2}{(P^2 - p_s^2)(q_\alpha^2 - p_s^2)}. \end{aligned} \quad (5.16)$$

Putting $\varphi = \psi_\beta$ into (5.16), then

$$\frac{\delta_{\alpha\beta}}{u_\beta} = 2k\mathbf{i}\partial_{u_\alpha}\varphi_{l+1} - \frac{2\, c_{\alpha,m}}{\lambda''(\psi_\alpha)} \sum_{s=1}^{l} \frac{p_s^2\, P'(\varphi_s)^2}{(q_\beta^2 - p_s^2)(q_\alpha^2 - p_s^2)}.$$

Especially, taking $\varphi = \psi_{l+1}$, we obtain the desired formula of $\partial_{u_\alpha}\varphi_{l+1}$. □

Lemma 5.4. *For $\beta, \gamma = 1, \cdots, l$, we have*

$$S_{\beta,\gamma} := \sum_{\alpha=1}^{l+1} \frac{c_{\alpha,m}\, u_\alpha}{\lambda''(\psi_\alpha)\,(p_\beta^2 - q_\alpha^2)(p_\gamma^2 - q_\alpha^2)} = \frac{\delta_{\beta\gamma}}{2\, p_\beta^2\,(p_\beta^2 - 1)}. \quad (5.17)$$

Proof. Letting

$$\lambda(z) = (z-1)^{-m}(a_0 z^{k+m} + \cdots + a_l z^{-n}) = a_0\,(z-1)^{-m}\, z^{-n} \prod_{j=1}^{l}(z - p_j^2).$$

So, $\lambda(\varphi) = \lambda(z)|_{z=P^2}$ and

$$\frac{d\lambda(z)}{dz} = k\, a_0\,(z-1)^{-m-1} z^{-n-1} \prod_{\alpha=1}^{l+1}(z - q_\alpha^2), \quad \frac{\lambda(z)}{z\,(z-1)\frac{d\lambda(z)}{dz}} = \frac{\prod_{j=1}^{l}(z - p_j^2)}{\prod_{\alpha=1}^{l+1}(z - q_\alpha^2)},$$

which yields that if $q_\alpha^2 \neq 0$ (or 1) for all $\alpha = 1, \cdots, l+1$, then $z = 0$ (or 1) is not a pole of the function $\dfrac{\lambda(z)}{z\,(z-1)\,\frac{d\lambda(z)}{dz}}$.

With the use of (5.11) and $P'(\varphi)^2 = 1 - P^2$, we rewrite $S_{\beta,\gamma}$ as

$$\begin{aligned}
S_{\beta,\gamma} &= \sum_{\alpha=1}^{l+1} \frac{\lambda(\varphi)\,(P^2 - q_\alpha^2)}{P\,\lambda'(\varphi)\,P'(\varphi)\,(P^2-p_\beta^2)\,(P^2-p_\gamma^2)}|_{\varphi=\psi_\alpha} \\
&= -\frac{1}{2}\sum_{\alpha=1}^{l+1} \frac{\lambda(z)\,(z-q_\alpha^2)}{z\,(z-1)\,\frac{d\lambda(z)}{dz}\,(z-p_\beta^2)\,(z-p_\gamma^2)}|_{z=q_\alpha^2} \\
&= -\frac{1}{2}\sum_{\alpha=1}^{l+1} \operatorname*{res}_{z=q_\alpha^2} \frac{\lambda(z)}{z\,(z-1)\,\frac{d\lambda(z)}{dz}\,(z-p_\beta^2)\,(z-p_\gamma^2)}|_{z=q_\alpha^2} \\
&= \frac{1}{2}(\operatorname*{res}_{z=\infty} + \operatorname*{res}_{z=p_\beta^2} + \operatorname*{res}_{z=p_\gamma^2}) \frac{\lambda(z)}{\frac{d\lambda(z)}{dz}\,z(z-1)\,(z-p_\beta^2)(z-p_\gamma^2)}dz \\
&= \frac{\delta_{\beta\gamma}}{2} \operatorname*{res}_{z=p_\beta^2} \frac{\lambda(z)}{\frac{d\lambda(z)}{dz}\,z(z-1)\,(z-p_\beta^2)^2}dz = \frac{\delta_{\beta\gamma}}{2\,p_\beta^2(p_\beta^2-1)}.
\end{aligned}$$

