**PERTURBATION TECHNIQUES APPLIED TO THE REAL VANISHING VISCOSITY APPROXIMATION OF AN INITIAL BOUNDARY VALUE PROBLEM**

## 1. Analysis of ODE with one singular parameter

In this section we consider a singular ODE of the form

$$\dot{u} = \phi(\sigma, u), \quad u \in \mathbb{R}^N. \tag{1.1}$$

Our aim is to introduce the techniques which will be used in the next section in a simpler setting, i.e. when the parameter $\sigma$, which will play the role of singular parameter, does not depend on $u$. This assumption simplifies the problem of studying the existence of invariant manifolds for (1.1), and we can use a standard approach to singular ODE. The approach will follow closely the book [11], Chapter 6.

We assume that

(1) as $\sigma \to 0$, the function $\phi$ has the form

$$\phi(\sigma, u) = \frac{1}{\sigma}\phi^s(u) + \phi^n(\sigma, u); \tag{1.2}$$

(2) $\phi(\sigma, 0) = 0$, i.e. $u = 0$ is an equilibrium point of (1.1) for all $\sigma$;

(3) $u_\tau = \phi^s(u)$ has a center manifold with trivial dynamics, i.e. it is made only of equilibria near $u = 0$.

Before studying the non linear case, we consider the following simple example which introduces the ideas and results of this section.

*Example* 1.1. At a linear level we are studying the system

$$\dot{u} = \frac{1}{\sigma} Au + Bu, \tag{1.3}$$

assuming that $A$ has no eigenvalues on the imaginary axis different from 0. Let $n^-$, $n^+$ be the eigenvalues of $A$ with negative, positive real part respectively, and let $P_0 = R_0 \otimes L_0$ be the projection on the kernel of $A$. We will write

$$A = \left[ \begin{array}{ccc} A^- & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A^+ \end{array} \right] \tag{1.4}$$

as the block decomposition of $A$.

By a standard perturbation techniques ([6], page 74 and followings), the eigenvalues of (1.3) are distributed as follows for $\sigma > 0$:

(1) $n^-$ eigenvalues with negative real part $-\mathcal{O}(1)/\sigma$;

(2) $n^+$ eigenvalues with positive real part $\mathcal{O}(1)/\sigma$;

(3) $n^0 = N - n^- - n^+$ eigenvalues which converge to the eigenvalues of $L_0 B R_0$.

In particular, the singular eigenvalues behaves like $1/\sigma$, while the others remain bounded. A completely similar situation occurs for $\sigma < 0$, exchanging $n^-$ with $n^+$. We consider now the case $\sigma > 0$.

To obtain the singular dynamics, it is thus sufficient to multiply (1.3) by $\sigma$

$$u_\tau = Au + \sigma Bu, \qquad t = \sigma\tau, \tag{1.5}$$

and take the limit $\sigma \to 0$: we thus obtain the projectors $P^-(\sigma)$, $P^+(\sigma)$, which correspond to the stable, unstable eigenspace $M^-(\sigma)$, $M^+(\sigma)$ of (1.3). The equation on these eigenspace can be written as

$$\dot{x}^- = \frac{1}{\sigma}A^- x^- + \mathcal{O}(1)x^-, \quad x^- \in \mathbb{R}^{n^-}, \qquad \dot{x}^+ = \frac{1}{\sigma}A^+ x^+ + \mathcal{O}(1)x^+, \quad x^+ \in \mathbb{R}^{n^+}. \tag{1.6}$$

Observe that since $\det A^- \neq 0$, $\det A^+ \neq 0$, then the term $\mathcal{O}(1)$ is negligible for $\sigma \to 0$.

The non singular dynamics corresponds to the reduction on the eigenspace generated by the eigenvalues which remains bounded. Using the asymptotic expansion of the projectior $P_0(\sigma)$ w.r.t. $\sigma$, one can obtain for all $\sigma$ small an invariant $n_0$ dimensional eigenspace $M^0(\sigma)$ for (1.3), on which the ODE becomes

$$\dot{x}^0 = L_0 B R_0 x^0 + \mathcal{O}(\sigma), \quad x^0 \in \mathbb{R}^{n^0}. \tag{1.7}$$

For this reduced equation, we can decompose the space $M^0(\sigma)$ into a uniformly stable, uniformly unstable and center space: the uniformly stable (unstable/center) space is the eigenspace of the eigenvalues whose real part is $\leq -c$ ($\geq c$, $\mathcal{O}(\sigma)$) for some $c > 0$. We denote their dimension as $n^{0-}$, $n^{00}$ and $n^{0+}$, respectively. The center space is actually the fiber of the center manifold of (1.7) together with $\dot{\sigma} = 0$, and for $\sigma \to 0$ it converges to the center eigenspace of $L_0 B R_0$. Similarly, as $\sigma \to$, the uniformly stable/unstable converges to the eigenspace of $L_0 B R_0$ generated by the eigenvalues with positive/negative real part.

We collect thus the various projectors which we constructed, to which there correspond invariant linear spaces of (1.3):

1. $n^-$ dimensional linear space $M^-(\sigma)$ corresponding to the eigenvalues with negative real part $-\mathcal{O}(1)/\sigma$. This manifold converges to the stable eigenspace of $A$ as $\sigma \to 0$;
2. $n^{0-}$ dimensional linear space $M^{0-}(\sigma)$ corresponding to the eigenvalues bounded and uniformly negative. This manifold converges to the stable eigenspace of $L_0 B R_0$ as $\sigma \to 0$, when considered embedded in $\mathbb{R}^N$;
3. $n^{00}$ dimensional linear space $M^{00}(\sigma)$ corresponding to the eigenvalues real part is of order of $\sigma$. This manifold converges to the center eigenspace of $L_0 B R_0$ as $\sigma \to 0$, when considered embedded in $\mathbb{R}^N$;
4. $n^{0+}$ dimensional linear space $M^{0+}(\sigma)$ corresponding to the eigenvalues uniformly positive but bounded. This manifold converges to the unstable eigenspace of $L_0 B R_0$ as $\sigma \to 0$, when considered embedded in $\mathbb{R}^N$;
5. $n^+$ dimensional linear space $M^+(\sigma)$ corresponding to the eigenvalues with positive real part $\mathcal{O}(1)/\sigma$. This manifold converges to the unstable manifold of $A$ as $\sigma \to 0$.

Finally, for $\sigma > 0$, there is a uniformly stable manifold $M^s(\sigma)$ of $x = 0$ which is the sum of the stable eigenspace of (1.7) and the stable eigenspace of (1.5): the projector of this manifold is the projector corresponding to the eigenvalues with negative real part of (1.3), and its dimension is $n^s = n^- + n^{0-}$. Similarly one can define the $n^u = n^+ + n^{0+}$ dimensional uniformly unstable manifold $\mathcal{M}^u(\sigma)$ as the sum of the unstable manifold of (1.7) and the unstable space of (1.5).

Note that when $\sigma$ crosses 0, the stable part of (1.7) remains stable, while the stable part of (1.5) becomes unstable.

We want to prove that these manifold exist also to the non linear equation (1.1), in a neighborhood of an equilibrium point. The diameter of the neighborhood does not depend on the paramater $\sigma$ (while a direct application of the theorems on existence of invariant manifolds around the equilibrium $u = 0$ would yield a diameter of order $\sigma$). More precisely, we are interested in the existence of the stable manifold and a center manifold of an equilibrium point, which we will identify with $u = 0$. Since the study of the center-unstable manifold of (1.1) is completely similar to the study of the center-stable one, we will only consider this last case.

The arguments we use are based on the construction of invariant manifolds of slow and fast dynamics. It follows that the study of the center manifold is somehow easier since it lays on this slow dynamic manifold, so that we just need to apply the standard center manifold theorem.

Conversely, in the study of the stable manifold, first we assume that for $\sigma > 0$ the singular part of the equation is stable (hence for $\sigma < 0$ is unstable). Then we need a perturbation technique which shows that, roughly speaking, for $\sigma > 0$ this stable manifold is the sum of the stable manifold on the slow dynamics manifold and the fast dynamics, while for $\sigma < 0$ only the stable slow dynamics survives. The main difficulty is exactly the proof of this non linear sum of invariant manifolds.

1.1. **Canonical form of the singular ODE.** In this section we rewrite (1.1) in a more convenient form for studying the dependence of the solution w.r.t. the parameter $\sigma$. We will obtain that the equilibrium $u = 0$ has 3 invariants manifolds: 2 correspond to the fast or singular stable/unstable dynamics, while the other is the slow or non singular dynamics. We will use standard results on invariant manifolds for non singular ODE from [4, 5, 7].

The change of variable $\tau = t/\sigma$ gives the rescaled equation

$$\left\{ \begin{array}{lcl} u_\tau & = & \phi^s(u) + \sigma\phi^n(\sigma, u) \\ \sigma_\tau & = & 0 \end{array} \right. \tag{1.8}$$

For $\sigma = 0$ we obtain the limiting system

$$\left\{ \begin{array}{lcl} u_\tau & = & \phi^s(u) \\ \sigma_\tau & = & 0 \end{array} \right. \tag{1.9}$$

Let $\mathcal{M}^-(\sigma)$, $\mathcal{M}^+(\sigma)$ be the uniformly stable/unstable manifold of $u = 0$, with dimension $n^-$, $n^+$, for (1.8) and $\sigma \geq 0$. These manifolds are defined as the set of all orbits converging/diverging from the equilibrium $u = 0$ with uniform exponential speed for all speeds close to 0.

The existence of these manifolds is assured by the exponential splitting of the equilibrium $u = 0$: in fact, following the analysis of Example 1.1, the eigenvalues of (1.8) can be separated into 3 groups: for some strictly positive constant $c$,

(1) $n^-$ eigenvalues with real part $\leq -c$;
(2) $n^+$ eigenvalues with real part $\geq c$;
(3) $n^0 + 1$ eigenvalues with real part of the order $\sigma$.

Using well known results on invariant manifold corresponding to exponential splitting (see for example [7], page 242), the existence of $\mathcal{M}^-(\sigma)$, $\mathcal{M}^+(\sigma)$ follows. For an explicit proof, see Section 3 of [1].

The (rescaled) dynamics on these manifold is thus $\mathcal{O}(1)e^{\pm c\tau}$ for some constant $c > 0$, i.e. they correspond in the original time scale to exponential fast decay to/blowup from $u = 0$ of order $1/\sigma$.

The center manifold $\mathcal{M}^0$ of (1.8) is a smooth manifold of dimension $n^0 + 1 = N - n^- - n^+ + 1$ tangent to the center eigenspace of $D\phi(0)$. By construction, the center manifold contains all the equilibria close to $u = 0$, $\sigma = 0$, in particular the manifold of equilibria $\{\sigma = 0, \phi^s(u) = 0\}$. Moreover, since the dimension of the intersection of the center manifold with $\{\sigma = 0\}$ is equal to the dimension of the equilibrium manifold $\{\phi^s(u) = 0\}$, it follows that the dynamics on the center manifold is 0 if $\sigma = 0$, i.e. we can write

$$\left\{ \begin{array}{lcl} u^0_\tau & = & \sigma\psi(\sigma, u^0) \\ \sigma_\tau & = & 0 \end{array} \right. \qquad u^0 \in \mathbb{R}^{n^0}, \tag{1.10}$$

where $u^0$ is a parametrization of $\mathcal{M}^0$ and $\psi$ a smooth function. In particular we obtain an invariant manifold for the original equation, which does not contain any singular dynamics: by scaling back time,

$$\dot{u}_0 = \psi(\sigma, u_0), \quad u \in \mathcal{M}^{00}(\sigma). \tag{1.11}$$

We have thus the following lemma:

**Lemma 1.2.** *The equilibrium $u = 0$ has three invariant manifold for $\sigma$ close to 0: for a positive constant $c > 0$,*

(1) *the $n^-$ dimensional manifold $\mathcal{M}^-(\sigma)$, on which the dynamics is $e^{-ct/\sigma}$;*
(2) *the $n^+$ dimensional manifold $\mathcal{M}^+(\sigma)$, on which the dynamics is $e^{ct/\sigma}$;*
(3) *the $n^0$ dimensional manifold $\mathcal{M}^0(\sigma)$, on which the dynamics, given by* (1.10)*, does not contain the singular parameter.*

*These manifolds depend smoothly on $\sigma$.*

On the reduced $n^0$ dimensional system on $\mathcal{M}^0(\sigma)$, we can define the uniformly stable manifold $\mathcal{M}^-(\sigma)$, the uniformly unstable manifold $\mathcal{M}^+(\sigma)$ and the ($\sigma$ fiber of the) center manifold $\mathcal{M}^0(\sigma)$. By the previous lemma, the dynamics on the manifold $\mathcal{M}^0(\sigma)$ is non singular, so that all the manifolds $\mathcal{M}^-(\sigma)$, $\mathcal{M}^+(\sigma)$, $\mathcal{M}^0(\sigma)$ of (1.1) exist in a neighborhood of radius $4\delta_0$ to the equilibrium point $u = 0$ for all $\sigma \in (-4\delta_0, 4\delta_0)$, and depends smoothly on $\sigma$.

The next step is to prove the existence of a global (fast + slow) uniformly stable manifold for the equilibrium $u = 0$. This is the most technical part, since we cannot just rescale time and use the center manifold theorem to get rid of the singular parameter.

We assume that

(4) the singular part $\phi^s(u)$ does not have the unstable part, i.e. the eigenvalues $D\phi^s(0)$ have negative real part.

This can be accomplished by reducing on the center stable manifold of (1.8).

In this case, for all $\sigma$ small we can assume that the uniformly stable manifold and center manifold of (1.8) are given by

$$\mathcal{M}^-(\sigma) = \{u^0 = 0\}, \ \mathcal{M}^0(\sigma) = \{u^- = 0\}, \qquad (u^-, u^0) \in \mathbb{R}^{n^-+n^0}. \tag{1.12}$$

Moreover, we consider the stable manifold of the equilibria $\{u^- = 0, \sigma = 0\}$ for (1.8): these manifolds are smooth and can be parameterized by

$$\mathcal{M}^-(\bar{u}^0) = \Big\{ u^0 = \mathcal{M}^-(\bar{u}^0, u^-) \Big\}, \quad \mathcal{M}^-(\bar{u}^0 = 0) = \mathcal{M}^-(\sigma = 0),$$

where $\bar{u}^0$ is the limit of all orbits of $\mathcal{M}^-(\bar{u}^0)$. With a slight abuse of notation we have written $\mathcal{M}^-(\bar{u}^0)$ as the stable manifold of the equilibrium $(u^-, u^0, \sigma) = (0, \bar{u}^0, 0)$, and $\mathcal{M}^-(\bar{u}^0, u^-)$ as a parameterization of the manifold using the variable $u^-$.

Since $\mathcal{M}^-(\bar{u}^0, u^-)$ is smooth and $\mathcal{M}^-(\bar{u}^0, u^-) - \bar{u}^0 = \mathcal{O}(1)u^-$, we can thus make a change of coordinates and assume that

$$\mathcal{M}^-(\bar{u}^0) = \Big\{ u = (u^-, u^0), u^0 = \bar{u}^0 \Big\}. \tag{1.13}$$

Using these coordinates, the system takes the form

$$\begin{cases} \sigma \dot{u}^- &= \phi(\sigma, u^-, u^0)u^- \\ \dot{u}^0 &= \psi(\sigma, u^-, u^0)u^0 \end{cases} \tag{1.14}$$

with $\phi$ strictly negative definite in a neighborhood of 0, i.e.

$$|e^{\phi(\sigma,0,0)t}X| \le e^{-2ct}|X|, \quad X \in \mathbb{R}^{n^-}, \ \forall \sigma \in (-4\delta_0, 4\delta_0). \tag{1.15}$$

We have used the fact that these manifolds are invariant for the flow, i.e.

(1) if $u^-(t=0) = 0$ then $u^-(t) = 0$ for all $t \ge 0$, so that in the equation for $u^-$ we can factorize $u^-$;
(2) if $u^0(t=0) = 0$ then $u^0(t) = 0$ for all $t \ge 0$, so that in the equation for $u^0$ we can factorize $u^0$;
(3) for $\sigma = 0$ the vector $u^0_\tau$ is equal to 0 even if $u^- \neq 0$, so that in the equation for $u^0$ we can factorize $\sigma$.

A remark on the notation: $c > 0$ will be used for estimating the exponential of matrices, and depends only on the function $\phi$, $\psi$ an the radius $\delta_0$; $C$ (eventually with an index) will be a suitable large constant, which may change from line to line during the computations. Finally, we will always use $\tau$ to denote the fast time scale.

We collect the results of this section into the following proposition.

**Proposition 1.3.** *Consider the singular ODE*

$$\dot{u} = \phi(\sigma, u), \quad u \in \mathbb{R}^N,$$

*and assume that the conditions 1), 2), 3) of page 1 and condition 4) of page 3 hold. Then there is a smooth change of coordinates in* $|u|, |\sigma| \le 4\delta_0$, $\delta_0 \ll 1$,

$$\mathbb{R}^N \ni u \mapsto (u^-, u^0) \in \mathbb{R}^{n^-+n^0}, \qquad N = n^- + n^0,$$

*such that the singular ODE can be written as*

$$\begin{cases} \sigma \dot{u}^- &= \phi(\sigma, u^-, u^0)u^- \\ \dot{u}^0 &= \psi(\sigma, u^-, u^0)u^0 \end{cases} \tag{1.16}$$

*Moreover* $|e^{\phi(\sigma,0,0)t}X| \le e^{-2ct}|X|$ *for all* $X \in \mathbb{R}^{n^-}$, $\sigma \in (-4\delta_0, 4\delta_0)$.

From now on we will work with a system of the form (1.16).

By applying the standard center manifold theorem [4] to the non singular part of (1.16), we obtain immediately the following theorem.

**Theorem 1.4.** *Let conditions 1), 2), 3) of page 1 hold. Then, there exists an invariant manifold* $\mathcal{M}^{00}(\sigma)$ *with dimension* $n^{00}$ *for* (1.16) *containing all orbits whose speed of convergence/divergence from* $u = 0$ *is of order* $\sigma$, *where* $n^{00}$ *is the dimension of the null space of* $\psi(0,0,0)$. *This manifold is defined in a neighborhood of* $(\sigma, u) = 0$ *of radius* $4\delta_0$, *it is smooth and tangent at* $u = 0$ *to the eigenspace* $M^{00}(\sigma)$ *generated by the eigenvectors whose eigenvalues are of order* $\sigma$.

We next study the construction of the stable manifold $\mathcal{M}^s$, where the simultaneous presence of slow and fast dynamics makes the analysis more complicated.

1.2. **Asymptotic expansion.** Aim of this section is to study the $n^s$ dimensional stable manifold of (1.16) for $\sigma$ close to 0, under the assumption 4) of page 3. For $\sigma < 0$, clearly this manifold coincides with the $n^{0-}$ stable manifold of the nonsingular equation (1.11). For $\sigma > 0$, roughly speaking one expect that this manifold will be the sum of the stable manifold of (1.11) and the singular dynamics in $u^-$, i.e. $n^s = n^- + n^{0-}$. These dynamics are weakly interfering each other, for $\sigma > 0$, while for $\sigma = 0$ we can imagine an instantaneous jump along the fast dynamics $u^-$, and then the exponential decaying orbit on the stable manifold of the reduced slow ODE.

The idea (O'Malley/Hoppensteadt construction [11], page 177) is that each orbit on this manifold can be constructed as the sum of:

- a term $X^-$ exponentially decaying as $e^{-2ct/\sigma}$ in $u^-$ for $\sigma > 0$, solution to

$$\sigma \dot{X}^- = \phi(\sigma, X^-, X^0(0))X^-. \tag{1.17}$$

  The parameter $X^0(0)$ means that the principal term in the fast dynamics is the exponential decay to 0: the time 0 is chosen because the slow dynamics does not move in the limit $\sigma = 0$. For $\sigma \leq 0$ this term is not present;
- a term $X^0$ exponentially decaying as $e^{-2ct}$ in $u^0$, corresponding to the stable manifold of the equation

$$\dot{X}^0 = \psi(\sigma, 0, X^0)X^0. \tag{1.18}$$

  The initial data $X^0(0)$ is thus given on the stable $n^{0-}$ dimensional manifold $\mathcal{M}^-$ of the above equation, which can be parametrized by the stable eigenspace of $\psi(0,0,0)$. The coefficient $X^0(0)$ of (1.17) is the initial data of (1.18) on the stable manifold $\mathcal{M}^-$;
- rest terms $R$, $S$ to the equations (1.17), (1.18), respectively, which compensate the errors occurred in separating completely the dynamics of (1.14). These errors are exponentially decaying, and remain of order $\mathcal{O}(\delta_0\sigma)$. The initial data for $R$ is 0, while the initial data of $S$ correct $X^0(0)$ is such a way to remain on the stable manifold: to fix $S(0)$, we set the projection of $S(0)$ on the stable eigenspace of $\psi(\sigma,0,0)$ to be 0.

For $\sigma = 0$, the rest terms $R$, $S$ will be identically with 0, i.e. the solution can be thought as the sum of an instantaneous jump along $X^-$ and the stable part of $X^0$.

A similar contruction can be performed also for studying the stable manifold corresponding to an exponential splitting of (1.18). In this case $X^0$ is on the invariant manifold corresponding to the exponential splitting of (1.18), and $S$ is again the correction term.

Our construction is (up to small variations) the first order expansion of the O'Malley/Hoppensteadt expansion ([11], page 177): write the function $u^-$, $u^0$ in series of the form

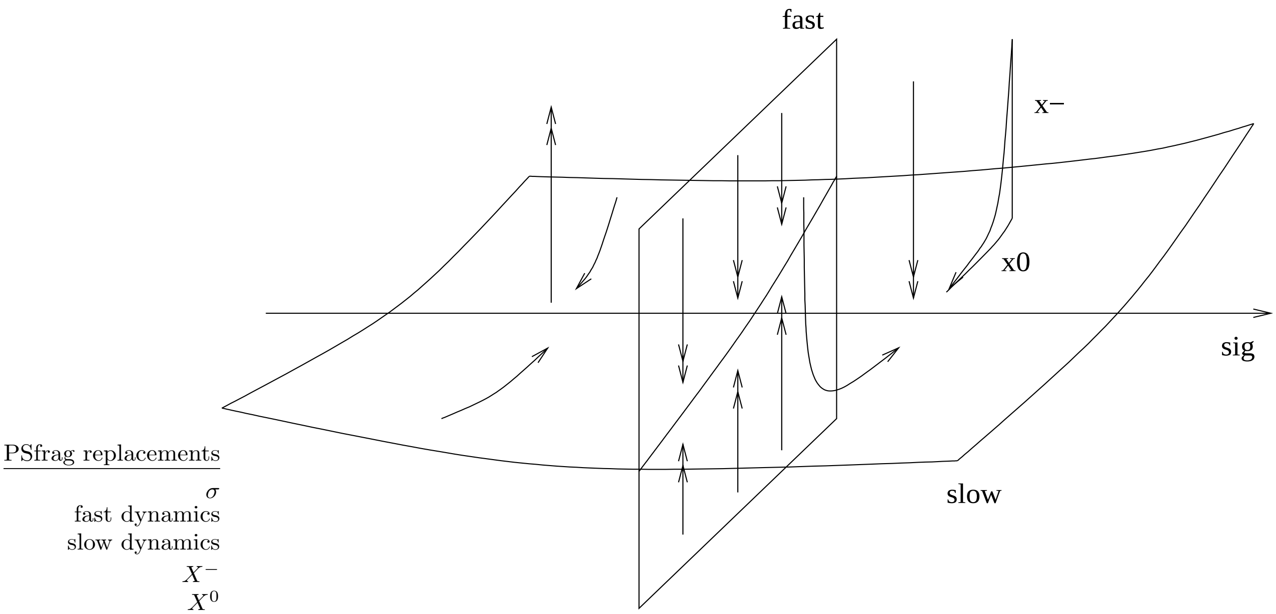
$$u^-(\sigma,t) = X_0^M(\sigma,t) + X_1^M(\sigma,t/\sigma) + R^M(\sigma,t) = \sum_{i=0}^M \sigma^i X_{0i}(t) + \sum_{i=0}^M \sigma^i X_{1i}(t/\sigma) + R^M(t,\sigma),$$

$$u^0(\sigma,t) = Y_0^M(\sigma,t) + Y_1^M(\sigma,t/\sigma) + R^M(\sigma,t) = \sum_{i=0}^M \sigma^i Y_{0i}(t) + \sum_{i=0}^M \sigma^i Y_{1i}(t/\sigma) + S^M(x,\sigma), \tag{1.19}$$

where $(X_0, Y_0)$ is the outer function (slow/non singular dynamics) and $(X_1, Y_1)$ are the inner functions (fast/singular dynamics). The index $M$ is the order of approximation, and the equations for the $i$-th order terms are obtained recursively. The goal is to show that $R^M$, $S^M$ remain of order $\sigma^{M+1}$.