We thus prove the identity (5.17). □

Lemma 5.5.

$$\frac{\lambda''(\psi_{l+1})}{\lambda(\psi_{l+1})} = -2\left(k + \sum_{s=1}^{l} \frac{p_s^2\,P'(\varphi_s)^2}{(q_{l+1}^2 - p_s^2)^2}\right). \tag{5.18}$$

Proof. Observe that

$$\lambda'(\varphi) = 2PP'(\varphi)\lambda(\varphi)\left(\sum_{s=1}^{l} \frac{1}{P^2-p_s^2} - \frac{m}{P^2-1} - \frac{n}{P^2}\right), \tag{5.19}$$

which yields

$$P'(\varphi)\left(\sum_{s=1}^{l} \frac{1}{P^2-p_s^2} - \frac{m}{P^2-1} - \frac{n}{P^2}\right)|_{\varphi=\psi_{l+1}} = 0. \tag{5.20}$$

[**Case 1.** $m = 0$]. In this case, $P'(\psi_{l+1}) = 0$. Using (5.19) and (5.20), we have

$$\frac{\lambda''(\varphi)}{\lambda(\varphi)}|_{\varphi=\psi_{l+1}} = 2(l-k) - \sum_{s=1}^{l} \frac{2q_{l+1}^2}{q_{l+1}^2 - p_s^2} = -2k - \sum_{s=1}^{l} \frac{2\,p_s^2}{q_{l+1}^2 - p_s^2},$$

which is exactly the formula (5.18) because of $q_{l+1} = 1$ and $P'(\varphi_s)^2 = 1 - q_s^2$.

[**Case 2.** $m \neq 0$]. In this case, $P'(\psi_{l+1}) \neq 0$. By using (5.20),

$$\sum_{s=1}^{l} \frac{1}{q_{l+1}^2 - p_s^2} = \frac{m}{q_{l+1}^2 - 1} + \frac{n}{q_{l+1}^2}. \tag{5.21}$$

So, using (5.19) and (5.21), we get

$$\begin{aligned}\frac{\lambda''(\varphi)}{\lambda(\varphi)}|_{\varphi=\psi_{l+1}} &= 2PP'(\varphi)\frac{d}{d\varphi}\left(\sum_{s=1}^{l}\frac{1}{P^2-p_j^2}-\frac{m}{p^2-1}-\frac{n}{p^2}\right)|_{\varphi=\psi_{l+1}}\\ &= -2\left(k+\sum_{s=1}^{l}\frac{p_s^2\,P'(\varphi_s)^2}{(q_{l+1}^2-p_s^2)^2}\right).\end{aligned}$$

□

We are now in a position to state our main theorem in this section.

Theorem 5.6. *Let*

$$\mathfrak{f}:(x_1,\cdots,x_{l+1})\mapsto(\varphi_1,\cdots,\varphi_{l+1}), \tag{5.22}$$

be a map defined by

$$\varphi_j=\pi(x_j-x_{j-1}),\quad x_0=0,\quad j=1,\cdots,l,\quad \varphi_{l+1}=\pi x_{l+1}.$$

Then

(1). the map (5.22) *establishes a diffeomorphism of the orbit space of* $\widetilde{W}^{(k)}(C_l)$ *to the space* $\mathfrak{M}_{k,m,n}$*. Moreover,*

(2). the induced diffeomorphism (5.22) *is an isomorphism of Frobenius manifolds.*

Proof. (1). The first part follows from the explicit formulae for y^r and a_r, that is,

$$a_0=e^{ky^{l+1}},\ a_j=(-1)^j\left(\sum_{s=1}^{j}2^{j-s}\,e^{(k-d_s)y^{l+1}}y^s(x)+e^{ky^{l+1}}\right),\ j=1,\cdots,l, \tag{5.23}$$

where $d_s=s$ for $s=1,\cdots,k$ and $d_s=k$ for $s=k+1,\cdots,l$.

(2). It is not difficult to check that the Euler vector fields (5.4) and (3.79) coincide. So it suffices to prove that the intersection form (5.3) coincides with the intersection form of the orbit space, and the metric (5.2) coincides with the metric (3.18).

By definition of η in (5.2) and using (5.13), we get

$$\begin{aligned}
\eta_{\alpha\beta}(u) &:= \eta(\partial_{u_\alpha}, \partial_{u_\beta}) = (-1)^{k+1} \sum_{|\lambda|<\infty} \operatorname*{res}_{d\lambda=0} \frac{\partial_{u_\alpha}(\lambda(\varphi)d\varphi)\partial_{u_\beta}(\lambda(\varphi)d\varphi)}{d\lambda(\varphi)} \\
&= (-1)^{k+1} \sum_{\gamma=1}^{l+1} \operatorname*{res}_{\varphi=[\psi_\gamma]} \frac{c_{\alpha,m}c_{\beta,m}P^2P'(\varphi)^2}{(P^2-q_\alpha^2)(P^2-q_\beta^2)} \frac{\lambda'(\varphi)}{\lambda''(\psi_\alpha)\lambda''(\psi_\beta)} d\varphi.
\end{aligned}$$

We remark that $[\psi_\gamma]$ represents four different points $\pm\psi_\gamma$ and $\pm\psi_\gamma+\pi$ satisfying $q_\gamma^2 = (e^{\mathbf{i}[\psi_\gamma]} + e^{-\mathbf{i}[\psi_\gamma]})^2$. Obviously, when $\alpha \neq \beta$, $\eta_{\alpha\beta}(u) = 0$. So,

$$\begin{aligned}
\eta_{\alpha\alpha}(u) &= (-1)^{k+1} \operatorname*{res}_{\varphi=[\psi_\alpha]} \frac{c_{\alpha,m}^2 P^2 P'(\varphi)^2}{(q_\alpha^2-P^2)^2} \frac{\lambda'(\varphi)}{\lambda''(\psi_\alpha)^2} d\varphi \\
&= (-1)^k \frac{2c_{\alpha,m}^2}{\lambda''(\psi_\alpha)^2} \operatorname*{res}_{P=\pm q_\alpha} \frac{P^2}{P^2-q_\alpha^2} \frac{\dot\lambda(P)(P^2-1)}{P^2-q_\alpha^2} dP \\
&= (-1)^k \frac{2\,c_{\alpha,m}^2}{\lambda''(\psi_\alpha)^2} \operatorname*{res}_{P=\pm q_\alpha} \frac{P^2}{P^2-q_\alpha^2} \frac{\dot\lambda(P)(P^2-1)}{P^2-q_\alpha^2} dP \\
&= (-1)^{k+1} \frac{2\,c_{\alpha,m}}{\lambda''(\psi_\alpha)}.
\end{aligned}$$

We thus obtain

$$\eta_{\alpha\beta}(u) = (-1)^{k+1} \frac{2\,c_{\alpha,m}\delta_{\alpha\beta}}{\lambda''(\psi_\alpha)}.$$

Similarly, we can obtain the formula of $g_{\alpha\beta}(u) := g(\partial_{u_\alpha}, \partial_{u_\beta})$ as

$$g_{\alpha\beta}(u) = -\frac{2\,c_{\alpha,m}\delta_{\alpha\beta}}{u_\alpha \lambda''(\psi_\alpha)}.$$

Observe that the vector field $e = \sum_{j=k}^{l} c_j \frac{\partial}{\partial y^j}$ in (3.20) in the coordinates $a_1, \cdots, a_{l+1}$ coincides with $e = (-1)^k \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \frac{\partial}{\partial a_{k+m-s}}$. The shifting

$$a_{k+m-s} \longmapsto a_{k+m-s} + c\,(-1)^{m-s}\binom{m}{s}, \quad s = 0, \cdots, m,$$

produces the corresponding shift

$$u_\alpha \longmapsto u_\alpha + c, \quad \alpha = 1, \cdots, l+1$$

of the critical values. This shift does not change the critical points ψ_α neither the values of the second derivative $\lambda''(\psi_\alpha)$. So

$$\mathcal{L}_e g^{\alpha\beta} = \mathcal{L}_e(-\frac{u_\alpha \lambda''(\psi_\alpha)}{2\, c_{\alpha,m}\delta_{\alpha\beta}}) = (-1)^{k+1}\frac{\lambda''(\psi_\alpha)}{2\, c_{\alpha,m}\delta_{\alpha\beta}} = \eta^{\alpha\beta}. \tag{5.24}$$