Let $M^s(\sigma)$ be the $n^s = n^- + n^{0-}$ dimensional eigenspace of the uniform negative eigenvalues of the linearization of (1.16): this subspace is defined in Example 1.1. We have the following result:

**Theorem 1.5.** *Let $\sigma > 0$ and let conditions 1), 2), 3) of page 1 and 4) of page 3 hold. Then, there exists an invariant manifold $\mathcal{M}^s(\sigma)$ with dimension $n^s = n^- + n^{0-}$ for (1.16) containing all orbits converging to $u = 0$ with uniform exponential speed. This manifold is defined in a neighborhood of $(\sigma, u) = 0$ of radius $4\delta_0$, it is smooth and tangent to the eigenspace $M^s(\sigma)$ at $u = 0$. The orbits on this manifold can*




*be written as*

$$\left\{ \begin{array}{rcl} u^-(t,\sigma) & = & X^-(t,\sigma) + R(t,\sigma) \\ u^0(t,\sigma) & = & X^0(t,\sigma) + S(t,\sigma) \end{array} \right. \tag{1.20}$$

*where* $X^-$, $X^0$ *satisfy* (1.17), (1.18)*, respectively, and the rest terms* $R$, $S$ *are of order* $\delta_0\sigma$.

*When* $\sigma \to 0^+$, $\mathcal{M}^s(\sigma)$ *converges to* $(-4\delta_0, 4\delta_0)^{n^-} \times \mathcal{M}^{0-}(0)$*, i.e. the direct product of the stable manifold of* (1.11) *and the space* $u^-$.

We will not prove this theorem, because in the next section we will study the case where the parameter $\sigma$ depends on the solution.

We thus have that the stable manifold $\mathcal{M}^s(\sigma)$ of (1.14) of dimension $n^- + n^{0-}$ depends smoothly on the parameter $\sigma > 0$, and its diameter do not depend on $\sigma$. We recall that the stable manifold of the slow dynamics depends smoothly on $\sigma$, since it corresponds to the stable part of (1.18). For $\sigma < 0$ we have that the stable part of (1.18) depends smoothly on $\sigma$, while the stable part of (1.17) disappears. Observe that the solution cannot depends smoothly on $\sigma$, as it can be seen in the following simple example: this is a consequence of the presence of the fast scale $t/\sigma$.

*Example* 1.6. Consider the system

$$\left\{ \begin{array}{rcl} \dot{x} & = & -x/\sigma \\ \dot{y} & = & -y \\ \dot{z} & = & z + xy \end{array} \right.$$

The solution converging to 0 is given by

$$x(t) = x(0)e^{-t/\sigma}, \quad y(t) = y(0)e^{-t}, \quad z(t) = -\frac{\sigma x(0)y(0)}{2+\sigma} e^{-(1+1/\sigma)t},$$

and for $\sigma > 0$, the stable manifold is given by

$$z(0) = -\frac{\sigma x(0)y(0)}{1+2\sigma},$$

while for $\sigma < 0$ is only $y = 0$, $z = 0$. Observe that while the solution does not deends smoothly on $\sigma$, the manifold $\mathcal{M}^s$ does for $\sigma > 0$.

In Figure 1.2 it is represented a simple case in which $u^-$, $u^0$ are one dimensional and $\psi(\sigma, 0, 0) < 0$.

*Remark* 1.7. We observe that the same methods can be used to study all the orbits in the slow manifold, showing that these can be represented as the sum of the fast dynamics along $u^-$ and the part of the orbit which remains in the neighborhood around the origin. However for our purpose this analysis is not needed.

## 2. Singular ODE with parameter depending on the solution

We turn now to the more difficult case: we consider again the system

$$\dot{u} = \phi(u, \sigma(u)) = \frac{1}{\sigma(u)} \phi^s(u) + \phi^n(u), \qquad u \in \mathbb{R}^{n^- + n^0 + 1}, \tag{2.1}$$

but now the singular parameter $\sigma$ depends on the solution $u$. We assume the following.

(1) There is a smooth $N_{\mathrm{eq}}$ dimensional manifold of equilibria $\mathcal{M}_{\mathrm{eq}} = \{\phi^s(u) = \phi^n(u) = 0\}$, transversal to the singular surface $\sigma(u) = 0$: this means that its intersection with the level sets $\{\sigma(u) = \sigma\}$ has constant dimension $N_{\mathrm{eq}} - 1$. It is not restrictive to assume that $\phi^s(0) = \phi^n(0) = 0$ and $\sigma(0) = 0$, i.e. $u = 0$ is an equilibrium and it belongs to the singular surface. Rewrite thus the ODE as

$$\dot{u} = \frac{1}{\sigma(u)} \phi^s(u) u + \phi^n(u) u, \tag{2.2}$$

(2) Define the singular part of (2.1) as the ODE

$$u_\tau = \phi^s(u) u. \tag{2.3}$$

Then we assume that $\phi^s(u)$ is negative definite in a neighborhood of the equilibrium $u = 0$ and the $n^0 + 1$ dimensional center manifold of (2.3) is made only by equilibria and it is transversal to the singular surface $\{\sigma(u) = 0\}$. The dimension of the stable manifold of $u = 0$ is thus $n^-$.

We rescale time by writing the ODE in the following form

$$\left\{ \begin{array}{rcl} u_\tau &=& \phi^s(u) u + \sigma(u) \phi^n(u) u \\ t_\tau &=& \sigma(u) \end{array} \right. \tag{2.4}$$

This rescaling in general changes the ODE, because when $\sigma(u)$ changes sign the solutions to the above equation are not solutions to (2.2). Here and in the following, $\tau$ will denote the fast time scale.

We can repeat the decomposition into the center and center uniformly stable manifold of Section 1.1: however in this case the parameter $\sigma$ depends on the solution, so that it will have its own non trivial. By a change of variable, we can assume that $u = (y, u^-, u^0)$ such that

- the $n^- + n^0$ dimensional singular surface is $\{u : y = 0\}$, and the set $\{u : u^- = 0, u^0 = 0\}$ are equilibria;
- the $n^- + 1$ uniformly stable manifold $\mathcal{M}^-$ of $\{u : u^- = 0, u^0 = 0\}$ is given by $\{u : u^0 = 0\}$;
- the $n^0 + 1$ center manifold $\mathcal{M}^0$ of $\{u : u^- = 0, u^0 = 0\}$ is given by $\{u : u^- = 0\}$.

Note that, differently from the previous section, these manifolds contains also the $y$ axis.

We thus can rewrite the ODE (2.4) as

$$\left\{ \begin{array}{rcl} y_\tau &=& \tilde{\xi}(y, u^-, u^0) u^- + \tilde{\eta}(y, u^-, u^0) u^0 \\ u^-_\tau &=& \phi(y, u^-, u^0) u^- \\ u^0_\tau &=& \tilde{\psi}(y, u^-, u^0) u^0 \\ t_\tau &=& y \end{array} \right. \qquad (y, u^-, u^0) \in \mathbb{R}^{1 + n^- + n^0} \tag{2.5}$$

In fact, if $u^-(0) = 0/u^0(0) = 0$ then it remains 0 so that we can factorize $u^-/u^0$ in the equation for $u^-/u^0$, and if $(u^-, u^0) = 0$ we are on an equilibrium.

The assumption that the center manifold of $u_\tau = \phi^s(u) u$ is made only of equilibria implies that

$$\tilde{\psi}(0, 0, u^0) = 0, \qquad \tilde{\eta}(0, 0, u^0) = 0$$

so that we can rewrite the equations as

$$\left\{ \begin{array}{rcl} y_\tau &=& \hat{\xi}(y, u^-, u^0) u^- + y \hat{\eta}(y, u^0) u^0 \\ u^-_\tau &=& \phi(y, u^-, u^0) u^- \\ u^0_\tau &=& y \hat{\psi}(y, u^0) u^0 + \check{\psi}(y, u^-, u^0) u^- u^0 \\ t_\tau &=& y \end{array} \right. \qquad (y, u^-, u^0) \in \mathbb{R}^{1 + n^- + n^0} \tag{2.6}$$

We now prove the following lemma, which shows that invariance of $\mathcal{M}^0$ even in the non rescaled equation.

**Lemma 2.1.** *The center manifold $\mathcal{M}^0$ of* (2.4) *is invariant also for* (2.1)*. Moreover, the projection of the ODE $\dot u = \phi^N(u)$ on the manifold $\{\phi^s(u) = 0\}$ coincides on the set $\{\sigma(u) = 0\} \cap \{\phi^s(u) = 0\}$ with*

$$\left\{ \begin{array}{rcl} \dot y &=& \hat\eta(0, u^0)u^0 \\ \dot u^- &=& 0 \\ \dot u^0 &=& \hat\psi(0, u^0)u^0 \end{array} \right. \tag{2.7}$$

*in the coordinates $u = (y, u^-, u^0)$.*

*Proof.* By rescaling back time an using the change of coordinates leading to (2.6) (which is possible for $y \neq 0$), we can write the ODE (2.1) as

$$\begin{pmatrix} y \\ u^- \\ u^0 \end{pmatrix} = \frac{1}{y} \begin{pmatrix} \hat\xi(y, u^-, u^0)u^- \\ \phi(y, u^-, u^0)u^- \\ \check\psi(y, u^-, u^0)u^- u^0 \end{pmatrix} + \begin{pmatrix} \hat\eta(y, u^0)u^0 \\ 0 \\ \hat\psi(y, u^0)u^0 \end{pmatrix}, \tag{2.8}$$

so that (2.7) follows by taking $u^- = 0$ and letting $y \to 0$. Since the above vector field and the manifold $\mathcal{M}^0$ are smooth and $\mathcal{M}^0$ is invariant for

$$\left\{ \begin{array}{rcl} \dot y &=& \hat\eta(y, u^0)u^0 \\ \dot u^- &=& 0 \\ \dot u^0 &=& \hat\psi(y, u^0)u^0 \end{array} \right.$$

then the invariance for $y = 0$ again follows by letting $y \to 0$. □

A consequence of this lemma is that the manifold of slow dynamics exists under only the above two assumptions.

Before introducing the next assumption, we consider the following important remark.

*Remark* 2.2. A major problem when the singular parameter depends on the solution is that $u(t)$ may cross $\{y = 0\}$ in finite time: if we start in the region where $y > 0$ and at a certain time $\bar t$ $y$ becomes negative, then the fast dynamics part of (2.1) disappears. From the point of view of boundary profiles of hyperbolic-parabolic conservation laws, this corresponds to a non smooth boundary profile: an example can be found in [3], Example 2.1 of page 13.

Here we consider two simple examples which show twe completely different behaviors: on one hand the system

$$\left\{ \begin{array}{rcl} \dot y &=& -x \\ \dot x &=& -x/y \end{array} \right.$$

has a smooth solution for all $t \geq 0$, $y \geq 0$. On the other hand, the ODE

$$\left\{ \begin{array}{rcl} \dot y &=& -x/y \\ \dot x &=& -x \end{array} \right.$$

has the $y$ component disappearing in finite time like $\sqrt{t}$.

We observe that the discriminant condition is that the "singular" dynamics (formally obtained by rescaling $\tau = t/y$ and taking $y \to 0$, as in Condition 2)) in the first case is

$$y_\tau = 0, \quad x_\tau = -x,$$

i.e. $y$ remains constant, while in the second is $\dot y = -x$, so that $\{y = 0\}$ is not invariant.

We next consider the system

$$\left\{ \begin{array}{rcl} \dot y &=& -z \\ \dot x &=& -x/y \\ \dot z &=& -z \end{array} \right.$$

whose solution with initial data $(a, x(0), 2a)$ is

$$\left\{ \begin{array}{rcl} y &=& 2ae^{-t} - a \\ x &=& x(0) \exp\{-\frac{1}{a} \int_0^t (2e^{-s} - 1)^{-1} ds\} \\ z &=& 2ae^{-t} \end{array} \right.$$

It is clear that the solution has a loss of regularity like $t^{1/a}$ when crossing the singular surface $\{y = 0\}$. In this case the problem is that $\{y = 0\}$ is not invariant for the slow dynamics

$$\begin{cases} \dot{y} &= -z \\ \dot{z} &= -z \end{cases}$$

By the above considerations, we thus make the following assumption.

(3) The singular surface $\{y = 0\}$ is invariant for the fast flow and for the slow flow defined respectively by

$$(2.9) \qquad \begin{cases} y_\tau &= \tilde{\xi}(y, u^-, u^0)u^- \\ u^-_\tau &= \phi(y, u^-, u^0)u^- \\ u^0_\tau &= \check{\psi}(y, u^-, u^0)u^- u^0 \end{cases} \quad \text{and} \quad \begin{cases} y_\tau &= \hat{\eta}(y, u^0)u^0 \\ u^-_\tau &= 0 \\ u^0_\tau &= \hat{\psi}(y, u^0)u^0 \end{cases}$$

*Remark* 2.3. The above condition is equivalent to require that the singular surface $\{\sigma(u) = 0\}$ is invariant for the ODE $u_\tau = \phi^s(u)u$ and for the projection of $u_t = \phi^n(u)u$ on the manifold $\phi^s(u) = 0$, by Lemma 2.1. We also observe that if we want to contruct the invariant manifold corresponding to an exponential gap of (2.2), it is sufficient to assume the invariance of the singular surface for the slow flow restricted to the manifold corresponding to the exponential gap.

By a change of variable analogous to (1.13), we can assume that the stable manifold of the equilibrium $u = (0, 0, \bar{u}^0)$ is given by $\{u_0 = \bar{u}_0, y = 0\}$, so that it follows using also the condition 3) that

$$\check{\psi}(0, u^-, u^0) = 0, \qquad \tilde{\xi}(0, u^-, u^0) = 0, \qquad \hat{\eta}(0, u^0) = 0.$$

By rescaling time back, we can write (2.2) as

$$(2.10) \qquad \begin{cases} \dot{y} &= \xi(y, u^-, u^0)u^- + y\eta(y, u^0)u^0 \\ \dot{u}^- &= \frac{1}{y}\phi(y, u^-, u^0)u^- \\ \dot{u}^0 &= \psi(y, u^-, u^0)u^0 \end{cases}$$

We collect these observation as well as the diagonalization of the ODE (2.1) to (2.10) in the following proposition.

**Proposition 2.4.** *Under the assumptions 1), 2) of page 7, there exists a smooth manifold* $\mathcal{M}^0 = \{u = (y, u^-, u^0), u^- = 0\}$*, defined for* $|u| \leq 4\delta_0$*, invariant for* (2.1)*, such that the dynamics is given by the non singular ODE*

$$(2.11) \qquad \begin{cases} \dot{y} &= y\eta(yu^0)u^0 \\ \dot{u}^0 &= \psi(y, u^0)u^0 \end{cases}$$

*Similarly, if we assume also condition 3) of page 9, there is an invariant manifold* $\mathcal{M}^- = \{u = (y, u^, u^0), u^0 = 0\}$ *where the dynamics is given by*

$$(2.12) \qquad \begin{cases} \dot{y} &= \xi(y, u^-, 0)u^- \\ \dot{u}^- &= \frac{1}{y}\phi(y, u^-, 0)u^- \end{cases}$$

After having found the invariant manifold of slow dynamics, the center manifold for (2.1) is easily found as the center manifold of the reduced equation (2.11). As in the previous case, we assume that the equilibrium $(y, u^0) = (0, 0)$ of the ODE (2.11) has a $n^{0-}$ stable manifold, and a $n^{00} + 1$ center manifold.

**Theorem 2.5.** *Under the assumptions 1), 2) of page 7, there exists a* $n^{00} + 1$ *dimensional smooth center manifold* $\mathcal{M}^{00}$ *of* (2.1)*, contained in the* $n^0 + 1$ *dimensional invariant manifold* $\mathcal{M}^0$ *of slow dynamics and given by the center manifold of the ODE* (2.11)*. This manifold is defined and smooth in a neighborhood of* $u = 0$ *of radius* $4\delta_0$*.*

More work is required to prove that there exists a smooth uniformly stable manifold, defined for $y > 0$ and characterized by the fact that all orbits on this manifold converges to some equilibrium $(y, 0, 0)$, $y > 0$, with uniform exponential speed. This is considered in the next section.

2.1. **Analysis of the stable manifold.** In this section we prove the following theorem.

**Theorem 2.6.** *Under the assumptions 1), 2), 3) of page 9, there exists a $n^- + n^{0-} + 1$ dimensional invariant manifold $\mathcal{M}^s$, defined and smooth in the set $\{(y, u^-, u^0), 0 < y \leq 4\delta_0, |u^-|, |u^0| \leq 4\delta_0\} \subset \mathbb{R}^{1+n^-+n^0}$, which contains all the orbits converging to the equilibria $(y, u^-, u^0)$ with uniform exponential speed $e^{-ct}$. This manifold can be parameterized by $y$, $u^-$ and the component $u^{0-} = P^{0-}u^0$ of $u^0$ on the stable eigenspace of (2.11) at $u = 0$.*

*Moreover the solutions on the manifold $\mathcal{M}^s$ can be written as*

$$y(t) = y(0) + Y(t) + Q(t), \quad u^-(t) = X^-(t) + R(t), \quad u^0(t) = X^0(t) + S(t), \tag{2.13}$$

*where $X^-$, $(Y, X^0)$ are solutions to*

$$X^-_\tau = \phi(0, X^-, X^0(0))X^0, \tag{2.14}$$

$$\begin{cases} \dot{Y} &= Y\xi^0(t, X^0)X^0 \\ \dot{X}^0 &= \psi(Y, 0, X^0)X^0 \end{cases} \tag{2.15}$$

*The functions $(Y, X^0)$ belongs to the uniformly stable manifold of (2.15), and rest terms $Q$, $R$, $S$ can be estimated as*

$$|Q(\tau)|, |R(\tau)|, |S(\tau)| \leq \mathcal{O}(1)\delta_0 y(0)e^{-c\tau}. \tag{2.16}$$

*Remark* 2.7. From (2.13), (2.16) and the results we will prove on (2.15), we will see that

$$|y(t) - y(0)| \leq \mathcal{O}(1)\delta_0 y(0). \tag{2.17}$$

The above estimate holds only on this stable manifold $\mathcal{M}^s$, because in general it is not true for other orbits, as one can prove with the following simple example with no singular dynamics:

$$\begin{cases} \dot{y} &= -yx \\ \dot{x} &= -x^2 \end{cases} \qquad (y, u^0) \in \mathbb{R} \times \mathbb{R}^2, \ \alpha \in \mathbb{R}^+.$$

In fact, a key ingredient is the uniform exponential decay (or at least to have an integrable decay). We will return on this point when we study the characteristic boundary profiles, where the particular structure of the ODE yieds again an estimate of the form (2.17).

*Proof.* The proof is divided into 4 steps.

1) *Refined change of coordinates.* Since there is a uniformly stable manifold $\mathcal{M}^{0-}$ of dimension $n^{0-}+1$ contained in the slow manifold $\mathcal{M}^0$, by a change of coordinates we can assume that $u^0 = (u^{0-}, u^{00}) \in \mathbb{R}^{n^{0-}+n^{00}}$, and this manifolds are given by $\mathcal{M}^{0-} = \{u^{00} = 0\}$ (abusing of notations, in this proof the vector $u^{00}$ will denote only the complementary vector to $u^{0-}$ on $\mathcal{M}^0$ and $n^{00}$ its dimension, not the coordinates on the center manifold $\mathcal{M}^{00}$). This diagonalization can be done for $u^- = 0$, so that the system (2.10) can be decomposed again as

$$\begin{cases} \dot{y} &= \xi(y, u^-, u^0)u^- + y\eta(y, u^0)u^0 \\ \dot{u}^- &= \frac{1}{y}\phi(y, u^-, u^0)u^- \\ \dot{u}^{0-} &= \psi^{0-}(y, u^0)u^{0-} + \psi^-(y, u^-, u^0)u^-u^0 \\ \dot{u}^{00} &= \psi^{00}(y, u^0)u^{00} + \psi^0(y, u^-, u^0)u^-u^0 \end{cases} \qquad u^0 = (u^{0-}, u^{00}). \tag{2.18}$$

To avoid the singular parameter $1/y$, we rescale back (2.18):

$$\begin{cases} y_\tau &= y\xi(y, u^-, u^0)u^- + y^2\eta(y, u^0)u^0 \\ u^-_\tau &= \phi(y, u^-, u^0)u^- \\ u^{0-}_\tau &= y[\psi^{0-}(y, u^0)u^{0-} + \psi^-(y, u^-, u^0)u^-u^0] \\ u^{00}_\tau &= y[\psi^{00}(y, u^0)u^{00} + \psi^0(y, u^-, u^0)u^-u^0] \end{cases} \qquad u^0 = (u^{0-}, u^{00}). \tag{2.19}$$

The matrices $A^-$, $A^{0-}$, $A^{00}$ are defined by

$$A^- = \phi(0, 0, 0), \quad A^{0-} = \psi^{0-}(0, 0), \quad A^{00} = \psi^{00}(0, 0). \tag{2.20}$$

By the choice of the decomposition, we have the estimates

$$|e^{A^-\tau}|, |e^{A^{0-}\tau}| \leq Ce^{-(c+a)\tau}, \quad |e^{-A^{00}\tau}| \leq Ce^{(c-a)\tau}, \qquad \tau \geq 0, \tag{2.21}$$

for some constants $C, c, a > 0$. In the following, for any $1 + n^- + n^{0-}$ multi-index $\beta$, we will write $\beta = (\beta^y, \beta^-, \beta^{0-})$ with

$$\partial^\beta = \partial_y^{\beta^y} \partial_{u^-}^{\beta^-} \partial_{u^{0-}}^{\beta^{0-}}.$$

2) *Analysis of the unperturbed equations.* We study the equations for the unpertubed systems, namely (2.14) and (2.15). We recall first the following simple lemma.

**Lemma 2.8.** *If $\mathcal{T}_1$, $\mathcal{T}_2$ are contractions in the metric space $X$,*

$$d(\mathcal{T}_i x, \mathcal{T}_i y) \leq c d(x, y), \qquad i = 1, 2,$$

*then the distance from the two fixrd points $x_1 = \mathcal{T}_1 x_1$, $x_2 = \mathcal{T}_2 x_2$ can be estimate by*

$$d(x_1, x_2) \leq \frac{1}{1-c} \sup_{x \in \mathcal{X}} (\mathcal{T}_1 x, \mathcal{T}_2 x). \tag{2.22}$$

*Proof.* The proof follows from

$$d(x_1, x_)) = d(\mathcal{T}_1 x_1, \mathcal{T}_2 x_2) \leq d(\mathcal{T}_1 x_1, \mathcal{T}_1 x_2) + d(\mathcal{T}_1 x_2, \mathcal{T}_1 x_2) \leq c d(x_1, x_2) + \sup_{x \in \mathcal{X}} (\mathcal{T}_1 x, \mathcal{T}_2 x)$$

□

a) *Fast dynamics.* The equation for the fast dynamics, given by

$$X^-_\tau = \phi(0, X^-, \bar{X}^0) X^- \tag{2.23}$$

with initial data $X^-(0) = \bar{X}^-$. By using the estimate (2.21) on $\phi(0, 0, 0)$, the following lemma can be proved by standard ODE analysis:

**Lemma 2.9.** *The solution $X^- = X^-(\tau, \bar{X}^-, \bar{X}^0)$ to* (2.23) *depends smoothly on time and the initial data $\bar{X}^-$, $\bar{X}^{0-}$: if $\beta$ is a multi-index, then for $0 \leq \tau_1 \leq \tau_2$*

$$\begin{aligned} &\Big| \partial^\beta X^-(\tau_1, \bar{X}^-_1, \bar{X}^0_1) - \partial^\beta X^-(\tau_2, \bar{X}^-_2, \bar{X}^0_2) \Big| \\ &\qquad \leq C_{|\beta|} e^{-(c+a/2)\tau_1} \Big( |\tau_1 - \tau_2| + |\bar{X}^-_1 - \bar{X}^-_2| + |\bar{X}^0_1 - \bar{X}^0_2| \Big). \end{aligned} \tag{2.24}$$

We prove this lemma as an introduction to the parts 3, 4 the proof.

*Proof.* By writing the solution to (2.23) as

$$X^-(\tau) = e^{A^- \tau} \bar{X}^- + \int_0^\tau e^{A^-(\tau - \varsigma)} \big( \phi(0, X^-, \bar{X}^0) - A^- \big) X^-(\varsigma) d\varsigma, \tag{2.25}$$

it is easy to check that the metric space

$$\mathcal{X}^- = \Big\{ |X^-(0)| \leq 4\delta_0, \| e^{(c+a/2)\tau} X^-(\tau) \|_{\mathrm{Lip}(\mathbb{R}^+)} \leq 8C\delta_0, \ d(X^-_1, X^-_2) = \sup_{\tau \in \mathbb{R}^+} \big| e^{(c+a/2)\tau} (X^-_1 - X^-_2) \big| \Big\}.$$

is invariant if $|\bar{X}^{0-}| \leq 4\delta_0$, for $\delta_0 \ll 1$ and $C$ sufficiently large. Moreover the integral in the r.h.s. of (2.25) defines a contraction in $\mathcal{X}^-$, so that (2.24) is verified for $\beta = 0$, $\tau_1 = \tau_2$, $\bar{X}^0_! = \bar{X}^0_2$ and taking as second solution $X^- = 0$.

The general case is done by induction. The equation for $\partial^\beta X^-$ are (with the abuse of notation $\partial_{u^-} = \nabla_{u^-}$)

$$\partial^\beta X^-_\tau = (\phi(0, X^-, \bar{X}^0) + \partial_{u^-} \phi(0, X^-, \bar{X}^0) X^-) \partial^\beta X^- + \mathcal{N}^\beta(\tau, \bar{X}^-, \bar{X}^0) \tag{2.26}$$

We assume by induction that for $0 \leq \tau_1 \leq \tau_2$

$$\big| \mathcal{N}^\beta(\tau_1, \bar{X}^-_1, \bar{X}^0_1) - \mathcal{N}^\beta(\tau_2, \bar{X}^-_2, \bar{X}^0_2) \big| \leq D_\beta e^{-(c+a/2)\tau_1} \Big( |\tau_1 - \tau_2| + |\bar{X}^-_1 - \bar{X}^-_2| + |\bar{X}^0_1 - \bar{X}^0_2| \Big). \tag{2.27}$$

The solution to (2.26) can be written as

$$\begin{aligned} \partial^\beta X^-(\tau) = {} & e^{A^- \tau} \partial^\beta X^-(0) + \int_0^\tau e^{A^-(\tau - \varsigma)} \mathcal{N}^\beta(\varsigma) d\varsigma \\ & + \int_0^\tau e^{A^-(\tau - \varsigma)} \big( \phi(0, X^-, \bar{X}^0) - A^- + \partial_{u^-} \phi(0, X^-, \bar{X}^0) X^- \big) \partial^\beta X^-(\varsigma) d\varsigma \end{aligned} \tag{2.28}$$

The initial data $\partial^\beta X^-(0)$ is either 0 or a unit vector $e \in \mathbb{R}^{n^-}$, if $\beta = (0, \beta^-, 0)$ and $|\beta^- = 1|$.

We have the estimates

$$\bigg| \int_0^\tau e^{A^-(\tau-\varsigma)} \Big( \mathcal{N}^\beta(\varsigma, \bar{X}_1^-, \bar{X}_1^0) - \mathcal{N}^\beta(\varsigma, \bar{X}_1^-, \bar{X}_1^0) \Big) d\varsigma \bigg| \leq \tilde{D}_\beta \tau e^{-(c+a/2)\tau} \Big( |\bar{X}_1^- - \bar{X}_2^-| + |\bar{X}_1^0 - \bar{X}_2^0| \Big)$$

$$\begin{aligned} \bigg| \int_0^\tau e^{A^-(\tau-\varsigma)} \Big( & \big(\phi(0, X_1^-, \bar{X}_1^0) - A^- + \partial_{u^-}\phi(0, X_1^-, \bar{X}_1^0) X_1^-\big) \partial^\beta X_1^-(\varsigma) \\ & - \big(\phi(0, X_2^-, \bar{X}_2^0) - A^- + \partial_{u^-}\phi(0, X_2^-, \bar{X}_2^0) X_2^-\big) \partial^\beta X_2^-(\varsigma) \Big) d\varsigma \bigg| \\ \leq & \int_0^\tau C^2 \delta_0 e^{-(c+a)\varsigma} |\partial^\beta X_1^-(\varsigma) - \partial^\beta X_2^-(\varsigma)| d\varsigma + \int_0^\tau C^2 e^{-(c+a)\varsigma} |X_1^-(\varsigma) - X_2^-(\varsigma)| d\varsigma \\ & + \int_0^\tau C^2 e^{-(c+a)\varsigma} (|X_1^-(\varsigma)| + |\partial^\beta X_1^-(\varsigma)|) |\bar{X}_1^0 - \bar{X}_2^0| d\varsigma \end{aligned}$$

Introducing the metric space

$$\mathcal{X}^{-,\beta} = \Big\{ e^{(c+a/2)\tau} \partial^\beta X^-(\tau) \in \mathrm{Lip}(\mathbb{R}^+),\ d(\partial^\beta X_1^-, \partial^\beta X_2^-) = \sup_{\tau \in \mathbb{R}^+} \big| e^{(c+a/2)\tau} (\partial^\beta X_1^- - \partial^\beta X_2^-) \big| \Big\},$$

the second integral can be estimated as

$$\begin{aligned} \bigg| \int_0^\tau e^{A^-(\tau-\varsigma)} \Big( & \big(\phi(0, X_1^-, \bar{X}_1^0) - A^- + \partial_{u^-}\phi(0, X_1^-, \bar{X}_1^0) X_1^-\big) \partial^\beta X_1^-(\varsigma) \\ & - \big(\phi(0, X_2^-, \bar{X}_2^0) - A^- + \partial_{u^-}\phi(0, X_2^-, \bar{X}_2^0) X_2^-\big) \partial^\beta X_2^-(\varsigma) \Big) d\varsigma \bigg| \\ & \leq \tilde{C}_{|\beta|} \tau e^{-(c+a/2)\tau} \Big( \delta_0 d(\partial^\beta X_1^-, \partial^\beta X_2^-) + |\bar{X}_1^- - \bar{X}_2^-| + |\bar{X}_1^0 - \bar{X}_2^0| \Big). \end{aligned}$$

It thus follows that $\mathcal{X}^{-,\beta}$ is mapped into itself (by taking the special solution $\partial^\beta X^- = e^{A^-\tau} \partial^\beta X^-(0)$ for $\bar{X}_2^0 = 0$, $\bar{X}_1^- = \bar{X}_2^-$), and that the two integrals define a contraction in $\mathcal{X}^{-,\beta}$. Thus (2.24) follows from (2.22), when $\tau_1 = \tau_2$.

The next source terms is for $|\beta'| = |\beta| + 1$

$$\begin{aligned} \mathcal{N}^{\beta'} &= \partial^\alpha \partial^\beta (\phi(0, X^-, \bar{X}^0) X^-) - (\phi(0, X^-, \bar{X}^0) + \partial_{u^-}\phi(0, X^-, \bar{X}^0) X^-) \partial^\alpha \partial^\beta X^- \\ &= \sum_{|\gamma| = |\beta|} E^\gamma(X^-, \bar{X}^0) \partial^\gamma X^- + F^{\beta'}(X^-, \bar{X}^0), \end{aligned}$$

where $E^\gamma$ ($F^{\beta'}$) contains only terms in which only terms $\partial^\gamma X^-$ for $|\gamma| \leq 1$ ($|\gamma| < |\beta|$) appears. Thus the estimates of up to $|\beta| - 1$ yield

$$\begin{aligned} E^\gamma(X_1^-, \bar{X}_1^0) - E^\gamma(X_2^-, \bar{X}_2^0) &= \mathcal{O}(1) \Big( |\bar{X}_1^- - \bar{X}_2^-| + |\bar{X}^0 - \bar{X}^0| \Big), \\ F^{\beta'}(X_1^-, \bar{X}_1^0) - F^{\beta'}(X_2^-, \bar{X}_2^0) &= \mathcal{O}(1) e^{-(c+a/2)\tau} \Big( |\bar{X}_1^- - \bar{X}_2^-| + |\bar{X}^0 - \bar{X}^0| \Big). \end{aligned}$$

We have used that $F^{\beta'}(0, \bar{X}^0) = 0$, consequnce of the factor $X^-$ in the r.h.s. of (2.23). Using the estimates for $\partial^\beta X^-$ obtained in the $\beta$ step of the induction, it follows that (2.27) holds also for $|\beta'| = |\beta| + 1$.

The time dependence follows by taking the initial data for $X_2^-$ to be $\bar{X}_2^- = X_1^-(\tau_2 - \tau_1)$, and noticing that due to Lipschitz continuity this is of order $\tau_2 - \tau_1$. □

A more refined estimate shows that

$$\begin{aligned} & \Big| \partial^\beta X^-(\tau_1, \bar{X}_1^-, \bar{X}_1^{0-}) - \partial^\beta X^-(\tau_2, \bar{X}_2^-, \bar{X}_2^{0-}) \Big| \\ & \leq C_{|\beta|} e^{-(c+a/2)\tau_1} \Big( |\bar{X}^-|^{1-\min\{1,|\beta^-|\}} |\tau_1 - \tau_2| + |\bar{X}_1^- - \bar{X}_2^-| + |\bar{X}^-|^{1-\min\{1,|\beta^-|\}} |\bar{X}_1^{0-} - \bar{X}_2^{0-}| \Big), \end{aligned} \tag{2.29}$$

where the presence of $|\bar{X}^-|$ in front of the $\tau$, $\bar{X}^0$ terms follows from the fact that in the equation (2.23) one can factorize $X^-$: this terms disappears as soon as we take a derivative w.r.t. the coordinates $u^-$.

By taking as special solutions the functions $X^- = 0$, $e^{A^-\tau}\partial_{u^-}^{\beta^-} X^-(0)$ and using the more refined estimate (2.29), we recover the uniform exponential decay of $\partial^\beta X^-$:

$$\left|\partial^\beta X^-(\tau,\bar X^-,\bar X^-)\right| \le C_{|\beta|} \begin{cases} |X^-|e^{-(c+a/2)\tau} & \partial^\beta = \partial_{u^{0-}}^{\beta^{0-}} \\ e^{-(c+a/2)\tau} & \text{otherwise} \end{cases} \tag{2.30}$$

*Remark* 2.10. Again one can improve the analysis to obtain

$$\left|\partial^\beta X^-(\tau,\bar X^-,\bar X^-)\right| \le C_{|\beta|} \begin{cases} \tau^{\min\{1,|\beta|\}}|X^-|e^{-(c+a/2)\tau} & \partial^\beta = \partial_{u^{0-}}^{\beta^{0-}} \\ e^{-(c+a/2)\tau} & \partial^\beta = \partial_{u^-}^{\beta^-},\ |\beta^-|=1 \\ \tau e^{-(c+a/2)\tau} & \text{otherwise} \end{cases} \tag{2.31}$$

a) *Slow dynamics.* The equation for the slow dynamics are given by

$$\left\{ \begin{array}{ccc} Y_\tau & = & Y^2\eta(Y,X^0)X^0 \\ X^-_\tau & = & 0 \\ X^{0-}_\tau & = & Y\psi^{0-}(Y,X^0)X^{0-} \\ X^{00}_\tau & = & Y\psi^{00}(Y,X^0)X^{00} \end{array} \right. \qquad X^0 = (X^{0-},X^{00}). \tag{2.32}$$

with data $Y(+\infty) = \bar Y$, $X^-(0) = 0$, $X^{0-}(0) = \bar X^{0-}$, $X^{00}(0) = 0$ (hence also the last equation is redundant). For the rescaled equations

$$\left\{ \begin{array}{ccc} \dot Y & = & Y\eta(Y,X^{0-},0)(X^0,0) \\ \dot X^{0-} & = & \psi^{0-}(Y,X^{0-},0)X^{0-} \end{array} \right. \tag{2.33}$$

the solution is written as

$$\left\{ \begin{array}{ccc} Y(t) & = & \bar Y - \int_t^{+\infty}\eta(Y,X^{0-},0)(X^{0-}(\varsigma),0)d\varsigma \\ X^{0-}(t) & = & e^{A^{0-}t}\bar X^{0-} + \int_0^t e^{A^{0-}\varsigma}(\psi^{0-}(Y,X^{0-},0)-A^{0-})X^{0-}(\varsigma)d\varsigma \end{array} \right. \tag{2.34}$$

Using (2.21) and techniques completely similar to the proof of Lemma 2.9, one can prove the following lemma:

**Lemma 2.11.** *The solution* $(Y,X^{0-}) = (Y,X^{0-})(\tau,\bar Y,\bar X^{0-})$ *to* (2.34) *depends smoothly on time and the data* $Y(+\infty)=\bar y$, $X^{0-}(0) = \bar X^{0-}$*: if* $\beta$ *is a multi-index,* $m$ *an integer, then for* $0 \le t_1 \le t_2$

$$\begin{aligned} &\left|\frac{d^m}{dt^m}\partial^\beta Y(t_1,\bar Y,\bar X_1^{0-}) - \frac{d^m}{dt^m}\partial^\beta Y(t_2,\bar Y_2,\bar X_2^{0-})\right| \le C_{|\beta|+m}e^{-ct_1} \\ &\qquad\Big(|\bar X_1^{0-}|^{1-\min\{1,|\beta^{0-}|\}}|\bar Y_1|^{1-\min\{1,|\beta^y|\}}|t_1-t_2| + |\bar X_1^{0-}|^{1-\min\{1,|\beta^{0-}|\}}|\bar Y_1-\bar Y_2| \\ &\qquad\quad + |\bar Y_2|^{1-\min\{1,|\beta^y|\}}|\bar X_1^{0-}-\bar X_2^{0-}|\Big), \\ &\left|\frac{d^m}{dt^m}\partial^\beta X^{0-}(t_1,\bar Y_1,\bar X_1^{0-}) - \frac{d^m}{dt^m}\partial^\beta X^{0-}(t_2,\bar Y_2,\bar X_2^{0-})\right| \le C_{|\beta|+m}e^{-ct_1} \\ &\qquad\Big(|\bar X_1^{0-}|^{1-\min\{1,|\beta^{0-}|\}}|t_1-t_2| + |\bar X_1^{0-}|^{1-\min\{1,|\beta^{0-}|\}}|\bar Y_1-\bar Y_2| + |\bar X_1^{0-}-\bar X_2^{0-}|\Big). \end{aligned} \tag{2.35}$$

By taking as special solution $Y(t) = \bar Y$, $X^{0-}(t) = 0$, one obtain the decay estimates

$$\left|\frac{d^m}{dt^m}\partial^\beta Y(t,\bar Y,\bar X^{0-}) - \partial^\beta \bar Y\right| \le C_{|\beta|+m} \begin{cases} \bar Y|\bar X^{0-}|e^{-ct} & \beta = 0 \\ |\bar X^{0-}|e^{-ct} & \partial^\beta = \partial_y^{\beta^y} \\ \bar Y e^{-ct} & \partial^\beta = \partial_{X^{0-}}^{\beta^-} \\ e^{-ct} & \text{otherwise} \end{cases} \tag{2.36}$$

$$\left|\frac{d^m}{dt^m}\partial^\beta X^0(t,\bar Y,\bar X^{0-})\right| \le C_{|\beta|+m} \begin{cases} |\bar X^{0-}|e^{-ct} & \partial^\beta = \partial_y^{\beta^y} \\ e^{-ct} & \text{otherwise} \end{cases} \tag{2.37}$$

In particular, for $\bar X^{0-}$ sufficiently small, $Y(t)$ satisfies an estimate of the form (2.17), in particular it remains positive if $\bar Y > 0$.

To come back to the $\tau$ variable, we assume $\bar{Y} > 0$ and we solve the ODE

$$t_\tau = \bar{Y} + (Y(t, \bar{Y}, \bar{X}^{0-}) - \bar{Y}) = \bar{Y}(1 + |\bar{X}^{0-}|\zeta(t, \bar{Y}, \bar{X}^{0-})e^{-ct}), \tag{2.38}$$

with $\zeta$ bounded by a constant and regular (we have used (2.36)). We solve this ODE as follows:

$$\bar{Y}\tau = t + \int_t^{+\infty} \frac{|\bar{X}^{0-}|\zeta(s, \bar{Y}, \bar{X}^{0-})e^{-cs}}{1 + |\bar{X}^{0-}|\zeta(s, \bar{Y}, \bar{X}^{0-})e^{-cs}} ds. \tag{2.39}$$

The r.h.s. defines a smooth and invertible function such that for $0 \le t_1 \le t_2$ and using Lemma 2.11

$$\begin{aligned} &\left| \partial^\beta \int_{t_1}^{+\infty} \frac{|\bar{X}_1^{0-}|\zeta(s, \bar{Y}_1, \bar{X}_1^{0-})e^{-cs}}{1 + |\bar{X}_1^{0-}|\zeta(s, \bar{Y}_1, \bar{X}_1^{0-})e^{-cs}} ds - \partial^\beta \int_{t_2}^{+\infty} \frac{|\bar{X}_2^{0-}|\zeta(s, \bar{Y}_2, \bar{X}_2^{0-})e^{-cs}}{1 + |\bar{X}_2^{0-}|\zeta(s, \bar{Y}_2, \bar{X}_2^{0-})e^{-cs}} ds \right| \\ &\qquad \le C_{|\beta|} e^{-ct_1} \Big( |t_1 - t_2| + |\bar{Y}_1 - \bar{Y}_2| + |\bar{X}_1^{0-} - \bar{X}_2^{0-}| \Big). \end{aligned} \tag{2.40}$$

In particular, it follows that the r.h.s. is invertible, so that by inverting (2.39) we obtain that $t = t(\bar{y}\tau)$ defines a smooth function such that

$$\left| \partial^\beta t(\bar{Y}_1\tau_1, \bar{Y}_1, \bar{X}_1^{0-}) - \partial^\beta t(\bar{Y}_2\tau_2, \bar{Y}_2, \bar{X}_2^{0-}) \right| \le C_{|\beta|} \Big( |\bar{Y}_1\tau_1 - \bar{Y}_2\tau_2| + |\bar{Y}_1 - \bar{Y}_2| + |\bar{X}_1^{0-} - \bar{X}_2^{0-}| \Big). \tag{2.41}$$

Using the decay estimate (2.40), we obtain also

$$\left| t(\bar{Y}\tau, \bar{Y}, \bar{X}^{0-}) - \bar{Y}\tau \right| \le C|\bar{X}^{0-}|e^{-c\bar{Y}\tau}, \tag{2.42}$$

so that we see that $\bar{Y}\tau$ is the asymptote to $t$.

We now rewrite Lemma 2.11 in the time variable $\tau$:

**Lemma 2.12.** *The solution* $(Y, X^{0-}) = (Y, X^-)(\tau, \bar{Y}, \bar{X}^{0-})$ *to*

$$\left\{ \begin{array}{rcl} Y_\tau &=& Y^2\eta(Y, X^{0-}, 0)(X^0, 0) \\ X_\tau^{0-} &=& Y\psi^{0-}(Y, X^{0-}, 0)X^{0-} \end{array} \right. \tag{2.43}$$

*depends smoothly on time and the data* $Y(+\infty) = \bar{Y} > 0$, $X^{0-}(0) = \bar{X}^{0-}$*: if* $\beta$ *is a multi-index, then for* $0 \le \tau_1 \le \tau_2$

$$\begin{aligned} &\left| \partial^\beta Y(\tau_1, \bar{Y}, \bar{X}_1^{0-}) - \partial^\beta Y(\tau_2, \bar{Y}_2, \bar{X}_2^{0-}) \right| \le C_{|\beta|} e^{-c\bar{Y}_1\tau_1} \\ &\qquad \Big( |\bar{X}_1^{0-}|^{1-\min\{1,|\beta^{0-}|\}} \bar{Y}_1^{1-\min\{1,|\beta^y|\}} |\bar{Y}_1\tau_1 - \bar{Y}_2\tau_2| + |\bar{X}_1^{0-}|^{1-\min\{1,|\beta^{0-}|\}} |\bar{Y}_1 - \bar{Y}_2| \\ &\qquad\quad + \bar{Y}_2^{1-\min\{1,|\beta^y|\}} |\bar{X}_1^{0-} - \bar{X}_2^{0-}| \Big), \\ &\left| \partial^\beta X^{0-}(\tau_1, \bar{Y}_1, \bar{X}_1^{0-}) - \partial^\beta X^{0-}(\tau_2, \bar{Y}_2, \bar{X}_2^{0-}) \right| \le C_{|\beta|} e^{-c\bar{Y}_1\tau_1} \\ &\qquad \Big( |\bar{X}_1^{0-}|^{1-\min\{1,|\beta^{0-}|\}} |\bar{Y}_1\tau_1 - \bar{Y}_2\tau_2| + |\bar{X}_1^{0-}|^{1-\min\{1,|\beta^{0-}|\}} |\bar{Y}_1 - \bar{Y}_2| + |\bar{X}_1^{0-} - \bar{X}_2^{0-}| \Big). \end{aligned} \tag{2.44}$$

By taking as special solution $Y(\tau) = \bar{Y}$, $X^{0-}(\tau) = 0$, one obtain the decay estimates

$$\left| \partial^\beta Y(\tau, \bar{Y}, \bar{X}^{0-}) - \partial^\beta \bar{Y} \right| \le C_{|\beta|} \begin{cases} \bar{Y}|\bar{X}^{0-}|e^{-c\bar{Y}\tau} & \beta = 0 \\ |\bar{X}^{0-}|e^{-c\bar{Y}\tau} & \partial^\beta = \partial_y^{\beta^y} \\ \bar{Y}e^{-c\bar{Y}\tau} & \partial^\beta = \partial_{X^{0-}}^{\beta^-} \\ e^{-c\bar{Y}\tau} & \text{otherwise} \end{cases} \tag{2.45}$$

$$\left| \partial^\beta X^0(\tau, \bar{Y}, \bar{X}^{0-}) \right| \le C_{|\beta|} \begin{cases} |\bar{X}^{0-}|e^{-c\bar{Y}\tau} & \partial^\beta = \partial_y^{\beta^y} \\ e^{-c\bar{Y}\tau} & \text{otherwise} \end{cases} \tag{2.46}$$

In the following we will not made use of the exponential decay $e^{-c\bar{Y}\tau}$.

3) *Equation for the rest terms and basic estimates.* The rest terms $Q$, $R$, $S^-$, $S^0$ satisfy the ODE

$$\left\{\begin{array}{rcl} Q_\tau &=& (Y+Q)\xi(Y+Q,X^-+R,X^0+S)(X^-+R) \\ && \quad +\big[(Y+Q)^2\eta(Y+Q,X^0+S)(X^0+S)-Y^2\eta(Y,X^0)X^0\big] \\ R_\tau &=& \phi(Y+Q,X^-+R,X^0+S)(X^-+R)-\phi(0,X^-,\bar X^{0-})X^- \\ S^-_\tau &=& \big[(Y+Q)\psi^{0-}(Y+Q,X^0+S)(X^{0-}+S^-)-Y\psi^{0-}(Y,X^0)X^{0-}\big] \\ && \quad +(Y+Q)\psi^-(Y+Q,X^-+R,X^0+S)(X^-+R)(X^0+S) \\ S^0_\tau &=& (Y+Q)\psi^{00}(Y+Q,S)S^0+(Y+Q)\psi^0(Y+Q,X^-+R,X^0+S)(X^-+R)(X^0+S) \end{array}\right. \tag{2.47}$$

where for notation we write $X^0=(X^{0-},0)$, $S=(S^-,S^0)$, vectors in $\mathbb{R}^{n^{0-}+n^{00}}$.