Finally, we want to compute the metric $g^{\beta\gamma}(\varphi)$ given by

$$g^{\beta\gamma}(\varphi) : \;=\; (d\varphi_\beta, d\varphi_\gamma) = \sum_{\alpha,\kappa=1}^{l+1} \frac{1}{g_{\alpha\kappa}(u)} \frac{\partial \varphi_\beta}{\partial u_\alpha} \frac{\partial \varphi_\gamma}{\partial u_\kappa} = \sum_{\alpha=1}^{l+1} \frac{1}{g_{\alpha\alpha}(u)} \partial_{u_\alpha}\varphi_\beta\, \partial_{u_\alpha}\varphi_\gamma.$$

Using (5.14), (5.17) and (5.18), we have

Case 1. $1 \leq \beta, \gamma \leq l$.

$$g^{\beta\gamma}(\varphi) = -\frac{p_\beta\, p_\gamma\, P'(\varphi_\beta)\, P'(\varphi_\gamma)}{2} \sum_{\alpha=1}^{l+1} \frac{c_{\alpha,m}\, u_\alpha}{\lambda''(\psi_\alpha)\,(p_\beta^2 - q_\alpha^2)(p_\gamma^2 - q_\alpha^2)} = \frac{1}{4}\delta_{\beta\gamma}.$$

Case 2. $1 \leq \beta \leq l$ **and** $\gamma = l+1$.

$$\begin{aligned} g^{\beta,l+1}(\varphi) &= \frac{p_\beta\, P'(\varphi_\beta)}{4k\mathbf{i}} \sum_{\alpha=1}^{l+1} \frac{1}{p_\beta^2 - q_\alpha^2} \left(\delta_{\alpha,l+1} + \frac{2\, c_{\alpha,m}\, u_\alpha}{\lambda''(\psi_\alpha)} \sum_{s=1}^{l} \frac{p_s^2\, P'(\varphi_s)^2}{(q_{l+1}^2 - p_s^2)(q_\alpha^2 - p_s^2)} \right) \\ &= \frac{p_\beta\, P'(\varphi_\beta)}{4k\mathbf{i}} \left(\frac{1}{p_\beta^2 - q_{l+1}^2} - \sum_{s=1}^{l} \frac{p_s^2\, P'(\varphi_s)^2}{q_{l+1}^2 - p_s^2} \sum_{\alpha=1}^{l+1} \frac{2\, c_{\alpha,m}\, u_\alpha}{\lambda''(\psi_\alpha)(q_\alpha^2 - p_s^2)(q_\alpha^2 - p_\beta^2)} \right) \\ &= 0. \end{aligned}$$

<u>**Case 3.** $\beta = \gamma = l+1$.</u>

$$
\begin{aligned}
g^{l+1,l+1}(\varphi) &= \sum_{\alpha=1}^{l+1} \frac{u_\alpha \, \lambda''(\psi_\alpha)}{8k^2 \, c_{\alpha,m}} \left(\frac{\delta_{\alpha,l+1}}{\lambda(\psi_\alpha)} + \frac{2\, c_{\alpha,m}}{\lambda''(\psi_\alpha)} \sum_{s=1}^{l} \frac{p_s^2 \, P'(\varphi_s)^2}{(q_{l+1}^2 - p_s^2)(q_\alpha^2 - p_s^2)} \right)^2 \\
&= \frac{1}{8k^2} \frac{\lambda''(\psi_{l+1})}{\lambda(\psi_{l+1})} + \frac{1}{2k^2} \sum_{s=1}^{l} \frac{p_s^2 \, P'(\varphi_s)^2}{(q_{l+1}^2 - p_s^2)^2} \\
&+ \frac{1}{2k^2} \sum_{s,j=1}^{l} \frac{p_s^2 \, p_j^2 \, P'(\varphi_s)^2 \, P'(\varphi_j)^2}{(q_{l+1}^2 - p_s^2)\,(q_{l+1}^2 - p_j^2)} \sum_{\alpha=1}^{l+1} \frac{c_{\alpha,m} \, u_\alpha}{\lambda''(\psi_\alpha)(q_\alpha^2 - p_s^2)(q_\alpha^2 - p_j^2)} \\
&= -\frac{1}{4k^2} \left(k + \sum_{s=1}^{l} \frac{p_s^2 \, P'(\varphi_s)^2}{(q_{l+1}^2 - p_s^2)^2} \right) + \frac{1}{2k^2} \sum_{s=1}^{l} \frac{p_s^2 \, P'(\varphi_s)^2}{(q_{l+1}^2 - p_s^2)^2} \\
&+ \frac{1}{4k^2} \sum_{s,j=1}^{l} \frac{p_s^2 \, p_j^2 \, P'(\varphi_s)^2 \, P'(\varphi_j)^2}{(q_{l+1}^2 - p_s^2)\,(q_{l+1}^2 - p_j^2)} \frac{\delta_{sj}}{p_j^2 \, (p_j^2 - q_{l+1}^2)} \\
&= -\frac{1}{4k}.
\end{aligned}
$$