We solve them by considering the map $\mathcal{T}$

$$\mathcal{T}:\left\{\begin{array}{l} -\int_\tau^{+\infty}(Y+Q)\xi(Y+Q,X^-+R,X^0+S)(X^-+R)(\varsigma)d\varsigma \\ \quad -\int_\tau^{+\infty}\big[(Y+Q)^2\eta(Y+Q,X^0+S)(X^0+S)(\varsigma)-Y^2\eta(Y,X^0)X^0(\varsigma)\big]d\varsigma \\ -\int_\tau^{+\infty}\big[\phi(Y+Q,X^-+R,X^0+S)(X^-+R)(\varsigma)-\phi(0,X^-,\bar X^{0-})X^-(\varsigma)\big]d\varsigma \\ -\int_\tau^{+\infty}\big[(Y+Q)\psi^{0-}(Y+Q,X^0+S)(X^{0-}+S^-)(\varsigma)-Y\psi^{0-}(Y,X^0)X^{0-}(\varsigma)\big]d\varsigma \\ \quad -\int_\tau^{+\infty}(Y+Q)\psi^-(Y+Q,X^-+R,X^0+S)(X^-+R)(X^0+S)(\varsigma)d\varsigma \\ -\int_\tau^{+\infty}(Y+Q)\psi^{00}(Y+Q,S)S^0(\varsigma)d\varsigma \\ \quad +\int_0^\tau(Y+Q)\psi^0(Y+Q,X^-+R,X^0+S)(X^-+R)(X^0+S)(\varsigma)d\varsigma \end{array}\right. \tag{2.48}$$

Define the metric space $\mathcal{X}$ as

$$\mathcal{X}_0=\Big\{(Q,R,S):\mathrm{Lip}\Big(e^{c\tau}Q(\tau),e^{c\tau}R(\tau),e^{c\tau}S(\tau)\Big)\le\tilde C_0\delta_0\bar Y\Big\}, \tag{2.49}$$

with the distance

$$d_0\Big((Q_1,R_1,S_1),(Q_2,R_2,S_2)\Big)\le\max\Big\{\|e^{c\tau}(Q_1-Q_2)\|_{C^0},\|e^{c\tau}(R_1-R_2)\|_{C^0},\|e^{c\tau}(S_1-S_2)\|_{C^0}\Big\}. \tag{2.50}$$

We now show that the map $\mathcal{T}$ is a contraction in $(\mathcal{X}_0,d_0)$. This follows from the following computations: for $(Q,R,S)\in\mathcal{X}_0$, $0<\bar Y_1\le\bar Y_2$,

$$\begin{aligned} &\Big|\mathcal{T}_{(Y_1,X_1^-,X_1^{0-})}(Q_1,R_1,S_1)-\mathcal{T}_{(Y_2,X_2^-,X_2^{0-})}(Q_2,R_2,S_2)\Big| \\ &\quad\le C\delta_0e^{-c\tau}d_0\Big((Q_1,R_1,S_1),(Q_2,R_2,S_2)\Big)+C\delta_0e^{-c\tau}\Big(|\bar Y_1-\bar Y_2|+\bar Y_2|\bar X_1^--\bar X_2^-|+\bar Y_2|\bar X_1^{0-}-\bar X_2^{0-}|\Big). \end{aligned} \tag{2.51}$$

In fact, the above estimate shows that (2.48) is a contraction in $(\mathcal{X}_0,d_0)$, and by comparing with the solution $Q,R,S=0$ obtained for $\bar X^-=0$, $\bar X^{0-}=0$, one shows that $\mathcal{T}$ maps $\mathcal{X}_0$ into itself, for $\tilde C_0\gg1$ an $\delta_0\ll1$. By replacing $(Y(\tau),X^-(\tau),X^{0-}(\tau))$ with $(Y(\tau+\delta\tau),X^-(\tau+\delta\tau),X^{0-}(\tau+\delta\tau))$, the Lipschitz dependence w.r.t. is again a consequence of the estimate (2.51) and Lemmas 2.9, 2.12.

We first observe that in all the terms of the integrals (2.48) an exponential term like $e^{-c\tau}$ appears: in fact, for the second integral of the first line and the first integral of the third line one has

$$\begin{gathered}\big[(Y+Q)^2\eta(Y+Q,X^0+S)(X^0+S)-Y^2\eta(Y,X^0)X^0\big]=\mathcal{O}(1)S+\mathcal{O}(1)Q \\ \big[(Y+Q)\psi^{0-}(Y+Q,X^0+S)(X^{0-}+S^-)-Y\psi^{0-}(Y,X^0)X^{0-}\big]=\mathcal{O}(1)S+\mathcal{O}(1)Q,\end{gathered}$$

so that all integrands are exponentially decaying. Moreover, their form is

$$\begin{gathered}\zeta_Q(Y,Q,X^-,R,X^{0-},S)Q+\zeta_R(Y,Q,X^-,R,X^{0-},S)R+\zeta_S(Y,Q,X^-,R,X^{0-},S)S \\ \zeta_{X^-}(Y,X^-,X^{0-})YX^-+(\phi(0,X^-,X^{0-})-\phi(0,X^-,\bar X^{0-}))X^-\end{gathered}$$

for some smoth functions $\zeta_Q$, $\zeta_R$, $\zeta_S$, such that

$$\zeta_Q(0),\zeta_R(0),\zeta_S(0)=0.$$

In fact the integrands are quadratic and the only terms not containing $Q$, $R$, $S$ are $Y\xi X^-$, $\phi(Y,X^-,X^0)-\phi(0,X^-,\bar X^0)X^-$, $Y\phi^-X^-X^0$, $Y\phi^0X^-X^0$.

Using the fact that $Y$, $X^-$, $X^{0-}$ are of order $\delta_0$, $Y$ satisfies 2.45 and $(Q,R,S) \in \mathcal{X}_0$, it follows for $0 < \bar{Y}_1 \leq \bar{Y}_2$

$$\begin{aligned}
&\int_\tau^{+\infty} \Big|\zeta_Q(Y_1,Q_1,X_1^-,R_1,X_1^0,S_1)Q_1(\varsigma) - \zeta_Q(Y_2,Q_2,X_2^-,R_2,X_2^0,S_2)Q_2(\varsigma)\Big| d\varsigma \\
&\qquad \leq \int_\tau^{+\infty} C\delta_0\Big(|Q_1-Q_2| + |R_1-R_2| + |S_1-S_2|\Big)d\varsigma \\
&\qquad\quad + \int_\tau^{+\infty} C\delta_0 e^{-c\tau}\Big(|Y_1-Y_2| + \bar{Y}_2|X_1^{0-}-X_2^{0-}|\Big)d\varsigma + \int_\tau^{+\infty} C\delta_0\bar{Y}_2|X_1^- - X_2^-|d\varsigma \\
&\qquad \leq C\delta_0 e^{-c\varsigma} d_0\Big((Q_1,R_1,S_1),(Q_2,R_2,S_2)\Big) + C\delta_0 e^{-c\tau}\Big(|\bar{Y}_1-\bar{Y}_2| + \bar{Y}_2|\bar{X}_1^- - \bar{X}_2^-| + \bar{Y}_2|\bar{X}_1^{0-}-\bar{X}_2^{0-}|\Big),
\end{aligned}$$

and the same estimates can be obtained for $\zeta_R$, $\zeta_S$. For the terms $\zeta_{X^-}YX^-$, $(\phi(0,X^-,X^{0-})-\phi(0,X^-,\bar{X}^{0-}))X^-$ we have

$$\begin{aligned}
&\Big|\int_\tau^{+\infty}\Big(\zeta_{X^-}(Y_1,X_1^-,X_1^{0-})Y_1X_1^-(\varsigma) - \zeta_{X^-}(Y_2,X_2^-,X_2^{0-})Y_2X_2^-(\varsigma)\Big)d\varsigma\Big| \\
&\qquad \leq \int_\tau^{+\infty} C\delta_0 e^{-c\varsigma}\Big(|Y_1-Y_2| + Y_2|X_1^{0-}-X_2^{0-}|\Big)d\varsigma + \int_\tau^{+\infty} CY_2|X_1^- - X_2^-|d\varsigma \\
&\qquad \leq C\delta_0 e^{-c\tau}\Big(|\bar{Y}_1-\bar{Y}_2| + \bar{Y}_2|\bar{X}_1^- - \bar{X}_2^-| + \bar{Y}_2|\bar{X}_1^{0-}-\bar{X}_2^{0-}|\Big),
\end{aligned}$$

$$\begin{aligned}
&\Big|\int_\tau^{+\infty}\Big(\big(\phi(0,X_1^-,X_1^{0-}) - \phi(0,X_1^-,\bar{X}_1^{0-})\big)X_1^-(\varsigma) - \big(\phi(0,X_2^-,X_2^{0-}) - \phi(0,X_2^-,\bar{X}_2^{0-})\big)X_2^-(\varsigma)\Big)d\varsigma\Big| \\
&\qquad \leq \int_\tau^{+\infty} C|X_1^{0-}-\bar{X}_1^{0-}||X_1^- - X_2^-|d\varsigma + \int_\tau^{+\infty} C\delta_0 e^{-c\varsigma}\varsigma\Big(|\bar{Y}_1-\bar{Y}_2| + \bar{Y}_2|\bar{X}_1^- - \bar{X}_2^-|\Big)d\varsigma \\
&\qquad \leq C\delta_0 e^{-c\tau}\Big(|\bar{Y}_1-\bar{Y}_2| + \bar{Y}_2|\bar{X}_1^- - \bar{X}_2^-| + \bar{Y}_2|\bar{X}_1^{0-}-\bar{X}_2^{0-}|\Big),
\end{aligned}$$

where we used the estimates

$$\frac{d}{d\tau}X^{0-} = \mathcal{O}(1)Y\tau, \qquad \int_\tau^{+\infty} e^{-(c+a/2)\varsigma}\varsigma d\varsigma \leq \mathcal{O}(1)e^{-c\tau}.$$

Finally, we observe that the initial data $Q(0)$, $R(0)$, $S(0)$ are parameterized by $(\bar{Y},\bar{X}^-,\bar{X}^{0-})$, and from (2.36) (for $m=0$, $\beta=0$) and the fact that $(Q,R,S)\in\mathcal{X}_0$ we have

$$|Y(0)+Q(0)-\bar{Y}|, |R(0)|, |S(0)| \leq C\delta_0. \tag{2.52}$$

It thus follows that the map

$$(\bar{Y},\bar{X}^-,\bar{X}^{0-}) \mapsto (Y(0)+Q(0), \bar{X}^- + R(0), \bar{X}^{0-} + S^-(0))$$

is invertible. The quadratic estimate in the definition of $\mathcal{X}^0$ yields that the manifold $S^0(0) = \mathcal{M}^2(\bar{Y},\bar{X}^-,\bar{X}^{0-})$ is tangent to the eigenspace $(Y,X^-,X^{0-})$ at $(0,0,0)$. This concludes the proof of the existence of the invariant manifold and the asymptotic expansion (2.16).

4) *Regularity estimates.* We are left with the regularity estimates. If $\beta$ is a multiindex, then the equations satisfied by the derivatives $\partial^\beta Q$, $\partial^\beta R$, $\partial^\beta S$ are

(2.53)

$$\begin{cases}
\partial^\beta Q_\tau &= \Big[(\xi+(Y+Q)\partial_y\xi)(X^-+R) + (2(Y+Q)\eta + (Y+Q)^2\partial_y\eta)(X^0+S)\Big]\partial^\beta Q \\
&\quad + \Big[(Y+Q)(\xi+\partial_{X^-}\xi(X^-+R))\Big]\partial^\beta R \\
&\quad + \Big[(Y+Q)\partial_{X^0}\xi(X^-+R) + (Y+Q)^2(\eta+\partial_{X^0}\eta(X^0+S))\Big]\partial^\beta S + \mathcal{Q}^\beta \\
\partial^\beta R_\tau &= [\partial_y\phi(X^-+R)]\partial^\beta Q + \big[\phi+\partial_{X^-}\phi(X^-+R)\big]\partial^\beta R + [\partial_{X^0}\phi(X^-+R)]\partial^\beta S + \mathcal{R}^\beta \\
\partial^\beta S_\tau^- &= \big[(\psi^{0-}+(Y+Q)\partial_y\psi^{0-})(X^{0-}+S^-)\big]\partial^\beta Q + \big[(Y+Q)(\psi^- + \partial_{X^-}\psi^-(X^-+R))(X^0+S)\big]\partial^\beta R \\
&\quad + \Big[(Y+Q)(\psi^{0-}+\partial_{X^0}\psi^{0-}(X^{0-}+S)) + (Y+Q)(\psi^{0-}+\partial_{X^0}\psi^{0-}(X^0+S))(X^-+R)\Big]\partial^\beta S + \mathcal{S}^{-,\beta} \\
\partial^\beta S_\tau^0 &= \big[(\psi^{00}+(Y+Q)\partial_y\psi^{00})S^0\big]\partial^\beta Q + \big[(Y+Q)(\psi^0+\partial_{X^-}\psi^0(X^-+R))(X^0+S)\big]\partial^\beta R \\
&\quad + \Big[(Y+Q)(\psi^{00}+\partial_{X^0}\psi^{00}S^0) + (Y+Q)(\psi^0+\partial_{X^0}\psi^0(X^0+S))(X^-+R)\Big]\partial^\beta S + \mathcal{S}^{0,\beta}
\end{cases}$$

We assume that the source terms satisfy

$$|\mathcal{Q}_1^\beta - \mathcal{Q}_2^\beta| \leq Ce^{-c\tau}\Big(|\bar{Y}_1 - \bar{Y}_2| + \bar{Y}_2|\bar{X}_1^- - \bar{X}_2^-| + \bar{Y}_2|\bar{X}_1^{0-} - \bar{X}_2^{0-}|\Big), \tag{2.54}$$

and similarly for the terms $\mathcal{R}^\beta$, $\mathcal{S}^\beta$. For $|\beta| = 1$, these estimates are correct.

We solve (2.53) by considering the map $\mathcal{T}^\beta$

(2.55)

$$\mathcal{T}^\beta : \begin{cases} -\int_\tau^{+\infty}\Big[(\xi + (Y+Q)\partial_y\xi)(X^- + R) + (2(Y+Q)\eta + (Y+Q)^2\partial_y\eta)(u^+S)\Big]\partial^\beta Q d\varsigma \\ \quad -\int_\tau^{+\infty}\Big[(Y+Q)(\xi + \partial_{X^-}\xi(X^- + R))\Big]\partial^\beta R d\varsigma \\ \quad -\int_\tau^{+\infty}\Big[(Y+Q)\partial_{X^0}\xi(X^- + R) + (Y+Q)^2(\eta + \eta_{X^0}(X^0 + S))\Big]\partial^\beta S d\varsigma - \int_\tau^{+\infty}\mathcal{Q}^\beta d\varsigma \\ -\int_\tau^{+\infty}[\partial_y\phi(X^- + R)]\partial^\beta Q + \big[\phi + \partial_{X^-}\phi(X^- + R)\big]\partial^\beta R + [\partial_{X^0}\phi(X^- + R)]\partial^\beta S d\varsigma - \int_\tau^{+\infty}\mathcal{R}^\beta d\varsigma \\ -\int_\tau^{+\infty}\big[(\psi^{0-} + (Y+Q)\partial_y\psi^{0-})(X^{0-} + S^-)\big]\partial^\beta Q + \big[(Y+Q)(\psi^- + \partial_{X^-}\psi^-(X^- + R))(X^0 + S)\big]\partial^\beta R d\varsigma \\ \quad -\int_\tau^{+\infty}\Big[(Y+Q)(\psi^{0-} + \partial_{X^0}\psi^{0-}(X^{0-} + S)) + (Y+Q)(\psi^{0-} + \partial_{X^0}\psi^{0-}(X^0 + S))(X^- + R)\Big]\partial^\beta S d\varsigma \\ \quad -\int_\tau^{+\infty}\mathcal{S}^{-,\beta}d\varsigma \\ -\int_\tau^{+\infty}\big[(\psi^{00} + (Y+Q)\psi_y^{00})S^0\big]\partial^\beta Q + \big[(Y+Q)(\psi^0 + \psi^0_{X^-}(X^- + R))(X^0 + S)\big]\partial^\beta R d\varsigma \\ \quad -\int_\tau^{+\infty}\Big[(Y+Q)(\psi^{00} + \psi_{X^0}S^0) + (Y+Q)(\psi^0 + \psi_{X^0}(X^0 + S))(X^- + R)\Big]\partial^\beta S d\varsigma - \int_\tau^{+\infty}\mathcal{S}^{0,\beta}d\varsigma \end{cases}$$

By repeating the computations leading to (2.51), one obtains the estimate

$$\begin{aligned} &\mathcal{T}^\beta_{(Y_1,X_1^-,X_1^0)}(\partial^\beta Q_1, \partial^\beta R_1, \partial^\beta S_1) - \mathcal{T}^\beta_{(Y_2,X_2^-,X_2^0)}(\partial^\beta Q_2, \partial^\beta R_2, \partial^\beta S_2) \\ &\quad \leq Ce^{-c\tau}\max\Big\{\|e^{ct}(\partial^\beta Q_1 - \partial^\beta Q_2)\|_{C^0}, \|e^{ct}(\partial^\beta R_1 - \partial^\beta R_2)\|_{C^0}, \|e^{ct}(\partial^\beta S_1 - \partial^\beta S_2)\|_{C^0}\Big\} \\ &\qquad + Ce^{-c\tau}\Big(|\bar{Y}_1 - \bar{Y}_2| + \bar{Y}_2|\bar{X}_1^- - \bar{X}_2^-| + \bar{Y}_2|\bar{X}_1^{0-} - \bar{X}_2^{0-}|\Big). \end{aligned} \tag{2.56}$$

It thus follows (by taking the special solution $(Y, X^-, X^0) = 0$, $(\partial^\alpha Q, \partial^\alpha R, \partial^\alpha S) = 0$ for $0 \leq \alpha \leq \beta$) that $(\partial^\beta Q, \partial^\beta R, \partial^\beta S)$ belongs to the space

$$\mathcal{X}_|\beta| = \Big\{(\partial^\beta Q, \partial^\beta R, \partial^\beta S) : \mathrm{Lip}\Big(e^{c\tau}\partial^\beta Q(\tau), e^{c\tau}\partial^\beta R(\tau), e^{c\tau}\partial^\beta S(\tau)\Big) \leq \tilde{C}_\beta \bar{Y}\Big\}, \tag{2.57}$$

with the distance

$$\begin{aligned} &d_0\Big((\partial^\beta Q_1, \partial^\beta R_1, \partial^\beta S_1), (\partial^\beta Q_2, \partial^\beta R_2, \partial^\beta S_2)\Big) \\ &\quad \leq \max\Big\{\|e^{c\tau}(\partial^\beta Q_1 - \partial^\beta Q_2)\|_{C^0}, \|e^{c\tau}(\partial^\beta R_1 - \partial^\beta R_2)\|_{C^0}, \|e^{c\tau}(\partial^\beta S_1 - \partial^\beta S_2)\|_{C^0}\Big\}. \end{aligned} \tag{2.58}$$

The Lipschitz regularity of $(\partial^\beta Q, \partial^\beta R, \partial^\beta S)$ is a consequence of the decay estimates of the source term.

To end the proof we need to estimate the source term for $\beta'$, $|\beta'| = |\beta| + 1$. As in the proof of Lemma 2.9, the source term $\mathcal{Q}^{\beta'}$ for the next step $\beta'$ can be written as

$$\begin{aligned} \mathcal{Q}^{\beta'} = &\sum_{|\gamma|=|\beta|} E^\gamma_{\mathcal{Q},Q}(Y, X^-, X^{0-}, Q, R, S)\partial^\gamma Q + \sum_{|\gamma|=|\beta|} E^\gamma_{\mathcal{Q},R}(Y, X^-, X^{0-}, Q, R, S)\partial^\gamma R \\ &+ \sum_{|\gamma|=|\beta|} E^\gamma_{\mathcal{Q},S}(Y, X^-, X^{0-}, Q, R, S)\partial^\gamma S + F^{\beta'}_{\mathcal{Q}}(Y, X^-, X^{0-}, Q, R, S), \end{aligned}$$

where $E^\gamma_{\mathcal{Q}}$ ($F^{\beta'}_{\mathcal{Q}}$) depends only on the derivatives of $(Y, X^-, X^{0-}, Q, R, S)$ up to order 1 $(\beta - 1)$. A similar expansion holds for the sources $\mathcal{R}^{\beta'}$, $\mathcal{S}^{\beta'}$.

Thus the estimates of up to $|\beta| - 1$ yield

$$E^\gamma_{\mathcal{Q}}(Y, X^-, X^{0-}, Q, R, S) - E^\gamma_{\mathcal{Q}}(Y, X^-, X^{0-}, Q, R, S) = \mathcal{O}(1)\Big(|\bar{Y}_1 - \bar{Y}_2| + \bar{Y}_2|\bar{X}_1^- - \bar{X}_2^-| + \bar{Y}_2|\bar{X}_1^{0-} - \bar{X}_2^{0-}|\Big),$$

$$F^{\beta'}_{\mathcal{Q}}(Y, X^-, X^{0-}, Q, R, S) - F^{\beta'}_{\mathcal{Q}}(Y, X^-, X^{0-}, Q, R, S) = \mathcal{O}(1)e^{-c\tau}\Big(|\bar{Y}_1 - \bar{Y}_2| + \bar{Y}_2|\bar{X}_1^- - \bar{X}_2^-| + \bar{Y}_2|\bar{X}_1^{0-} - \bar{X}_2^{0-}|\Big).$$

We have used that $F^{\beta'}(0, 0, 0, Q, R, S) = 0$, consequence of the fact that $(0, 0, 0)$ is an exact solution to 2.19. Using the estimates for $(\partial^\beta Q, \partial^\beta R, \partial^\beta S)$ obtained in the $\beta$ step of the induction, it follows that (2.54) holds also for $|\beta'| = |\beta| + 1$.

This concludes the proof of Theorem 2.6. □

## 3. Boundary profiles and Riemann solver

In this section we consider a hyperbolic-parabolic system of the form

$$E(u)u_t + A(u,u_x)u_x = B(u)u_{xx}, \quad u \in R^N \tag{3.1}$$

with the standard dissipativity assumptions, see [8, 9] and [3], Section 2.2:

(1) $E(u)$ symmetric strictly positive definite;
(2) $E^{-1}(u)A(u,0)$ symmetric and strictly hyperbolic, with

$$A(u,u_x) = \left[\begin{array}{cc} A_{11}(u) & A_{21}(u)^T \\ A_{21}(u) & A_{22}(u,u_x) \end{array}\right]; \tag{3.2}$$

(3) $B(u)$ is of the form

$$B(u) = \left[\begin{array}{cc} 0 & 0 \\ 0 & b(u) \end{array}\right], \tag{3.3}$$

with $b(u)$ $r \times r$ strictly positive definite matrix;
(4) Kawashima condition,

$$\ker B(u) \cap \{\text{eigenvectors of } E^{-1}(u)A(u,u_x)\} = 0. \tag{3.4}$$

Differently from [3], Section 2.2 and Remark 4.2, we allow the speed of the boundary or of the traveling wave to be one of the eigenvalues of the hyperbolic part of (3.1),

$$E_{11}(u)u_{1,t} + A_{11}(u)u_{1,x} = 0, \quad u = (u_1,u_2) \in \mathbb{R}^{N-r} \times \mathbb{R}^r. \tag{3.5}$$

In this case, one can check that the reduction given in [3], Section 4.1.2, generates a singular ODE. Without any loss of generality we can assume that (3.5) holds for $u = 0$, i.e.

$$\det\left(E_{11}^{-1}(0)A_{11}(0) - \sigma \mathbb{I}_{N-r}\right) = 0, \tag{3.6}$$

where $\mathbb{I}_{N-r}$ is the $N-r$ dimensional identity matrix.