Using the isomorphism (5.22), it is easy to know that the intersection form $g^{\alpha\beta}(\varphi)$ coincides with $(\ ,\ )^{\sim}$ defined in (3.1) and (3.2). The coincidence of the metric (5.2) with the metric (3.18) follows (5.24). We thus complete the proof of the theorem. □

Remark 5.7. *(1). When $m = 0$, $\mathfrak{M}_{k,0,n} \simeq \mathcal{M}_{k,0}(C_l)$, which is the Frobenius manifold structure constructed in arXiv:052365v1 (*[11]*).*

(2). On the orbit space of the extended affined Weyl group $\widetilde{W}^{(k)}(D_{k+2})$, Dubrovin and Zhang constructed a weighted homogenous polynomial Frobenius structure, denoted by $\mathcal{M}_{\mathrm{DZ}}^{(k)}(D_{k+2})$ which is isomorphic to $\mathfrak{M}_{k,1,1}$. Actually, in this case, there is a tri-polynomial description introduced in [17, 19]*, also used in* [10].

6. Concluding Remarks

For the root systems of type B_l, C_l and D_l, we have constructed families of Frobenius manifold structures on the orbit spaces of the extended affine Weyl groups $\widetilde{W}^{(k)}(R)$ with respect to the choice of an arbitrary vertex on the Dynkin diagram, as it was suggested in [18] motivated by the results of [20, 13, 14].

It remains a challenging problem to understand whether the constructions of the present paper can be generalized to the root systems of the types E_6, E_7, E_8, F_4, G_2. Another open problem is to obtain an explicit realization of the integrable hierarchies associated with the Frobenius manifolds of the type $\widetilde{W}^{(k)}(R)$. So far this problem was solved only for $R = A_l$, see [3, 4, 8, 9, 15, 16] for details. We plan to study these problems in subsequent publications.

To end up this section, we remark that the potential of the semisimple Frobenius manifold structures constructed above from the root systems of type $(C_l, k, m = 0)$ has the form

$$F = \frac{1}{2}\,(t^k)^2 t^{l+1} + \frac{1}{2} t^k \sum_{\alpha,\beta\neq k} \eta_{\alpha\beta} t^\alpha t^\beta + \sum_{j=0}^{n} f_j(t^2, t^3, \dots, t^l, \frac{1}{t^l})\, e^{j\, t^{l+1}},$$

where $f_j(t^2, t^3, \dots, t^l, \frac{1}{t^l}), j = 0, \dots, n$ are some polynomials of their independent variables. The Euler vector field has the form

$$E = \sum_{j=1}^{l} d_j \frac{\partial}{\partial t^j} + r \frac{\partial}{\partial t^{l+1}}.$$

Here $0 < d_j < 1, r > 0$, they also satisfy the duality relation given in (3.68), (3.69) for the case $m = 0$. We expect that these potentials of semisimple Frobenius manifolds together with the ones that are constructed in [7] exhaust all solutions of the above form, and we have verified this for the cases when $l = 1, 2, 3$ and $n \leq 6$.

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