Since the matrix $A(u) - \sigma E(u)$ satisfies the condition 2) above, in the following we will take $\sigma = 0$ as the speed of the boundary. We then assume moreover that the boundary is characteristic, i.e.

$$\det A(0,0) = 0. \tag{3.7}$$

### 3.1. Notations, assumptions on $E_{11}^{-1}(u)A_{11}(u)$ and basic transversality results.

3.1.1. *Analysis of the reduced hyperbolic part.* We denote with $\eta_i(u)$, $\zeta_i(u)$, $i = 1,\dots,N-r$ the eigenvalues, eigenvectors of

$$A_{11}(u)\zeta_i(u) = \eta_i(u)E_{11}(u)\zeta_i(u), \tag{3.8}$$

counted with their multiplicity. Due to the symmetry assumptions on $A$, $E$, the eigenvectors $\zeta_i(u)$ can be taken to form an orthonomal base of $\mathbb{R}^{N-r}$ w.r.t. the scalar product $\langle a,b\rangle_{E_{11}} = a^T E_{11}(u)b$, $a,b \in \mathbb{R}^{N-r}$. With elementary computations of linear algebra, the eigenvectors of the matrix $E_{11}^{-1/2}(u)A_{11}(u)E_{11}^{-1/2}(u)$ are given by $E_{11}^{1/2}\zeta_i$, while the eigenvalues are again the $\eta_i(u)$. In the following we denote with $Z_i$ the vectors

$$Z_i(u) = \left(\begin{array}{c} \zeta_i(u) \\ 0 \end{array}\right), \tag{3.9}$$

where $\zeta_i(u)$ is an eigenvector of (3.8).

We make the following assumption:

(5) The eigenvalues of (3.8) can be ordered as follows:
- $\eta_1(u) \le \cdots \le \eta_{n_{11}}(u) < 0$;
- $\eta_{n_{11}+1}(u) = \cdots = \eta_{n_{11}+q}(u)$, and $\eta_i(0) = \sigma$ for $i = n_{11}+1,\dots,n_{11}+q$;
- $0 < \eta_{n_{11}+q+1}(u) \le \cdots \le \eta_{N-r}(u)$.

Moreover $D\eta_{n_{11}+1}(u) \neq 0$.

The assumption that the gradient is not equal to 0 ensures that the singular set $\{\eta_{n_{11}+1}(u) = 0\}$ is a $n-1$ dimensional surface, and that $\eta_{n_{11}+1}(u)$ is equivalent to the distance of $u$ from $\{\eta_{n_{11}+1}(u) = 0\}$.

By using the base of eigenvectors $\zeta_i(u)$ of (3.8), it follows that we can write

$$A_{11}(u) - \sigma E_{11}(u) = \sum_{i=1}^{N-r} E_{11}(u)\zeta_i(u)(\eta_i(u) - \sigma)\zeta_i^T(u)E_{11}(u), \tag{3.10}$$

where we used the relation $\sum_i \zeta_i(u)\zeta_i^T(u)E_{11}(u) = \mathbb{I}_{N-r}$. If $\eta_i(u) \neq \sigma$ for all $i = 1, \dots, N_r$, then we can compute the inverse function to $A_{11}(u) - \sigma E_{11}(u)$:

$$\big(A_{11}(u) - \sigma E_{11}(u)\big)^{-1} = \sum_{i=1}^{N-r} (\eta_i(u) - \sigma)^{-1}\zeta_i(u)\zeta_i^T(u). \tag{3.11}$$

Condition 5 thus corresponds to the fact that $(A_{11} - \sigma E_{11})^{-1}$ has a block of fixed dimension $q$ whose eigenvalue behaves like $(\eta_i(u) - \sigma)^{-1}$.

Observe that when $\sigma = 0$, then (3.6) reduces to

$$\det A_{11}(u) = 0,$$

so that the number $q$ of characteristic eigenvalues of (3.8) can be deduced from the number of eigenvalues of $A_{11}(0)$ equal to 0. A standard perturbation argument shows that also the number $n_{11}$ is equal to the number of eigenvalues of $A_{11}(u)$ strictly less than $\sigma$, see for example [10] or repeat the part of the proof of the next lemma concernig this perturbation technique.

*Remark* 3.1. The reason of the assumption 5) is that we do not want to have more than one singular parameter, when writing the equations for the boundary profiles or the travelling wave: this is clearly the case under the forementioned assumption, as it is shown in (3.11).

3.1.2. *Analysis of the hyperbolic equation.* We denote with $\lambda_i(u)$, $r_i(u)$, $i = 1, \dots, N$ the eigenvalues, eigenvectors of

$$A(u,0)r_i(u) = \lambda_i(u)E(u)r_i(u). \tag{3.12}$$

The assumption 2) of page 18 implies that the $\lambda_i(u)$ are real and distinct, and from (3.7) we have that the eigenvalue $\lambda_k(u)$ (for a fixed $k \in \{1, \dots, N\}$) satisfies

$$\lambda_k(0) = 0. \tag{3.13}$$

This eigenvalue $\lambda_k(u)$ will be called the *boundary characteristic eigenvalue*, and the corresponding eigenvector $r_k(u)$ the *boundary characteristic eigenvector*. Since we will study the system in a small neighborhood of $u = 0$ of radius $4\delta_0$, we will always have that for a positive constant $\ell \gg \delta_0 > 0$

(1) the eigenvalues $\lambda_i(u)$, $i = 1, \dots, k-1$ are uniformly negative $\leq -\ell$ for $|u| \leq 4\delta_0$;
(2) $\lambda_k(u)$ may change sign but it remains of order $\delta_0$ in $|u| \leq 4\delta_0$;
(3) the eigenvalues $\lambda_i(u)$, $i = k+1, \dots, N$ are uniformly positive $\geq \ell$ for $|u| \leq 4\delta_0$.

3.1.3. *Analysis of the parabolic equation.* We denote with $\Theta_i(u)$, $\mu_i(u)$ the eigenvalues, eigenvectors of the linear system

$$A(u,0)\Theta_i(u) = \mu_i(u)B(u)\Theta_i(u). \tag{3.14}$$

Since $B(u)$ is not invertible, the equation $\det(A(u,0) - \mu B(u)) = 0$ is not a polynomial of order $N$. The description of the roots of the above system is given by the following Lemma, whose proof follows [3], Lemmas 4.6 and 4.7.

**Lemma 3.2.** *Under conditions 1), 2), 3, 4) of page 18 and 5) of page 18, the roots of*

$$\det\big(A(u,0) - \mu B(u)\big) = 0 \tag{3.15}$$

*are ordered as follows:*

(1) *the number of roots $\mu_i$ is $r$ for $\eta_{n_{11}+1}(u) \neq 0$, and $r - q$ when $\eta_{n_{11}+1}(u) = 0$;*
(2) *there is one root, denoted with $\mu_k(u)$, whose real (imaginary) part behaves like $\lambda_k(u)$ ($\lambda_k(u)|$), the boundary characteristic root of $A(u,0)$. In particular it remains of order $\delta_0$ and its real part has the same sign of $\lambda_k(u)$;*

(3) *if* $\eta_{n_{11}+1}(u) \leq 0$ *then there are* $\mu_i(u)$, $i = k - n_{11}, \dots, k-1$, *roots with real part uniformly negative* $\leq -\ell \ll -\delta_0 < 0$. *If* $\eta_{n_{11}+1} > 0$, *then there are* $q$ *additional negative roots* $\mu_i$, $i = k - n_{11} - q, \dots, k - n_{11} - 1$, *whose real part behaves like* $-1/\eta_{n_{11}+1}(u)$;

(4) *if* $\eta_{n_{11}+1}(u) \geq 0$ *then there are* $\mu_i(u)$, $i = k+1, \dots, k+n_{11}$, *roots with real part uniformly positive* $\geq \ell \gg \delta_0 > 0$. *If* $\eta_{n_{11}+1} < 0$, *then there are* $q$ *additional positive roots* $\mu_i$, $i = k+n_{11}+1, \dots, k+n_{11}+q$, *whose real part behaves like* $-1/\eta_{n_{11}+1}(u)$.

It is known that their number is $r$ and their sign can be computed by knowing the number of the eigenvalues of $A$ and $A_{11}$.

Finally, let $r_i(u)$, $\lambda_i(u)$, $i = 1, \dots, N$, be the eigenvalues, eigenvectors of the hyperbolic part

$$A(u) r_i(u) = \lambda_i(u) E(u) r_i(u). \tag{3.16}$$

Due to the strictly hyperbolicty assumption, we suppose them to be ordered and distinct.

We have the following transversality results:

- transversaality w.r.t. the others eigenvectors to form a base in $\mathbb{R}^{N-n_{11}(-q)}$

-transvesality w.r.t. the initial data to show solvability at the linear level

The problem of studying the boundary profiles or travelling waves for (3.1) depends on finding the center and stable manifold for a singular ODE, with the singular parameter depending on the solution. We then check which implications the condition 3 of page 9 has on the form of the system (3.1).

3.2. **Reduction to a singular ODE.** In this section we use the analysis carried out in [3], Section 4.1, to reduce the algebraic-differential equation for travelling waves/boundary profiles to a standard ODE.

In what follows, we will use the following projector on $\mathbb{R}^{N-r}$:

$$R_s(u) = \left[ \begin{array}{ccc} \zeta_{n_{11}+1}(u) & \dots & \zeta_{n_{11}+q}(u) \end{array} \right], \; L_s(u) = R_s(u)^T E_{11}(u), \quad P_s(u) = R_s(u) L_s(u), \tag{3.17}$$

where $\zeta_i(u)$ are the eigenvectors of (3.8). Similarly, we define also

$$R_n(u) = \left[ \begin{array}{cccccc} \zeta_1(u) & \dots & \zeta_{n_{11}}(u) & \zeta_{n_{11}+q+1} & \dots & \zeta_{N-r}(u) \end{array} \right], \; L_n(u) = R_n(u)^T E_{11}(u). \tag{3.18}$$

and $P_s(u) = R_s(u) L_s(u)$. Clearly by construction $L_s(u) R_s(u) = \mathbb{I}_q$, $L_n(u) R_n(u) = \mathbb{I}_{N-r-q}$, $L_s(u) R_n(u) = 0$, $L_n(u) R_s(u) = 0$.

Moreover, by Kawashima condition (Condition 4 of page 18) it follows ([3], Lemma 4.2) that the matrix $R_s^T(u)(A_{21}(u) - \sigma E_{21}(u))^T$ has rank $q$: let $P_{\mathrm{Kaw}}(u) = R_{\mathrm{Kaw}}(u) R_{\mathrm{Kaw}}(u)^T(u)$, $P_{\mathrm{res}}(u) = R_{\mathrm{res}}(u) R_{\mathrm{res}}^T(u)$ be the orthogonal projectors on (a $q$ dimensional subspace of) $\mathrm{span}\{R_s^T(u)(A_{21}(u) - \sigma E_{21}(u))^T\}$ and its orthogonal space, respectively. In particular, $R_{\mathrm{Kaw}}(u) \in \mathbb{R}^{q \times r}$ and $R_s^T(u)(A_{21}(u) - \sigma E_{21}(u))^T R_{\mathrm{Kaw}}(u)$ has rank $k$.

We thus can write the equation for travelling/boundary profiles

$$(A - \sigma E) u_x = u_{xx} \tag{3.19}$$

as

$$u_x = \left( \begin{array}{c} R_s(u) w_1 + R_n(u) w_2 \\ R_{\mathrm{Kaw}}(u) z_1 + R_{\mathrm{res}}(u) z_2 \end{array} \right), \qquad w_1, z_1 \in \mathbb{R}^q, \; w_2 \in \mathbb{R}^{N-r-q}, \; z_2 \in \mathbb{R}^{r-q}, \tag{3.20}$$

(3.21)

$$\left[ \begin{array}{cccc} (\eta_{n_{11}+1}(u) - \sigma)\mathbb{I}_q & 0 & a_{11}^T(u,\sigma) & a_{21}^T(u,\sigma) \\ 0 & \tilde{A}_{11}(u,\sigma) & a_{12}^T(u,\sigma) & a_{22}^T(u,\sigma) \\ a_{11}(u,\sigma) & a_{12}(u,\sigma) & \alpha_{11}(u,u_x,\sigma) & \alpha_{21}(u,u_x,\sigma) \\ a_{21}(u,\sigma) & a_{22}(u,\sigma) & \alpha_{21}(u,u_x,\sigma) & \alpha_{22}(u,u_x,\sigma) \end{array} \right] \left( \begin{array}{c} w_1 \\ w_2 \\ z_1 \\ z_2 \end{array} \right) = \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \\ b_{11}(u) & b_{12}(u) \\ b_{21}(u) & b_{22}(u) \end{array} \right] \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right)_x,$$

where we used the symmetry of the matrices $A(u)$, $E(u)$, and

$$\det(\tilde{A}_{11}(u,\sigma)) = \det\Big( R_n^T(u)(A_{11}(u) - \sigma E_{11}(u)) R_n(u) \Big) \neq 0,$$

$$\det(a_{11}(u,\sigma)) = \det\Big( R_{\mathrm{Kaw}}^T(u)(A_{21}(u) - \sigma E_{21}(u)) R_s(u) \Big) \neq 0. \tag{3.22}$$

The matrix

$$\left[ \begin{array}{cc} b_{11}(u) & b_{12}(u) \\ b_{21}(u) & b_{22}(u) \end{array} \right] = \left[ \begin{array}{cc} R_{\mathrm{Kaw}}^T(u) b(u) R_{\mathrm{Kaw}}(u) & R_{\mathrm{Kaw}}^T(u) b(u) R_{\mathrm{res}}(u) \\ R_{\mathrm{res}}^T(u) b(u) R_{\mathrm{Kaw}}(u) & R_{\mathrm{res}}^T(u) b(u) R_{\mathrm{res}}(u) \end{array} \right]$$

is clearly invertible. The reduction of (3.21) to an ODE follows the same steps of [3], Section 4.

For $\eta_{n_{11}+1}(u) \neq \sigma$, we can reduce the first two lines by writing

$$\begin{aligned} w_1 &= -\frac{1}{\eta_{n_{11}+1}(u)-\sigma}\big(a_{11}^T(u,\sigma)z_1 + a_{21}^T(u,\sigma)z_2\big) \\ w_2 &= -\tilde{A}_{11}^{-1}(u,\sigma)\big(a_{12}^T(u,\sigma)z_1 + a_{22}^T(u,\sigma)z_2\big). \end{aligned} \tag{3.23}$$

Substituting into (3.20), (3.21), one obtains ODE with singular parameter $\lambda_{n_{11}+1}(u)-\sigma$

$$u_x = \begin{bmatrix} -\frac{R_s(u)a_{11}^T(u,\sigma)}{\eta_{n_{11}+1}(u)-\sigma} & -\frac{R_s(u)a_{21}^T(u,\sigma)}{\eta_{n_{11}+1}(u)-\sigma} \\ -R_n(u)\tilde{A}_{11}^{-1}(u,\sigma)a_{12}^T(u,\sigma) & -R_n(u)\tilde{A}_{11}^{-1}(u,\sigma)a_{22}^T(u,\sigma) \\ R_{\mathrm{Kaw}}(u) & 0 \\ 0 & R_{\mathrm{res}}(u) \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{3.24}$$

$$\begin{aligned} \begin{bmatrix} b_{11}(u) & b_{12}(u) \\ b_{21}(u) & b_{22}(u) \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_x &= \begin{bmatrix} \alpha_{11}(u,u_x,\sigma) & \alpha_{21}^T(u,u_x,\sigma) \\ \alpha_{21}(u,u_x,\sigma) & \alpha_{22}(u,u_x,\sigma) \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ &- \begin{bmatrix} a_{11}(u,\sigma) & a_{12}(u,\sigma) \\ a_{21}(u,\sigma) & a_{22}(u,\sigma) \end{bmatrix} \begin{bmatrix} \frac{1}{\eta_{n_{11}+1}(u)-\sigma}\mathbb{I}_q & 0 \\ 0 & \tilde{A}_{11}^{-1}(u,\sigma) \end{bmatrix} \begin{bmatrix} a_{11}(u,\sigma) & a_{12}^T(u,\sigma) \\ a_{21}^T(u,\sigma) & a_{22}^T(u,\sigma) \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \end{aligned} \tag{3.25}$$

By adding the equation

$$\sigma_x = 0, \tag{3.26}$$

we obtain a system of the form (2.1). We now analyze separately the singular and the non singular part.

3.2.1. *Analysis of the singular part.* The singular part of system (3.24), (3.25), (3.26) is given by (3.26) and

$$u_x = \begin{bmatrix} -R_s(u)a_{11}^T(u,\sigma) & -R_s(u)a_{21}^T(u,\sigma) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{3.27}$$

$$\begin{bmatrix} b_{11}(u) & b_{12}(u) \\ b_{21}(u) & b_{22}(u) \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_x = -\begin{bmatrix} a_{11}(u,\sigma) & 0 \\ a_{21}(u,\sigma) & 0 \end{bmatrix} \begin{bmatrix} a_{11}(u,\sigma) & 0 \\ a_{21}(u,\sigma) & 0 \end{bmatrix}^T \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{3.28}$$

The center manifold is given by the set

$$\Big\{(u,z,\sigma) : a_{11}^T(u,\sigma)z_1 + a_{21}^T(u,\sigma)z_2 = 0\Big\}, \tag{3.29}$$

and it is trivial to observe that it is made only by equilibria. We have used the symmetric structure of the r.h.s. of (3.28).

The stable manifold is $q$ dimensional, and in the space $u \in \mathbb{R}^N$ the trajectory is tangent to the $q$ dimensional subspace $u = \mathrm{span}\{(R_s(u),0)\}$, where we recall that $R_s(u)$ is the eigenspace of $A_{11}^{-1}(u) - \zeta E_{11}(u)$ corresponding to $\eta_{n_{11}+1} = \cdots = \eta_{n_{11}+q} = \sigma$.

The condition of invariance of the singular surface (Condition 3 of page 9) thus reads $D\eta_{n_{11}+1}u_x = 0$: since the initial $z = (z_1,z_2)$ can have an arbitrary direction in $\mathbb{R}^r$ (we require only smallness in the construction of the manifolds, Section 2), it thus follows from the invertibility of $a_{11}^T(u)$ that

$$D\eta_{n_{11}+i}(u)\zeta_{n_{11}+j}(u) = 0 \qquad \forall i,j = 1,\dots,q. \tag{3.30}$$

We thus obtain the following condition on the hyperbolic/parabolic system (3.1):

**Condition L.** *If the reduced hyperbolic system*

$$E_{11}u_{1,t} + A_{11}(u)u_{1,x} = 0 \tag{3.31}$$

*has $q$ eigenvalues $\eta_{n_{11}+1}(u) = \cdots = \eta_{n_{11}+q}(u)$ with the same speed $\sigma$ of the boundary or the travelling wave, then this eigenvalue is linearly degenerate.*

Note that we do not require the strict hyperbolicity of (3.31), but that the dimension of the eigenspace does not change. We recall that for a hyperbolic system in conservation form $g(u)_t + f(u)_x = 0$, with $Dg(u)$ strictly positive definite, an eigenvalue with constant multiplicity $\geq 1$ is linearly degenerate.

In [3], Section 2.2.2, a counterexample is shown where the above condition does not hold and the boundary profiles and travelling waves are not smooth, see also [12].

3.2.2. *Analysis of the non singular part.* When $\eta_{n_{11}+1}(u) = \sigma$, we have to compute the equation of the slow dynamics: we recall that, when the singular parameter is 0, this dynamics is equal to the projection of the non singular part $\phi^n$ of the ODE (2.2) on the manifold of equilibria of the singular part $\Phi^s$.

In our setting the equilibrium manifold of $\Phi^s = 0$ corresponds to the algebraic relation

$$a_{11}^T(u,\sigma)z_1 + a_{21}^T(u,\sigma)z_2 = 0, \tag{3.32}$$

so that it follows that the computations are exactly the same done in [3], Section 4.1.2, to obtain the explicit form for the agebraic/differential system

$$u_x = \begin{pmatrix} R_s(u)w_1 + R_n(u)w_2 \\ R_{\text{Kaw}}(u)z_1 + R_{\text{res}}(u)z_2 \end{pmatrix}, \qquad w_1, z_1 \in \mathbb{R}^q,\ w_2 \in \mathbb{R}^{N-r-q},\ z_2 \in \mathbb{R}^{r-q}, \tag{3.33}$$

$$\begin{bmatrix} 0 & 0 & a_{11}^T(u,\sigma) & a_{21}^T(u,\sigma) \\ 0 & \tilde{A}_{11}(u,\sigma) & a_{12}^T(u,\sigma) & a_{22}^T(u,\sigma) \\ a_{11}(u,\sigma) & a_{12}(u,\sigma) & \alpha_{11}(u,u_x,\sigma) & \alpha_{21}^T(u,u_x,\sigma) \\ a_{21}(u,\sigma) & a_{22}(u,\sigma) & \alpha_{21}(u,u_x,\sigma) & \alpha_{22}(u,u_x,\sigma) \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ z_1 \\ z_2 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ b_{11}(u) & b_{12}(u) \\ b_{21}(u) & b_{22}(u) \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_x. \tag{3.34}$$

We will follow these computations closely.

We now use the fact that $a_{11}(u,\sigma) \neq 0$, by Kawashima condition 4) of page 18. We can write

$$z_1 = -a_{11}^{-T}a_{21}^T z_2, \quad w_2 = -A_{11}^{-1}(a_{12}^T z_1 + a_{22}^T z_2) = A_{11}^{-1}\big(a_{12}^T a_{11}^{-T} a_{21}^T - a_{22}^T\big) z_2,$$

$$\begin{aligned} w_1 = &-\big(a_{11} + b_{11}(D_{u_1}(a_{11}^{-T}a_{21}^T)\cdot)z_2 R_s\big)^{-1}\Big[\big(a_{12} - b_{11}(D_{u_1}(a_{11}^{-T}a_{21}^T)\cdot)z_2\big)R_n w_2 \\ &\qquad + \big(\alpha_{11} - b_{11}D_{u_2}(a_{11}^{-T}a_{21}^T\cdot)z_2\big)R_{\text{Kaw}}z_1 + \big(\alpha_{12} - b_{11}(D_{u_2}(a_{11}^{-T}a_{21}^T)\cdot)z_2\big)R_{\text{res}}z_2 \\ &\qquad\qquad - \big(b_{12} - b_{11}a_{11}^{-T}a_{21}^T\big)z_{2,x}\Big] \\ = &\ \big(a_{11} + b_{11}(D_{u_1}(a_{11}^{-T}a_{21}^T)\cdot)z_2 R_s\big)^{-1}\big(b_{12} - b_{11}a_{11}^{-T}a_{21}^T\big)z_{2,x} \\ &- \big(a_{11} + b_{11}(D_{u_1}(a_{11}^{-T}a_{21}^T)\cdot)z_2 R_s\big)^{-1}\Big[\big(a_{12} - b_{11}(D_{u_1}(a_{11}^{-T}a_{21}^T)\cdot)z_2\big)R_n A_{11}^{-1}\big(a_{12}^T a_{11}^{-T}a_{21}^T - a_{22}\big) \\ &\qquad - \big(\alpha_{11} - b_{11}D_{u_2}(a_{11}^{-T}a_{21}^T)z_2\big)R_{\text{Kaw}}a_{11}^{-T}a_{21}^T + \big(\alpha_{12} - b_{11}(D_{u_{22}}(a_{11}^{-T}a_{21}^T)\cdot)z_2\big)R_{\text{res}}\Big]z_2. \end{aligned}$$

The notation follows the decomposition

$$u = (u_1, u_2) \in \mathbb{R}^{N_r} \times \mathbb{R}^r,$$

and we use $D$ to denote the Jacobian matrix: $((Df\cdot)z)w$ means $\sum \partial_i f_j z_j w_i$.

We thus obtain the equation for $u$, $z_2$ of the form

$$\begin{aligned} u_{1,x} = &\ R_s\big(a_{11} + b_{11}(D_{u_1}(a_{11}^{-T}a_{21}^T)\cdot)z_2 R_s\big)^{-1}\big(b_{12} - b_{11}a_{11}^{-T}a_{21}^T\big)z_{2,x} \\ &- R_s\big(a_{11} + b_{11}(D_{u_1}(a_{11}^{-T}a_{21}^T)\cdot)z_2 R_s\big)^{-1}\Big[\big(a_{12} - b_{11}(D_{u_1}(a_{11}^{-T}a_{21}^T)\cdot)z_2\big)R_n A_{11}^{-1}\big(a_{12}^T a_{11}^{-T}a_{21}^T - a_{22}\big) \\ &- \big(\alpha_{11} - b_{11}D_{u_2}(a_{11}^{-T}a_{21}^T)z_2\big)R_{\text{Kaw}}a_{11}^{-T}a_{21}^T + \big(\alpha_{12} - b_{11}(D_{u_{22}}(a_{11}^{-T}a_{21}^T)\cdot)z_2\big)R_{\text{res}}\Big]z_2 \\ &+ R_n A_{11}^{-1}\big(a_{12}^T a_{11}^{-T}a_{21}^T - a_{22}^T\big)z_2, \end{aligned} \tag{3.35}$$

$$u_{2,x} = (R_{\text{res}} - R_{\text{Kaw}}a_{11}^{-T}a_{21}^T)z_2, \tag{3.36}$$

$$
\begin{aligned}
&\Big[b_{22} - b_{21}a_{11}^{-T}a_{21}^T - a_{21}\big(a_{11} + b_{11}(D_{u_1}(a_{11}^{-T}a_{21}^T)\cdot)z_2R_s\big)^{-1}\big(b_{12} - b_{11}a_{11}^{-T}a_{21}^T\big)\Big]z_{2,x} \\
&= \bigg\{ - a_{21}\big(a_{11} + b_{11}(D_{u_1}(a_{11}^{-T}a_{21}^T)\cdot)z_2R_s\big)^{-1}\Big[\big(a_{12} - b_{11}(D_{u_1}(a_{11}^{-T}a_{21}^T)\cdot)z_2\big)R_nA_{11}^{-1}\big(a_{12}^Ta_{11}^{-T}a_{21}^T - a_{22}\big) \\
&\qquad - \big(\alpha_{11} - b_{11}D_{u_2}(a_{11}^{-T}a_{21}^T)z_2\big)R_{\mathrm{Kaw}}a_{11}^{-T}a_{21}^T + \big(\alpha_{12} - b_{11}(D_{u_{22}}(a_{11}^{-T}a_{21}^T)\cdot)z_2\big)R_{\mathrm{res}}\Big] \\
&\quad + a_{22}A_{11}^{-1}\big(a_{12}^Ta_{11}^{-T}a_{21}^T - a_{22}^T\big) - \alpha_{21}a_{11}^{-T}a_{21}^T \\
&+ b_{21}D_{u_1}(a_{11}^{-T}a_{21}^T)\bigg(R_s\big(a_{11} + b_{11}(D_{u_1}(a_{11}^{-T}a_{21}^T)\cdot)z_2R_s\big)^{-1}\big(b_{12} - b_{11}a_{11}^{-T}a_{21}^T\big)z_{2,x} \\
&- R_s\big(a_{11} + b_{11}(D_{u_1}(a_{11}^{-T}a_{21}^T)\cdot)z_2R_s\big)^{-1}\Big[\big(a_{12} - b_{11}(D_{u_1}(a_{11}^{-T}a_{21}^T)\cdot)z_2\big)R_nA_{11}^{-1}\big(a_{12}^Ta_{11}^{-T}a_{21}^T - a_{22}\big) \\
&- \big(\alpha_{11} - b_{11}D_{u_2}(a_{11}^{-T}a_{21}^T)z_2\big)R_{\mathrm{Kaw}}a_{11}^{-T}a_{21}^T + \big(\alpha_{12} - b_{11}(D_{u_{22}}(a_{11}^{-T}a_{21}^T)\cdot)z_2\big)R_{\mathrm{res}}\Big]z_2 \\
&+ R_nA_{11}^{-1}\big(a_{12}^Ta_{11}^{-T}a_{21}^T - a_{22}^T\big)z_2\bigg) + b_{21}D_{u_1}(a_{11}^{-T}a_{21}^T)(R_{\mathrm{res}} - R_{\mathrm{Kaw}}a_{11}^{-T}a_{21}^T)z_2\bigg\}z_2.
\end{aligned}
\tag{3.37}
$$

We now recall a result of [3], Lemma 4.3, on the above system, which assures that it is a standard ODE:

**Lemma 3.3.** *The matrix*

$$
\Big[b_{22} - b_{21}a_{11}^{-T}a_{21}^T - a_{21}\big(a_{11} + b_{11}(D_{u_1}(a_{11}^{-T}a_{21}^T)\cdot)z_2R_s\big)^{-1}\big(b_{12} - b_{11}a_{11}^{-T}a_{21}^T\big)\Big]
\tag{3.38}
$$

*is invertible for $|z_2|$ sufficiently small.*

It follows that (3.35), (3.36), (3.37) and $\sigma_x = 0$ define a system of ODE, and condition 3) of page 9 requires that

**Condition M.** *The singular surface $\eta_{n_{11}+1}(u, \sigma) = 0$ is invariant for the system* (3.35), (3.37) *and* $\sigma_x = 0$.

If we assume that the singular surface is invariant for the fast dynamics, then the relation of invariance can be written as

$$
D\eta_{n_{11}+i}\begin{pmatrix} 0 \\ R_nA_{11}^{-1}(a_{12}^Ta_{11}^{-T}a_{21}^T - a_{22}^T) \\ R_{\mathrm{res}} - R_{\mathrm{Kaw}}a_{11}^{-T}a_{21}^T \end{pmatrix} = 0,
\tag{3.39}
$$

where we used the freedom in choosing the direction of $z_2$.

In the following we denote with $\Theta_i$ the eigenvectors of

$$
(A(0,0) - \mu_iB(0))\Theta_i = 0, \qquad \mu_i < 0.
\tag{3.40}
$$

As it is shown in [3], the numbers of $\Theta_i$ is equal to $k - 1 - n_{11} - q$. Moreover the stable manifold of the slow manifold can be parameterized by the coordinates on $\mathrm{span}\{\Theta_i, i = n_{11} + q, \dots, k-1\}$.

*Remark* 3.4. We note that the invariance of $\{\eta_{n_{11}+1}(u) = \sigma\}$ is not necessary for the existence of a smooth travelling waves solution, as implied by Proposition 2.4. In particular it follows that, for small travelling profiles, cases like example [3], Example 2.2, cannot occur. However, for large travelling profiles and for boundary profiles with the fast dynamics component, this is necessary.

## 4. Construction of the boundary Riemann solver

We split this section into three parts: the first part recalls just the basic idea concerning the construction of the admissible $i$-th curves $s \mapsto \mathcal{T}_i^s u$ for the hyperbolic system

$$
E(u)u_t + A(u)u_x = 0.
\tag{4.1}
$$

We recall that $u_1 = \mathcal{T}_i^s u_0$ means that $u_0$ can be connected to $u_1$ with shocks, contact discontinuities or rarefactions of the $i$-th family, and the total variation of all these waves is $s$. The form of the viscosity enters in selecting the admissible shocks. Basic references are [2, 3].

The second part considers the construction of the stable manifold of an equilibrium state $\bar{u}$. The dimension of this manifold depends on the relative position of $\bar{u}$ and the singular surface, i.e. the sign of $\eta_{n_{11}+i}(\bar{u}) - \sigma$, $i = 1, \dots, q$ (we recall that all the $\eta_{n_{11}+i}$ have the same value). For simplicity, in this analysis we will not consider the case of characteristic boundary: one can apply the construction of ??cite??cite?? to generalize these results to the boundary characteristic case.

The last part concerns the resolution of the boundary Riemann problem. The first observation is that the boundary profile or the travelling wave with the same speed of the boundary do not cross the singular surface: this means that the crossing may occurs only for travelling waves which have a speed different form $\sigma$. Thus the ODE describing them is clearly not singular. The second key observation is that the dimension of the stable manifold changes exactly by $q$ on the singular surface, and the dimension of the boundary data changes again by $q$. Moreover the characteristic directions are the same in both cases, i.e. the subspace generated by the eigenvectors

$$\begin{pmatrix} \zeta_{n_{11}+1} \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \zeta_{n_{11}+q} \\ 0 \end{pmatrix},$$

so that it follows that the boundary Riemann solver depends smoothly also during the transition.

4.1. **Construction of the $i$-th admissible curves.** Once the center manifolds are constructed, the generations of the admissible curve of the $i$-th family is an application of the contraction mapping principle [2], once the following conditions are verified.

(1) The reduced ODE on the center manifold around the equilibrium $\{u = \bar{u}, u_x = 0, \sigma = \lambda_i(\bar{u})\}$ takes the form

$$\begin{cases} u_x &= v_i \tilde{r}_i(u, v_i, \sigma) \\ v_{i,x} &= \tilde{\lambda}_i(u, v_i, \sigma) v_i \\ \sigma &= 0 \end{cases} \tag{4.2}$$

with $u \in \mathbb{R}^n$, $v_i, \sigma \in \mathbb{R}$.

(2) $\tilde{r}_i(u, 0, \lambda_i(u)) = r_i(u)$, the $i$-th eigenvalue of $E^{-1}(u)A(u)$.

(3) The following stability condition holds:

$$\frac{\partial \tilde{\lambda}_i}{\partial \sigma} < 0. \tag{4.3}$$

This last condition is related to the well posedness of system (3.1) forward in time.

In our case, the center manifold of the singular ODE $(A - \sigma E)u_x = B u_{xx}$ corresponds to the center manifold of the slow dynamics: we can thus follows the analysis of [3], Section 4.2.1, to verify that these conditions are verified on the center manifold defined for the reduced ODE on the slow manifold. Thus the construction of the admissible $i$-th curve $\mathcal{T}_i$ follows the analysis of [2].

The idea is to rewrite the above ODE into the integral map

$$\begin{cases} u(s) &= \bar{u}_i + \int_0^s \tilde{r}_i(u(\tau), v_i(\tau), \sigma(\tau)) d\tau \\ v_i(s) &= \mathrm{conc}_{[0,s_i]} f_i(\tau) - f_i(\tau) \\ \sigma_i(s) &= \frac{d}{d\tau} \mathrm{conc}_{[0,s_i]} f_i(\tau) \end{cases} \tag{4.4}$$

where

$$f_i(s) = \int_0^s \tilde{\lambda}_i(u(\tau), v_i(\tau), \sigma(\tau)) d\tau. \tag{4.5}$$

The notation $\mathrm{conc}_{[a,b]} f(\tau)$ means the concave envelope of $f$ in the interval $[a, b]$ evaluated at $\tau$. The map (4.4) is equivalent to the ODE (4.2) for travelling waves, i.e. where $v_i(s) > 0$, thanks to the transformation

$$x = \int^{s_i} v_i(\tau) d\tau. \tag{4.6}$$

but solution of this map can be also rarefactions or contact discontinuities.

In [2] it is shown that the map (4.4) is a contraction is the space of Lipschitz curves from $[0, s_i]$ to $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$: the intervals of $[0, s_i]$ where $v_i \neq 0$ corresponds to travelling waves, the remaining parts to rarefactions or contact discontinuities. Moreover its solution depends Lipschitz continuously on the initial data $\bar{u}_i$.

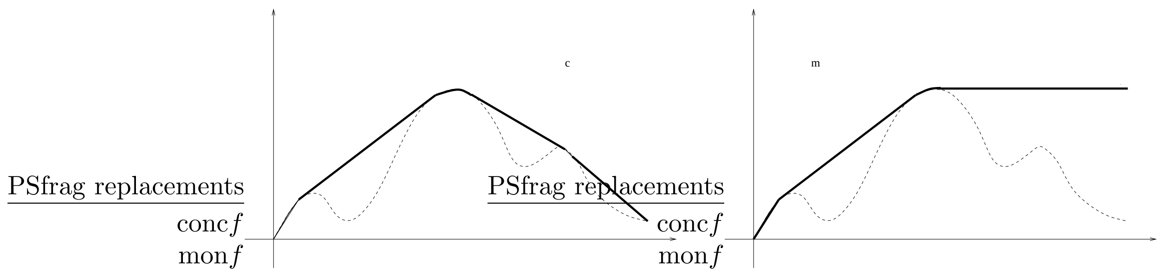



FIGURE 1. The concave and the monotone concave envelope of a given function $f$

It follows by definition that the final point $u(s_i)$ belongs to the admissible curve $\mathcal{T}_i^{s_i}\bar{u}_i$. In [2] it is proved that the set $u(s_i)$ defines a Lipschitz curve, parameterized by $s_i$. The curve can be prolonged on both sides of $\bar{u}_i$ either by taking the convex envelope for $s$ negative or just inverting the direction of the generalized eigenvector $\tilde{r}_i(u)$.

The precise result is the following proposition:

**Proposition 4.1.** *For the hyperbolic system* (4.1)*, limit under the rescaling* $(t,x) \mapsto (\epsilon t, \epsilon x)$ *of the parabolic system* (3.1) *satisfying assumptions 1),2),3),4),5) of pages 18, 18, there exists only one admissible curve* $\mathcal{T}_i^s\bar{u}_i$*,* $i = 1,\dots,N$*, for all* $s$*,* $\bar{u}_i$ *sufficiently small. This curve is Lipschitz continuous w.r.t.* $\bar{u}_i$*,* $s$ *and satisfies the estimate*

$$\mathcal{T}_i^0\bar{u}_i = \bar{u}_i, \qquad \left|\frac{d}{ds}\mathcal{T}_i^s\bar{u}_i - r_i(u)\right| \leq Cs. \tag{4.7}$$

*In particular* $\mathcal{T}_i^s u$ *is differentiable at* $s = 0$ *with derivative* $r_i(u)$*.*

We observe that for the construction of these curve only the existence of the slow manifold is necessary, as noticed in Remark 3.4. We finally observe that to know the exact composition of waves connection $\bar{u}_i$ to $\mathcal{T}_i^{s_i}\bar{u}_i$ one has to solve (4.4).

4.2. **Construction of the Boundary Riemann Solver.** We assume that the $k$-th characteristic field of (4.1) has the same speed of the boundary, i.e. $\lambda_k(0) = 0$. The idea is to contruct the map $\mathcal{U} : \mathbb{R}^n \times \mathbb{R}^n \ni (s_1,\dots,s_n) \times u_0 \mapsto \mathbb{R}^n$, where:

(1) $u_0$ is the initial data;
(2) $s_i$, $i = k+1,\dots,N$, is the length of the $i$-th admissible curve;
(3) $s_k$ is the length of the $k$-th admissible curve and characteristic part of the boundary profile;
(4) $(s_{\bar{i}},\dots,s_{k-1})$ is the parameterization of the non characteristic part of the boundary profile.

The value $\bar{i}$ can be $n_{11}$ or $n_{11}+q$, depending if the boundary profile is in the region where $\eta_{n_{11}+1} > 0$ or $\eta_{n_{11}+1} \leq 0$.

Using the admissible curves $\mathcal{T}_{i+1}^{s_i}$, $i = k+1,\dots,N$, starting from $u_0$, we reach a point $\bar{u}_k$. From $\bar{u}_k$, we cannot use the map (4.4), because it is possible that the speed $\sigma$, derivative of the concave envelope) becomes negative. We thus replace the concave envelope in $[0,s_k]$ with the monotone concave envelope (see figure 1:

$$\text{mon}_{[0,s_k]}f = \inf\left\{g : g \geq f, g \text{ concave}, \ g' \geq 0\right\}. \tag{4.8}$$

The map is the same as for the admissible curves: we only replace the concave envelope with the monotone concave envelope,

$$\left\{\begin{array}{rcl} u(s) & = & \bar{u}_k + \int_0^s \tilde{r}_k(u(\tau), v_k(\tau), \sigma(\tau))d\tau \\ v_k(s) & = & \text{mon}_{[0,s_k]}f_k(\tau) - f_k(\tau) \\ \sigma_k(s) & = & \frac{d}{d\tau}\text{mon}_{[0,s_k]}f_k(\tau) \end{array}\right. \tag{4.9}$$

where as before

$$f_k(s) = \int_0^s \tilde{\lambda}_k(u(\tau), v_k(\tau), \sigma(\tau))d\tau. \tag{4.10}$$

In [1, 3] it is shown that the above map is again a contraction in the space of Lipschitz curves. The fixed point is a curve whose structure is as follows:

(1) let $\bar{s}_k := \min\{s : \sigma_k(s) = 0\}$, and set $\hat{u}_k = u(\bar{s}_k)$;
(2) let $\underline{s}_k = \max\{s : \sigma_k(s) = 0, v_k(s) = 0\}$, and set $\underline{u}_k = u(\underline{s}_k)$.

The point $\hat{u}_k$ is the last point which can be connected to $\bar{u}_k$ by waves of the $k$-th family with strictly positive speed, while the point $\underline{u}_k$ is the last point which can be connected to $\bar{u}_k$ with waves of the $k$-th family with speed $\sigma_k(s) \geq 0$. The interval $[\underline{s}_k, s_k]$ corresponds to a characteristic boundary profile: since $v_k(s) > 0$ in $(\underline{s}_k, s_k]$, we can consider $u$ as a function of $x$ thanks to (4.6), and in general the decay of $u$ to $\underline{u}_k$ as $x \to +\infty$ is not exponential. We remark that the points $\bar{s}_k$, $\underline{s}_k$ depends on the total length $s_k$ and the initial point $\bar{u}_k$.

If $(s_{\bar{i}}, \dots, s_{k-1}) = 0$, then the contruction would be complete. In the general case, we have to add the non characteristic part of the boundary profile: this is the uniformly stable manifold of (3.19). Depending on the final point $\underline{u}_k$, the non characteristic part may containg the singular dynamics or not: this depends on the sign of $\eta_{n_{11}+1}(\underline{u}_k)$. More precisely:

(1) if $\eta_{n_{11}+1}(\underline{u}_k) > 0$, then the dimension of the non characteristic part is $k - 1 - n_{11}$;
(2) if $\eta_{n_{11}+1}(\underline{u}_k) < 0$, then the dimension of the non characteristic part is $k - 1 - n_{11} - q$.

We observe that, under Conditions L, M of pages 21, 23, the dimension of the non characteristic part cannot change on the boundary profiles: this is a consequence of the invariance of the singular surface $\{\eta_{n_{11}+1}(u) = 0\}$. The uniqueness of the characteristic part of the boundary profile shows also that the sign of $\eta_{n_{11}+1}(u)$ is constant along the characteristic part of the boundary profile.

The case where the singular dynamics is not present is studied in [1, 3]: here we will consider the case where $\eta_{n_{11}+1} < 0$. An important estimate is the estimate of the ratio between $\eta_{n_{11}+1}(\underline{u}_k)$ and $\eta_{n_{11}+1}$ along the profile: remember that in general this ratio is infinite, due to Remark 2.7. We estimate a similar ration among $\eta_{n_{11}+1}(\hat{u})$ and $\eta_{n_{11}+1}(\underline{u}_k)$.

**Lemma 4.2.** *If* $(u_k, v_k)$ *is the solution to* (4.9)*, then*

$$\big|\eta_{n_{11}+1}(u_k(s)) - \eta_{n_{11}+1}(\hat{u}_k)\big| \leq Cs\eta_{n_{11}+1}(\hat{u}_k), \qquad s \in [\bar{s}_k, \underline{s}_k]. \tag{4.11}$$

*Proof.* The main ingredient is that, due to the particular structure of the system (3.19), in the derivative of $\eta_{n_{11}+1}$ we gain the term $v_k$: by using the integral formulation (4.9),

$$\frac{d}{ds}\eta_{n_{11}+1}(u(s)) = D\eta_{n_{11}+1}\hat{r}_k(s) = \mathcal{O}(1)\eta_{n_{11}+1}(s),$$

where we have used the non degeneracy assumption $D\eta_{n_{11}+1} \neq 0$. It follows that

$$\big|\eta_{n_{11}+1}(u_k(s)) - \eta_{n_{11}+1}(\hat{u}_k)\big| \leq |\eta_{n_{11}+1}(\hat{u}_k)|(e^{Cs} - 1),$$

from which (4.11) follows. □

4.2.1. *Construction of the characteristic boundary profile.* The system for the boundary profile is

$$\begin{cases} u_x &= v \\ Bv_x &= (A(u, u_x) - \sigma E(u))v \\ \sigma_x &= 0 \end{cases} \tag{4.12}$$

where we use the reduction of Section 3.2 to obtain an explicit ODE. The speed $\sigma$ is the speed of the boundary: we will set it equal to 0.

We now use the three manifolds:

(1) By using also the rescaling $x \mapsto x/\eta_{n_{11}+1}$, we can find the $N+q$ dimensional manifold of singular dynamics $\mathcal{M}^{ss}$ and the $N + k - n_{11}$ manifold of slow dynamics $\mathcal{M}^{ns}$: the dynamics of the fast variable can be written as

$$\begin{cases} u_x &= R_{ss}(u, v_{ss})v_{ss} \\ v_{ss,x} &= \frac{1}{\eta n_{11}+1(u)}A_{ss}(u)v_{ss} \end{cases} \qquad v_{ss} \in \mathbb{R}^q. \tag{4.13}$$

(2) The uniformly stable manifold $\mathcal{M}^s$, generated by all orbits which converge to some equilibrium with uniform exponential speed. This $N+k-n_{11}-1$ dimensional manifold contains also the singular dynamics, and the ODE on it can be written as

$$\left\{ \begin{array}{rcl} u_x & = & R_s(u, v_s) v_s \\ v_{s,x} & = & A_s(u, v_s) v_s \end{array} \right. \tag{4.14}$$

with $A_s$ $(k-n_{11}-1)\times(k-n_{11}-1)$ dimensional strictly negative matrix, and $R_s$ is a $N\times(k-n_{11}-1)$ dimensional matrix with maximal rank. Its existence is given by Theorem 2.6. We recall also that on this manifold the solution may be splitted into a fast part and a slow part, as shown in (2.14).

(3) The $N+1$ dimensional center manifold $\mathcal{M}^c$, on which the ODE can be written as

$$\left\{ \begin{array}{rcl} u_x & = & v_k \tilde{r}_k(u, v_k) \\ v_{k,x} & = & \tilde{\lambda}_k(u, v_k) v_k \end{array} \right. \tag{4.15}$$

Its existence follows from Theorem 2.5.

Since these manifolds are invariant, it follows that the ODE on the $N+k-n_{11}$ dimensional center stable manifold can be written as

$$\left\{ \begin{array}{rcl} u_x & = & \hat{R}_{ss}(u, v_{ss}, v_{ns}, v_k) v_{ss} + \hat{R}_s(u, v_{ss}, v_{ns}, v_k) v_{ns} + v_k \hat{r}_k(u, v_{ss}, v_{ns}, v_k) \\ v_{ss,x} & = & \frac{1}{\eta_{n_{11}+1}(u)} \hat{A}_{ss}(u, v_{ss}, v_{ns}, v_k) v_{ss} \\ v_{ns,x} & = & \hat{A}_{ns}(u, v_{ss}, v_{ns}, v_k) v_{ns} + B_{ns}(u, v_{ss}, v_{ns}, v_k) v_{ss} v_k \\ v_{k,x} & = & \hat{\lambda}_k(u, v_{ss}, v_{ns}, v_k) v_k \end{array} \right. \tag{4.16}$$

whit the relations:

$$\begin{array}{lll} \hat{R}_{ss}(u, v_{ss}, 0, 0) = R_{ss}(u, v_{ss}), & \hat{R}_s(u, 0, v_s, 0) = R_s(u, v_s), & \hat{r}_k(u, 0, 0, v_k) = \tilde{r}_k(u, v_k, 0) \\ \hat{A}_{ss}(u, v_{ss}, 0, 0) = A_{ss}(u, v_{ss}), & \hat{A}_s(u, 0, v_s, 0) = A_s(u, v_s), & \hat{\lambda}_k(u, 0, 0, v_k) = \tilde{\lambda}_k(u, v_k, 0) \end{array} \tag{4.17}$$

In fact, if $v_{ss}(0) = 0$ then by the invariance of the slow manifold $v_{ss} \equiv 0$, and if $v_k(0) = 0$ then by the invariance of the uniformly stable manifold $v_k \equiv 0$. The only part which cannot be diagonalized is $v_s$.

We solve it by splitting into the three parts:

(1) The characteristic part:

$$\left\{ \begin{array}{rcl} u_k(s) & = & \underline{u}_k + \int_{\underline{s}_k}^s \hat{r}_k(u_k(\tau) + u_{ns}(\tau) + u_{ss}(\tau), v_{ss}(\tau), v_{ns}(\tau), v_k(\tau)) d\tau \\ v_k(s) & = & -f_k(\tau) \end{array} \right. \tag{4.18}$$

with

$$f_k(s) = \int_{\underline{s}_k}^s \hat{\lambda}_k(u_k(\tau) + u_{ns}(\tau) + u_{ss}, v_{ss}(\tau), v_{ns}(\tau), v_k(\tau)) d\tau. \tag{4.19}$$

The value $\underline{u}_k$ is obtained by point 2) of page 26, so that the change of variable

$$s_k = \underline{s}_k + \int_x^{+\infty} v_k(y) dy \tag{4.20}$$

is invertible in $(\underline{s}_k, s_k]$, at least for $u_s = 0$, $v_s = 0$. In particular the speed $\sigma$ is constant.

(2) uniformly stable non singular part:

$$\left\{ \begin{array}{rcl} u_{ns,x} & = & \hat{R}_{ns}(u_k + u_{ns} + u_{ss}, v_{ss}, v_{ns}, v_k) v_s \\ v_{ns,x} & = & \hat{A}_{ns}(u_k + u_{ns} + u_{ss}, v_{ss}, v_{ns}, v_k) v_s + B_{ns}(u_k + u_{ns} + u_{ss}, v_{ss}, v_{ns}, v_k) v_{ss} v_k \end{array} \right. \tag{4.21}$$

with initial data $\bar{v}_{ns}$;

(3) uniformly stable singular part:

$$\left\{ \begin{array}{rcl} u_{ss,x} & = & \hat{R}_{ss}(u_k + u_{ns} + u_{ss}, v_{ss}, v_{ns}, v_k) v_{ss} \\ v_{ss,x} & = & \hat{A}_{ss}(u_k + u_{ns} + u_{ss}, v_{ss}, v_{ns}, v_k) v_{ss} \end{array} \right. \tag{4.22}$$

with initial data $\bar{v}_{ss}$;

Our aim it to show that:

(1) it holds

$$\big| \eta_{n_{11}+1}(u_k + u_{ns} + u_{ss}) - \eta_{n_{11}+1}(\underline{u}_k) \big| \leq C \big( s_k - \underline{s}_k + |\bar{v}_{ns}| + |\bar{v}_{ss}| \big) \eta_{n_{11}+1}(\underline{u}_k); \tag{4.23}$$

(2) the change of variable (4.20) remains invertible if $v_s$ is exponentially decaying in $x$;
(3) the curve obtained by the first line of (4.18) can be joined to the $k$-th admissible curve to obtain again a Lipschitz curve satisfying Proposition 4.1;
(4) the solutions of system (4.21) converging to 0 define a manifold $\mathcal{M}^{ns}$ which can be parameterized by the non singular stable subspace of the linearized equation at $\underline{u}_k$ (hence at any $u$, due to the samllness assumptions, in particular by span$\{\Theta_i, i = n_{11}+q, \dots, k-1\}$). The dimension of this manifold is $k-1-n_{11}-q$. As $\eta_{n_{11}+1}(\underline{u}_k)$ converges to 0, the solution $(u_{ns}, v_{ns})$ converges to the solution of (4.16) when we assume $v_{ss} = 0$;
(5) the solutions of system (4.22) converging to 0 define a manifold $\mathcal{M}^{ss}$ which can be parameterized by the singular stable subspace of the linearized equation at $\underline{u}_k$ (hence at any $u$, due to the samllness assumptions, and also by span$\{Z_i, i = n_{11}+1, \dots, n_{11}+q\}$). The dimension of this manifold is $q$. As $\eta_{n_{11}+1}(\underline{u}_k)$ converges to 0, the $u$ component of the manifold $\mathcal{M}^{ss}$ converges to the manifold generated by the solutions to the ODE (3.27), (3.28), with as initial data the initial point of the solution of the non singular part.

4.2.2. *Construction of the solution to* (4.18). We prove the following proposition: for shortness we denote with $(u_s, v_s)$ the vector $((u_{ss}, u_{ns}), (v_{ss}, v_{ns}))$, and we set the distance

$$d\big((u_{s,1}, v_{s,1}), (u_{s,2}, v_{s,2})\big) = \sup_{x \in \mathbb{R}^+} \Big\{ e^{cx} \big|(u_{s,1}, v_{s,1}) - (u_{s,2}, v_{s,2})\big| \Big\}. \tag{4.24}$$

**Proposition 4.3.** *Let $u_s$, $v_s$ be exponentially decaying functions in $x$, with norm of order $\delta_0$. Assume that for $(u_s, v_s) = 0$ there exists a solution to* (4.18)*, where $f_k$, $s$ are given by* (4.19), (4.20)*, respectively, $0 \le s \le \delta_0$.*

*Then, there exists also a solution to* (4.18) *for $u_s, v_s \neq 0$ and moreover*

$$\big\|(u_{k,1}, v_{k,1}) - (u_{k,2}, v_{k,2})\big\|_{L^\infty} \le C\delta_0 d\big((u_{s,1}, v_{s,1}), (u_{s,2}, v_{s,2})\big). \tag{4.25}$$

*Finally, the curve $u_k(s)$ defined by the first line of* (4.18) *is a regular curve such that*

$$\left| \frac{du_k}{ds} - r_k(\underline{u}_k) \right| \le Cs, \qquad \big|\eta_{n_{11}+1}(u_k(s)) - \eta_{n_{11}+1}(\underline{u}_k)\big| \le Cs\eta_{n_{11}+1}\underline{u}_k). \tag{4.26}$$

The fact that if there is a solution for $u_s, v_s = 0$ then this solution survives is a consequence of the fact that if there is a trajectory $\gamma_0$ on the center manifold, then one can find the stable manifold of this trajectory, defined as the manifold generated by all trajectories converging exponentially to $\gamma_0$.

*Proof.* As in the previous proof, we consider the integral equations (4.18) as a integral map. By the change of variable (4.20), we can write

$$\begin{aligned}
&\left| \int_0^s \Big( \hat{r}_k(u_{s,1} + u_{k,1}, v_{s,1}, v_{k,1}) v_{k,1} - \hat{r}_k(u_{s,2} + u_{k,2}, v_{s,2}, v_{k,2}) v_{k,2} \Big) d\tau \right| \\
&\quad \le C \int_0^s |v_{k,1} - v_{k,2}| d\tau + C\delta_0 \int_0^s |u_{k,1} - u_{k,2}| d\tau \\
&\qquad + C\delta_0 d\big((u_{s,1}, v_{s,1}), (u_{s,2}, v_{s,2})\big) \int_0^s e^{-c \int_{s_k}^{\tau} \frac{d\varsigma}{-f_{k,1}(\varsigma)}} d\tau \\
&\quad = C \int_0^s |v_{k,1} - v_{k,2}| d\tau + C\delta_0 \int_0^s |u_{k,1} - u_{k,2}| d\tau \\
&\qquad + C\delta_0 d\big((u_{s,1}, v_{s,1}), (u_{s,2}, v_{s,2})\big) \int_0^s \lambda_{k,1}(\tau) e^{-c \int_{s_k}^{\tau} \frac{d\varsigma}{-f_{k,1}(\varsigma)}} d\tau \\
&\quad \le C \int_0^s |v_{k,1} - v_{k,2}| d\tau + C\delta_0 \int_0^s |u_{k,1} - u_{k,2}| d\tau + C\delta_0 d\big((u_{s,1}, v_{s,1}), (u_{s,2}, v_{s,2})\big) |f_{k,1}(s)|,
\end{aligned}$$

$$\begin{aligned}
&\bigg| \int_0^s \Big( \hat\lambda_k(u_{s,1}+u_{k,1},v_{s,1},v_{k,1}) - \hat\lambda_k(u_{s,2}+u_{k,2},v_{s,2},v_{k,2}) \Big) d\tau \bigg| \\
&\quad \le C \int_0^s \Big( |v_{k,1}-v_{k,2}| + |u_{k,1}-u_{k,2}| \Big) d\tau + Cd\big((u_{s,1},v_{s,1}),(u_{s,2},v_{s,2})\big) \int_0^s e^{-c\int_{s_k}^\tau \frac{d\varsigma}{-f_{k,1}(\varsigma)}} d\tau \\
&\quad \le C \int_0^s \Big( |v_{k,1}-v_{k,2}| + |u_{k,1}-u_{k,2}| \Big) d\tau + Cd\big((u_{s,1},v_{s,1}),(u_{s,2},v_{s,2})\big) |f_{k,1}(s)|.
\end{aligned}$$

We have used the apriori estimate $|f_k| \le \delta_0$.

Since $s \le \delta_0$, it follows that the above system define a contraction in $L^\infty((0,s_k),\mathbb{R}^{N+k-n_{11}-1})$, and the $L^\infty$ norms remains of order $\delta_0$. By taking as a special solution the functions $(u_k,v_k)$ obtained for $(u_s,v_s)=0$ and recalling that $\|(u_s,v_s)\|_{L^\infty} \le C\delta_0$, we recover the estimate

$$\big| f_k(s;u_s,v_s) - f_k(s;0,0) \big| \le C\delta_0 |f_k(s;0,0)|.$$

The first estimate (4.25) follows by noticing that $|f_k| \le \delta_0$, and (4.26) is a consequence of the fact that for $s=0$ one has $v_k=0$ and $(u_s,v_s)=0$, because $x=+\infty$. The second follows from the same computations of Lemma 4.2. □

4.2.3. *Construction of the solution to* (4.21). We prove the following proposition:

**Proposition 4.4.** *Let $(u_k,v_k)$, $e^{-cx/\eta_{n_{11}+1}}(u_{ss},v_{ss})$ be $L^\infty$ functions, with norm $\le C\delta_0$. Then for all $\bar v_s$ small, there exists a unique solution to*

$$(4.27)\qquad \left\{ \begin{array}{rcl} u_{ns,x} &=& \hat R_{ns}(u_k+u_{ns}+u_{ss},v_{ss},v_{ns},v_k) v_{ns} \\ v_{ns,x} &=& \hat A_{ns}(u_k+u_{ns}+u_{ss},v_{ss},v_{ns},v_k)v_{ns} + B_{ns}(u_k+u_{ns}+u_{ss},v_{ss},v_{ns},v_k)v_{ss}v_k \end{array}\right.$$

*such that $v_{ns}(0)=\bar v_{ns}$, converging to 0 as $x\to+\infty$.*

*The $k-1-n_{11}-q$ dimensional manifold $\mathcal{M}^{ns}(u_k,v_k,u_{ss},v_{ss})$ generated by these solutions can be parametrized by its projection on the eigenvectors of the stable manifold, in particular by span$\{\Theta_i, i=n_{11}+q+1,\dots,k-1\}$. Moreover we have the following estimate on the distance of the two orbits: if $(u_{k,1},v_{k,1})$, $(u_{k,2},v_{x,2})$, $e^{cx/\eta_{n_{11}+1}}(u_{ss,1},v_{ss,1})$, $e^{cx/\eta_{n_{11}+1}}(u_{ss,2},v_{ss,2})$ are $L^\infty$ functions, then*

$$\begin{aligned}
&\big|(u_{ns,1}(x),v_{ns,1}(x)) - (u_{ns,2}(x),v_{ns,2}(x))\big| \le Ce^{-cx}|\bar v_{ns,1}-\bar v_{ns,2}| + C\delta_0 e^{-cx}\big\|(u_{k,1},v_{k,1})-(u_{k,2},v_{k,2})\big\|_{L^\infty} \\
&\qquad\qquad + C\delta_0 e^{-cx}\eta_{n_{11}+1} \sup_{x\in\mathbb{R}^+} \Big\{ e^{cx/\eta_{n_{11}+1}} \big|(u_{ss,1},v_{ss,1})-(u_{ss,2},v_{ss,2})\big| \Big\}.
\end{aligned}\qquad (4.28)$$

*Finally, as $\eta_{n_{11}+1}\to 0$, the solution $(u_{ns},v_{ns})$ converges to the solution on the non singular stable manifold $\mathcal{M}^{ns}$ with the same initial data $\bar v_{ns}$.*

The constant $c$ is given by the estimate $|e^{A_{ns}x}| \le Ce^{-cx}$.

*Proof.* Let $\bar A_{ns} = \hat A_{ns}(0,0,0,0)$. We write the solution to (4.27) as the fixed point of the map

$$(4.29)\qquad \mathcal{T}(u_s,v_s) = \left\{ \begin{array}{l} -\int_x^{+\infty} \hat R_{ns}(0,0,0,0)e^{\bar A_{ns}y}\bar v_{ns} dy \\ \qquad -\int_x^{+\infty} \big(\hat R_{ns}(u_k+u_{ns}+u_{ss},v_{ss},v_{ns},v_k) - \hat R_{ns}(0,0,0,0)\big)v_{ns}(y)dy \\ e^{\bar A_{ns}x}\bar v_{ns} + \int_0^x e^{A_{ns}(x-y)}\big(\hat A_{ns}(u_k+u_{ns}+u_{ss},v_{ss},v_{ns},v_k) - \bar A_{ns}\big)v_{ns}(y)dy \\ \qquad + \int_0^x e^{A_{ns}(x-y)} B_{ns}(u_k+u_{ns}+u_{ss},v_{ss},v_{ns},v_k)v_{ss}v_k dy \end{array}\right.$$

We have the estimates: for $|u_{s,i}|, C|v_{s,2}| \le C^2\delta_0 e^{-cx}$, $i=1,2$,

$$\begin{aligned}
&\bigg| \int_x^{+\infty} \Big( \big(\hat R_{ns}(u_{k,1}+u_{ns,1}+u_{ss,1},v_{ns,1},v_{ss,1},v_{k,1}) - \hat R_{ns}(0,0,0,0)\big)v_{ns,1}(y) \\
&\qquad\qquad - \big(\hat R_{ns}(u_{k,2}+u_{ns,2}+u_{ss,2},v_{ns,2},v_{ss,2},v_{k,2}) - \hat R_{ns}(0,0,0,0)\big)v_{s,2}(y)\Big) dy \bigg| \\
&\quad \le C\int_x^{+\infty} |v_{s,1}-v_{s,2}|dy + C\delta_0 e^{-cx}\Big( \|u_{s,1}-u_{s,2}\|_{L^\infty} + \big\|(u_{k,1},v_{k,1})-(u_{k,2},v_{k,2})\big\|_{L^\infty} \Big) \\
&\qquad + C\delta_0 \eta_{n_{11}+1} e^{-cx/\eta_{n_{11}+1}} \sup_{x\in\mathbb{R}^+} \Big\{ e^{cx/\eta_{n_{11}+1}} \big|(u_{ss,1},v_{ss,1})-(u_{ss,2},v_{ss,2})\big| \Big\},
\end{aligned}$$

$$\bigg| \int_0^x \Big( e^{A_{ns}(x-y)} \big( \hat{A}_{ns}(u_{k,1}+u_{ns,1}+u_{ss,1}, v_{ns,1}, v_{ss,1}, v_{k,1}) - \bar{A}_{ns} \big) v_{s,1}(y)$$

$$- e^{A_{ns}(x-y)} \big( \hat{A}_{ns}(u_{k,2}+u_{ns,2}+uss,2, v_{ns,2}, v_{ss,2}, v_{k,2}) - \bar{A}_{ns} \big) v_{s,2}(y) \Big) dy \bigg|$$

$$\leq C\delta_0 \int_0^x e^{-c(x-y)} |v_{s,1}-v_{s,2}| dy + C\delta_0 e^{-cx} \Big( \|u_{s,1}-u_{s,2}\|_{L^\infty} + \big\|(u_{k,1},v_{s,1})-(u_{k,2},v_{s,2})\big\|_{L^\infty} \Big)$$

$$+ C\delta_0 \eta_{n_{11}+1} e^{-cx} \sup_{x\in\mathbb{R}^+} \Big\{ e^{cx/\eta_{n_{11}+1}} \big| (u_{ss,1},v_{ss,1}) - (u_{ss,2},v_{ss,2}) \big| \Big\}.$$

The $\eta_{n_{11}+1}$ appearing in front is beacuse

$$\int_0^x e^{-cy/\eta_{n_{11}+1}} dy = \mathcal{O}(\eta_{n_{11}+1}).$$

By taking as a particular solution $u_{ns}=0$, $v_{ns}=0$, we see that the metric space

$$(\mathcal{X},d) = \bigg\{ e^{cx}(u_{ns},v_{ns}) \in \mathrm{Lip}(\mathbb{R}^+, \mathbb{R}^{N+k-n_{11}-1}),$$

$$d\big((u_{s,1},v_{s,1}),(u_{s,2},v_{s,2})\big) = \sup_{x\in\mathbb{R}^+} \Big\{ e^{cx}\big(|u_{s,1}-u_{s,2}|+|v_{s,1}-v_{s,2}|\big) \Big\} \bigg\}$$

is invariant, and the map (4.29) is a contraction. Moreover it is easy to check that $(u_{ns}(x), v_{ns}(x))$ are of order $\bar{v}_{ns}e^{-cx}$.

Since we have

$$u_{ns}(0) = R_{ns}(0,0,0,0)\bar{A}_{ns}\bar{v}_{ns} + C\delta_0\bar{v}_{ns},$$

and $R_{ns}(0,0,0,0)$ has full rank, it follows that we can parametrized the manifold $\mathcal{M}^{us}$ by the coordinates of $u$ along span$\{R_{ns}(0,0,0,0)\}$. Note that by the decomposition, $R_{ns}(0,0,0,0)$ is the linear stable subspace of the ODE $Au_x = Bu_{xx}$.

The estimate (4.28) from the following well known estimate: if $\mathcal{T}_1$, $\mathcal{T}_2$ are contractions with contraction parameter $c\in[0,1)$, then the distance from the two fixrd points $x_1=\mathcal{T}_1x_1$, $x_2=\mathcal{T}_2x_2$ can be estimate by

$$d(x_1,x_2) \leq \frac{1}{1-c} \sup_{x\in\mathcal{X}} (\mathcal{T}_1x, \mathcal{T}_2x). \tag{4.30}$$

□

With exactly similar computations, we can prove the following proposition:

**Proposition 4.5.** *Let $(u_k,v_k)$, $e^{-cx}(u_{ns},v_{ns})$ be $L^\infty$ functions, with norm $\leq C\delta_0$. Then for all $\bar{v}_s$ small, there exists a unique solution to*

$$\begin{cases} u_{ns,x} = \hat{R}_{ss}(u_k+u_{ns}+u_{ss}, v_{ss}, v_{ns}, v_k)v_{ss} \\ v_{ns,x} = \hat{A}_{ss}(u_k+u_{ns}+u_{ss}, v_{ss}, v_{ns}, v_k)v_{ss} \end{cases} \tag{4.31}$$

*such that $v_{ss}(0)=\bar{v}_{ss}$, converging to $0$ as $x\to+\infty$.*

*The $q$ dimensional manifold $\mathcal{M}^{ss}(u_k,v_k,u_{ns},v_{ns})$ generated by these solutions can be parametrized by its projection on the eigenvectors of the stable manifold, in particular by span$\{Z_i, i=n_{11}+1,\dots,n_{11}+q\}$. Moreover we have the following estimate on the distance of the two orbits: if $(u_{k,1},v_{k,1})$, $(u_{k,2},v_{x,2})$, $e^{cx}(u_{ns,1},v_{ns,1})$, $e^{cx}(u_{ns,2},v_{ns,2})$ are $L^\infty$ functions, then*

$$\begin{aligned} \big|(u_{ss,1}(x),v_{ss,1}(x)) - (u_{ss,2}(x),v_{ss,2}(x))\big| &\leq Ce^{-cx/\eta_{n_{11}+1}}|\bar{v}_{ss,1}-\bar{v}_{ss,2}| \\ &+ C\delta_0\eta_{n_{11}+1}e^{-cx/\eta_{n_{11}+1}}\big\|(u_{k,1},v_{k,1})-(u_{k,2},v_{k,2})\big\|_{L^\infty} \\ &+ C\delta_0 e^{-cx/\eta_{n_{11}+1}}\eta_{n_{11}+1} \sup_{x\in\mathbb{R}^+}\Big\{e^{cx}\big|(u_{ns,1},v_{ns,1})-(u_{ns,2},v_{ns,2})\big|\Big\}. \end{aligned} \tag{4.32}$$

*Finally, as $\eta_{n_{11}+1}\to 0$, the solution $(u_{ss},v_{ss})$ converges to the (rescaled) solution on the singular stable manifold $\mathcal{M}^{ss}$ with the same initial data $\bar{v}_{ss}$.*

We can thus conclude with the following proposition, which gives the existence of the characteristic boundary profile, as well as the structure of the $k$-th characterstic curve.

**Proposition 4.6.** *Consider the ODE $(A(u) - \sigma E(u))u_x = Bu_{xx}$, and assume that the Conditions 1), 2), 3), 4) of page 18 and Condition 5) of page 18 are satisfied. Assume moreover that Condition L of page 21 and Condition M of page 23 are satisfied, and that the k-th eigenvalues of $A(u)$ is close to 0.*

*Then for all $\bar{u}_k$ close to 0, for all $(s_{n_{11}+1}, \dots, s_{k-1}) \in \mathbb{R}^{k-1-n_{11}-q}$ small, there exists a k-th Lipschitz curve $\mathcal{T}_k^{s_k}\bar{u}_k$, depending Lipschitz continuously on $\bar{u}_k$, such that:*

1. *the point $\mathcal{T}_k^{s_k}\bar{u}_k$ can be connected to $\bar{u}_k$ by rarefactions, contact discontinuities, admissible jumps and the chracteristic part of a boundary profile;*
2. *there exists two values $\bar{s}_k$, $\underline{s}_k$ such that*
   (a) *$\mathcal{T}_k^{\bar{s}_k}\bar{u}_k$ is connected to $\bar{u}_k$ with waves of strictly greater than 0;*
   (b) *$\mathcal{T}_k^{\underline{s}_k}\bar{u}_k$ is connected to $\mathcal{T}_k^{\bar{s}_k}\bar{u}_k$ with waves whose speed is exactly 0;*
   (c) *$\mathcal{T}_k^{s}\bar{u}_k$ is connected to $\mathcal{T}_k^{\underline{s}_k}\bar{u}_k$ with the characteristic part of the boundary profile.*
3. *$\mathcal{T}_k^{s}\bar{u}_k$ satisfies*

$$\mathcal{T}_k^{0}\bar{u}_k = \bar{u}_k, \qquad \left|\frac{d}{ds}\mathcal{T}_k^{s}\bar{u}_k - r_k(u)\right| \le Cs. \tag{4.33}$$

*Moreover, for all $(s_{n_{11}+1}, \dots, s_{k-1}) \in \mathbb{R}^{k-1-n_{11}-q}$ sufficiently small, there exist two Lipschitz maps $\mathcal{T}_{ss}^{(s_{n_{11}+1},\dots,s_{n_{11}+q})}$, $\mathcal{T}_{ns}^{(s_{n_{11}+q+1},\dots,s_{k-1})}$ such that:*

1. *$\mathcal{T}_{ss}^{(s_{n_{11}+1},\dots,s_{n_{11}+q})} \circ \mathcal{T}_{ns}^{(s_{n_{11}+q+1},\dots,s_{k-1})}\mathcal{T}_k^{s_k}\bar{u}_k$ can be connected to $\mathcal{T}_k^{\underline{s}_k}\bar{u}_k$ by a boundary profile;*
2. *the maps $\mathcal{T}_{ss}^{(s_{n_{11z}+1},\dots,s_{n_{11}+q})}$, $\mathcal{T}_{ns}^{(s_{n_{11}+q},\dots,s_{k-1})}$ depends Lipschitz continuously w.r.t. $(s_{n_{11}+1}, \dots, s_k)$;*
3. *as $\eta_{n_{11}+1}(\underline{u}_k) \to 0$, the map $\mathcal{T}_{ns}^{(s_{n_{11}+q},\dots,s_{k-1})}$ converges to the non singular stable manifold, while $\mathcal{T}_{ss}^{(s_{n_{11z}+1},\dots,s_{n_{11}+q})}$ converges to the singular one.*

*Proof.* The results concerning $\mathcal{T}_k$ when $(s_{n_{11}}, \dots, s_{k-1}) = 0$ follows from considerations similar to the case of the admissible curve, and are proved in [1, 3]. From $\underline{s}_k$, let $(u_k(s), u_s(s))$ be the first components of the solution to (4.21), (4.18): the existence and uniqueness of this solution follow from estimates (4.28), (4.25).

We define:

1. the map $\mathcal{T}_k^{s}\bar{u}_k$ as the end point $u_k(s_k)$ of the solution to (4.18);
2. the map $\mathcal{T}_s^{(s_{n_{11}},\dots,s_{k-1})}\mathcal{T}_k^{s_k}\bar{u}_k$ as the vector $u_s(s_{n_{11}}, \dots, s_{k-1}) + \mathcal{T}_k^{s_k}\bar{u}_k$, where $u_s(s_{n_{11}}, \dots, s_{k-1})$ is the end point of the solution to (4.27) with initial $\bar{v} = (s_{n_{11}}, \dots, s_{k-1})$.

The regularity of the map w.r.t. $(s_{n_{11}}, \dots, s_{k-1})$ is a consequence of the estimates (4.28), (4.25) and the fact that the maps (4.21), (4.18) define a contraction.

The Lipschitz continuity w.r.t. $s_k$ follows from the estimate

$$\int_0^s \left|(u_s, v_s)\left(-\int_{s_k}^{\tau} \frac{d\varsigma}{-f_k(\varsigma)}\right) - (u_s, v_s)\left(-\int_{s_k'}^{\tau} \frac{d\varsigma}{-f_k(\varsigma)}\right)\right| d\tau \le C\delta_0 s|s_k - s_k'|,$$

which describes the perturbation in (4.18) due to the variation of the extremals in the change of variable (4.20). The contraction of the maps (4.21), (4.18) yields that the same estimate is valid for the fixed point, i.e. the solution. □

Repeating the same analysis for the case when $\eta_{n_{11}}(\underline{u}_k) \le 0$, we obtain the following theorem, which gives the Riemann Solver maps.

**Theorem 4.7.** *For the hyperbolic system* (4.1) *there exists a unique map*

$$\mathbb{R}^{N-n_{11}} \times \mathbb{R}^N \ni ((s_{n_{11}}, \dots, s_N), u) \mapsto \left(\mathcal{T}_{ss}^{(s_{n_{11}+1},\dots,s_{n_{11}+q})} \circ \mathcal{T}_{ns}^{(s_{n_{11}+q+1},\dots,s_{k-1})} \circ \mathcal{T}_k^{s_k} \circ \mathcal{T}_k^{s_k} \circ \dots \circ \mathcal{T}_k^{s_k}\right)u, \tag{4.34}$$

*defined for $((s_{n_{11}}, \dots, s_N), u)$ close to 0, such that*

1. *$\mathcal{T}_i$, $i = k+1, \dots, N$ are the admissible curves defines in Proposition 4.1;*
2. *$\mathcal{T}_k$, $\mathcal{T}_{ss}^{(s_{n_{11}+1},\dots,s_{n_{11}+q})}$, $\mathcal{T}_{ns}^{(s_{n_{11}+q+1},\dots,s_{k-1})}$ are the maps defined in Proposition 4.6 if $\eta_{n_{11}+1}(\underline{u}_k) > 0$, where $\underline{u}_k$ is defined by Condition 2) of page 26. For $\eta_{n_{11}+1}(\underline{u}_k) < 0$ the map $\mathcal{T}_{ss}^{(s_{n_{11}+1},\dots,s_{n_{11}+q})}$ is not present;*
3. *The map is .....*

The last part mean that we loose control exactly on $q$ dimensions (fig. 2).

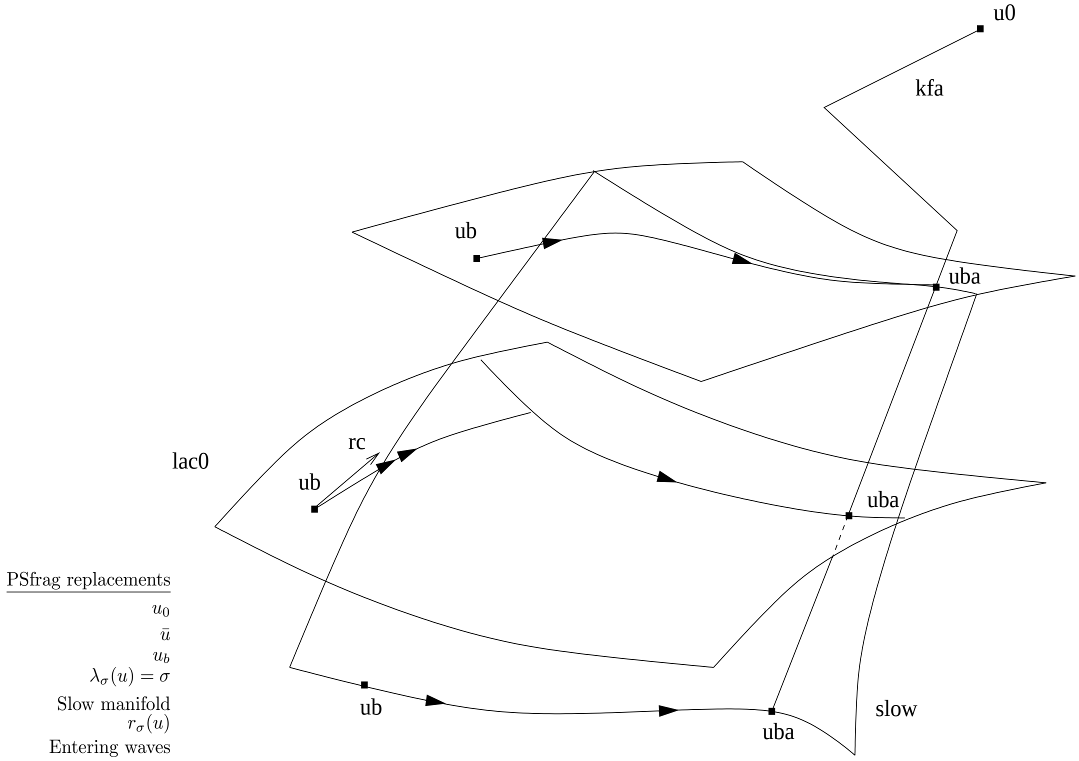



FIGURE 2. The reachable states of the boundary Riemann solver.

## 5. Resolution of the boundary Riemann problem

The boundary data are given in the following form. Let $\xi_\sigma(u), \dots, \xi_{N-r}(u)$ be the eigenvectors of $E_{11}^{-1}A_{11}$ corresponding to positive eigenvalues, and let $P_W(u)$ be the projection on the space

$$W(u) = \text{span}\left\{ \begin{pmatrix} \xi_\sigma(u) \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \xi_{N-r}(u) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ e_r \end{pmatrix} \right\}, \tag{5.1}$$

where $e_1, \dots, e_r$ is a base for $\mathbb{R}^r$. The boundary data for (3.1) are thus given by the equation

$$P_{W(u)} u(t, 0) = \bar{g}, \tag{5.2}$$

where $\bar{g}$ is a given vector in $\mathbb{R}^{n_+ + r}$ such that $|P_{W(u_0)}u_0 - \bar{g}| \ll 1$.

Due to the symmetry assumptions, there exists smooth unit eigenvectors $\eta_1(u), \dots, \eta_{N-r}(u)$, even if the eigenvalues are not separated ??cite??. Thus the boundary data can be given as

$$\{P_{W_2(u)} u = \bar{g}_2\} = \bigcap_i \{\langle \eta_i(u), u \rangle = g_i\}, \tag{5.3}$$

together with the critical part

$$\langle \eta_\sigma(u), u \rangle = g_1. \tag{5.4}$$

This last part is applied only when the end point has $\lambda_\sigma(u) > 0$.

Thus the dimension of the level sets of the function $P_{W(u)}u$ changes across the critical surface, and more precisely the intersection of $P_{W_2(u)}u = \bar{g}_2$ with $\{\lambda_\sigma(u) = 0\}$ is made by the integral curve of the vector field $(\xi_\sigma(u), 0)$ passing trough the intersection of $P_{W(u)}u = \bar{g}$ with $\{\lambda_\sigma(u) = 0\}$ (fig. 3). The invariance of the singular surface assures that the integral lines do not leave $\{\lambda_\sigma = 0\}$.

We first recall that the manifold generated by

$$(s_{n_{11}+q+1}, \dots, s_N), \bar{u} \mapsto \mathcal{T}_{ns}^{(s_{n_{11}+q+1}, \dots, s_{k-1}); s_k} \circ \mathcal{T}_k^{s_k} \circ \mathcal{T}_{k+1}^{s_{k+1}} \circ \dots \circ \mathcal{T}_N^{s_N} \bar{u}.$$

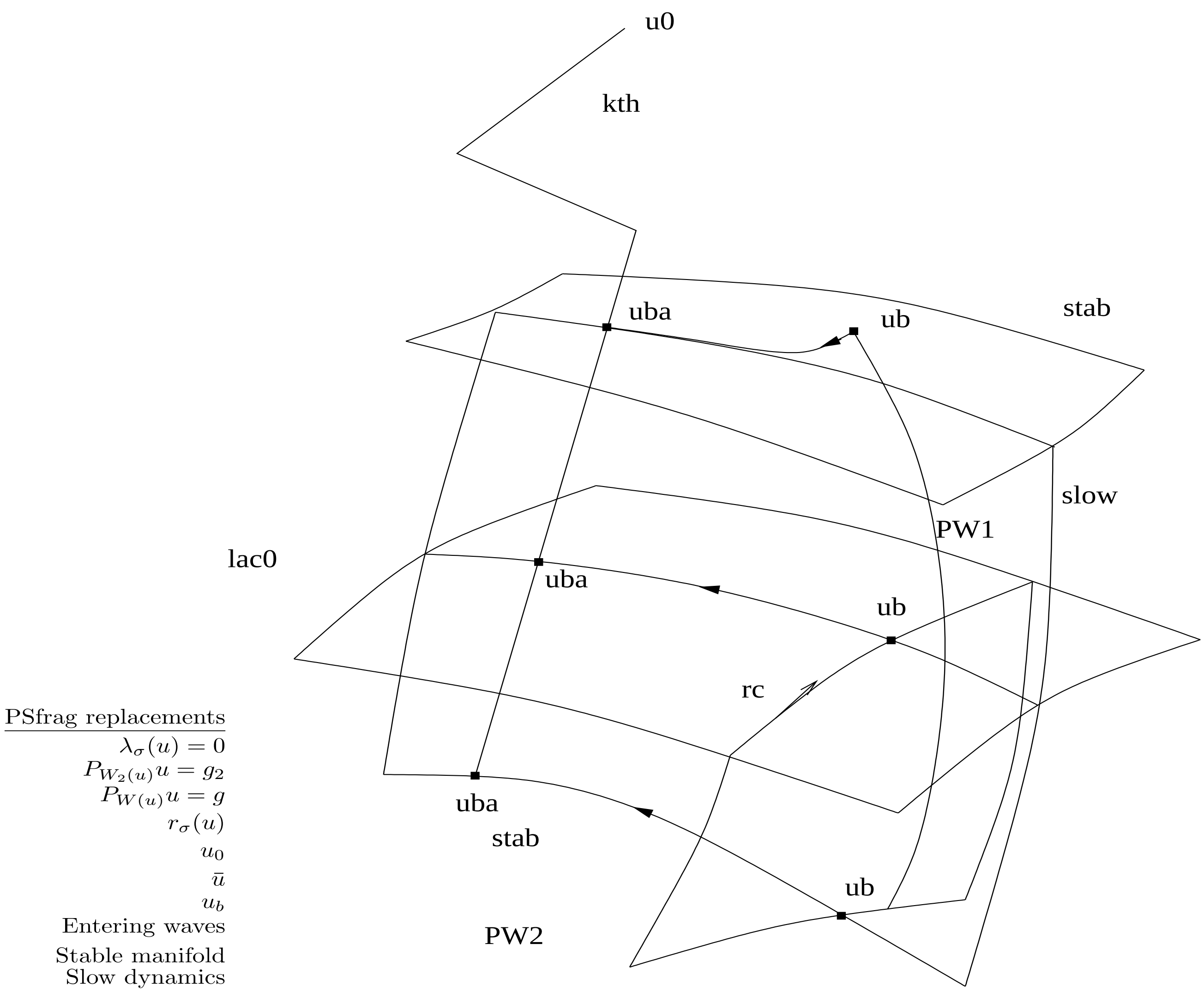



FIGURE 3. The change of dimension of the boundary data and the BRS.

is transversal to the boundary condition $P_{W_2}u = \bar{g}_2$, so that we can always invert the map to obtain the waves and the boundary profile. If $\eta_{n_{11}+1}(\underline{u}_k) \leq 0$ (or equivalently $\eta_{n_{11}+1}(\hat{u}_k) \leq 0$) then the solution is complete.

When $\eta_{n_{11}+1}(\underline{u}_k) > 0$, we need to use the singular part $\mathcal{T}_{ss}^{(s_{n_{11}+1},\dots,s_{n_{11}+q})}$ to match the additional conditions $P_{W_1}(u) = \bar{g}_1$.

If we are on the singular surface, then the dynamics is ompletely separated, and moreover since $u$ varies along the vectors $Z_i$ one has that the projection $P_{W_2}$ does not vary. In the geneal case, we have the estimates:

$$P_{W_2}u - P_{W_2}\bar{u} = \mathcal{O}(1)\eta_{n_{11}+1}(\underline{u}_k)\delta_0,$$

so that we see (by a contraction argument) that we remain again in $\eta_{n_{11}+1}(\underline{u}_k) > 0$. This completes the contruction.

## 6. Applications

In this section we consider two aplications: the construction of the BRS for Navier-Stokes and for the Magneto Hydro Dynamics equations in Eulerian coordinates.

6.1. **Navier-Stokes equations.** We consider the Navier-Stokes equations

$$\left\{\begin{array}{rcl} \rho_t + (\rho u)_x & = & 0 \\ (\rho u)_t + (\rho u^2 + P(\rho, e))_x & = & (\nu(\rho) u_x)_x \\ (\rho(e + u^2/2))_t + (\rho u e + \rho u^3/2 + uP(\rho, e))_x & = & (\nu(\rho) u u_x)_x + (\kappa(\rho)\theta_x)_x \end{array}\right. \tag{6.1}$$

with $\nu$ diffusion coefficient, $\kappa$ heat conductivity, and $e = R\theta/(\gamma-1)$ and $P = (\gamma-1)\rho e$, $\gamma > 1$. Writing the system in quasilinear form

$$\begin{pmatrix} \rho \\ u \\ e \end{pmatrix}_t + \begin{bmatrix} u & \rho & 0 \\ (\gamma-1)e/\rho & 2u - \nu'\rho_x & (\gamma-1) \\ 0 & (\gamma-1)e - \nu u_x/\rho & u - (\gamma-1)\kappa'\rho_x/(R\rho) \end{bmatrix} \begin{pmatrix} \rho \\ u \\ e \end{pmatrix}_x$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \nu/\rho & 0 \\ 0 & 0 & (\gamma-1)\kappa/(R\rho) \end{bmatrix} \begin{pmatrix} \rho \\ u \\ e \end{pmatrix}_{xx}$$

and taking the symmetrizer

$$\Sigma = \begin{bmatrix} (\gamma-1)e/\rho^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/e \end{bmatrix}, \tag{6.2}$$

we obtain the system

$$\begin{bmatrix} (\gamma-1)e/\rho^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/e \end{bmatrix} \begin{pmatrix} \rho \\ u \\ e \end{pmatrix}_t$$
$$+ \begin{bmatrix} (\gamma-1)eu/\rho^2 & (\gamma-1)e/\rho & 0 \\ (\gamma-1)e/\rho & 2u - \nu'\rho_x & (\gamma-1) \\ 0 & (\gamma-1) - \nu u_x/(\rho e) & u/e - (\gamma-1)\kappa'\rho_x/(R\rho e) \end{bmatrix} \begin{pmatrix} \rho \\ u \\ e \end{pmatrix}_x$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \nu/\rho & 0 \\ 0 & 0 & (\gamma-1)\kappa/(R\rho e) \end{bmatrix} \begin{pmatrix} \rho \\ u \\ e \end{pmatrix}_{xx} \tag{6.3}$$

It is standard to verify that the above systems satisfies the assumptions 1), 2), 3), 4) listed in page 18, and also condition 5) of page 18: in fact the hyperbolic part is one dimensional.

*Remark* 6.1. In this case the boundary is characteristic for the hyperbolic limit. Thus one should use the construction of the unifomly stable manifold given here together with the machinery used in ??cite??cite?? to construct the boundary layer, as said in Remark ??. However in our case the characteristic eigenvalues is linearly degenerate, so that the analysis is much easier.

The critical surface is $u = \sigma$, from which we obtain the singular system

$$\left\{\begin{array}{c} \rho_x = -\rho v_2, \quad u_x = 0, \quad e_x = 0 \\ v_{2,x} = -((\gamma-1)\rho e/\nu + (\nu'\rho/\nu)v_2)v_2 \\ v_{3,x} = -(\kappa'\rho v_2/\kappa)v_3 \end{array}\right. \tag{6.4}$$

It is clear from the above formula that the surface $\{u = \sigma\}$ is invariant. In particular $\lambda_c = u$ is linearly degenerate, as required.

The non singular system is obtained by taking $u = 0$ in (6.3),

$$\left\{\begin{array}{c} \rho_x = -(\rho/e), \quad u_x = 0, \quad e_x = v_3 \\ v_{3,x} = -(k'\rho v_3/(\kappa e))v_3 \end{array}\right. \tag{6.5}$$

Clearly the manifold $u = 0$ is again invariant.

The Riemann solver depends thus smoothly on the boundary data, also for the transition $u$ from positive to negative. In this particular case, the critical eigenvalue is assigned as a boundary data, so that the solution to the BRP can be explicited. We first consider the case $\sigma = 0$.

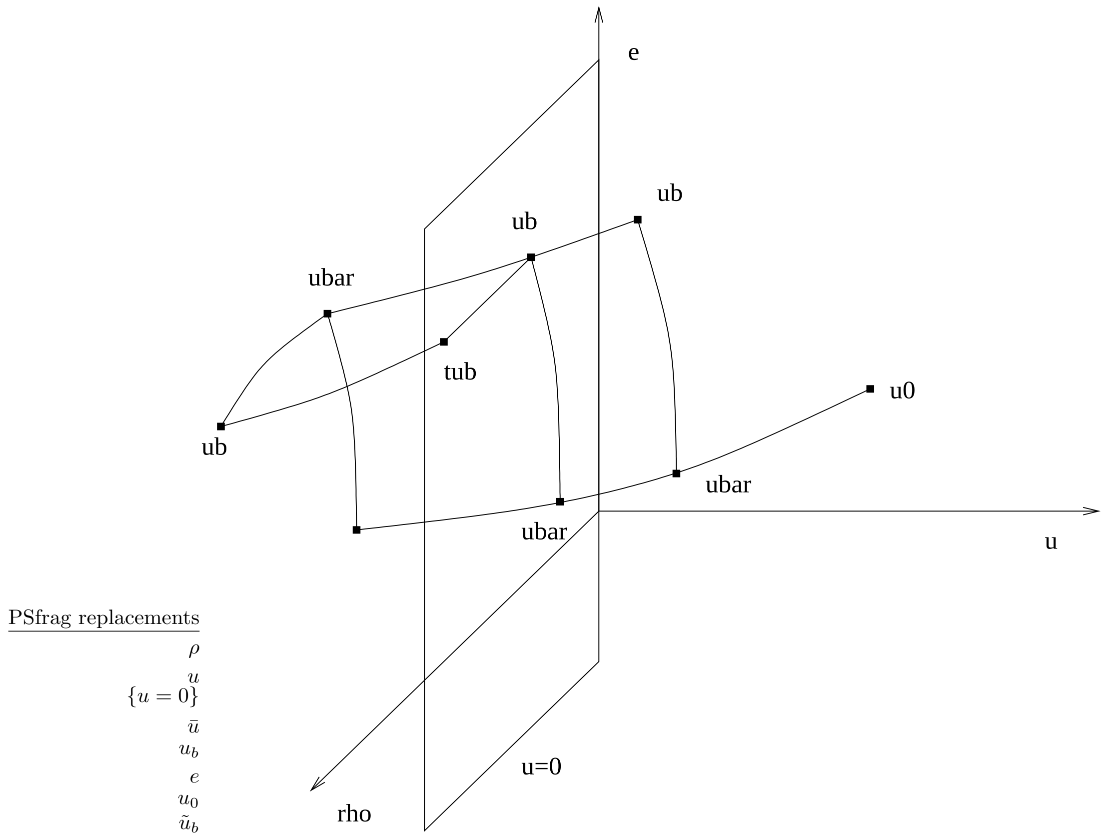



FIGURE 4. The BRS for NS.

(1) If $u > 0$, then the boundary data is the full vector $(\rho, u, e)$, so that for $u$ less than the sound speed the BRS is given by the entering sound wave (shock or rarefaction), the linearly degenerate characteristic field, and a boundary layer. As $u \to 0$ the boundary layer converges to a jump along $\rho$.
(2) If $u \leq 0$, then the boundary data is only $(u, e)$, and the BRS is the entering sound wave (shock or rarefaction) and the characteristic boundary layer, which in the case $u = 0$ degenerates into the linearly degenerate vector field with speed 0.

The connection of the to BRS is Lipschitz, since they coincide for $u = 0$: in fact, the limit of $u_b$ from $u < 0$ is the point $\tilde{u}_b$, which is connected to $u_b$ by a jump along the variable $\rho$, fig. 4.

6.2. **MHD equation.** The equations for MHD are

$$
\left\{
\begin{array}{ccc}
\rho_t + (\rho u)_x & = & 0 \\
(\rho u)_t + (\rho u^2 + P(\rho, e) + |b|^2/2)_x & = & (\nu u_x)_x \\
(\rho w)_t + (\rho u w - \beta b)_x & = & (\nu w_x)_x \\
b_t + (ub - \beta w)_x & = & (\eta b_x)_x \\
(\rho(e + (u^2 + |w|^2)/2) + |b|^2/2)_t & & \\
+(\rho u(e + (u^2 + |w|^2))/2 + u|b|^2)_x & & \\
+(P(\rho, e)u - \beta b \cdot w)_x & = & (\nu(u u_x + w^T w_x) + \kappa \theta_x + \eta b^T b_x)_x
\end{array}
\right.
\tag{6.6}
$$

where $(u, w) \in \mathbb{R} \times \mathbb{R}^2$ is the three dimensional speed, and $(\beta, b) \in \mathbb{R} \times \mathbb{R}^2$ is the magnetic field, with $\beta$ constant. Moreover

$$
P_\rho(\rho, e) > 0, \quad e = \frac{R\theta}{\gamma - 1}, \quad \gamma > 1,
\tag{6.7}
$$

and $\nu(\rho)$ is the viscosity, $\kappa(\rho)$ is the heat conductivity, $\eta$ is the conductivity.

The quasilinear form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ u & \rho & 0 & 0 & 0 \\ w & 0 & \rho\mathbb{I} & 0 & 0 \\ 0 & 0 & 0 & \mathbb{I} & 0 \\ e+(u^2+|w|^2)/2 & \rho u & \rho w^T & b^T & \rho \end{bmatrix} \begin{pmatrix} \rho \\ u \\ w \\ b \\ e \end{pmatrix}_t +$$

$$\begin{bmatrix} u & \rho & 0 & 0 & 0 \\ u^2+(\gamma-1)e & 2\rho u-\nu'\rho_x & 0 & b^T & (\gamma-1)\rho \\ uw & \rho w & (\rho u-\nu'\rho_x)\mathbb{I} & -\beta\mathbb{I} & 0 \\ 0 & b & -\beta\mathbb{I} & (u-\eta'\rho_x)\mathbb{I} & 0 \\ \begin{matrix} u(e+(u^2+|w|^2)/2) \\ +(\gamma-1)ue \end{matrix} & \begin{matrix} \rho e+3\rho u^2/2 \\ +\rho|w|^2/2+|b|^2 \\ +(\gamma-1)\rho e \\ -\nu'\rho_x u-\nu u_x \end{matrix} & \begin{matrix} \rho u w^T-\beta b^T \\ -\nu'\rho_x w^T-\nu w_x^T \end{matrix} & \begin{matrix} 2ub^T-\beta w^T \\ -\eta b_x^T \end{matrix} & \begin{matrix} \rho u+(\gamma-1)\rho u \\ +(\gamma-1)\kappa'\rho_x/R \end{matrix} \end{bmatrix} \begin{pmatrix} \rho \\ u \\ w \\ b \\ e \end{pmatrix}_x$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \nu & 0 & 0 & 0 \\ 0 & 0 & \nu\mathbb{I} & 0 & 0 \\ 0 & 0 & 0 & \eta\mathbb{I} & 0 \\ 0 & \nu u & \nu w^T & \eta b^T & (\gamma-1)\kappa/R \end{bmatrix} \begin{pmatrix} \rho \\ u \\ w \\ b \\ e \end{pmatrix}_{xx},$$

Considering the symmetrizer

$$\Sigma = \mathrm{diag}\Big((\gamma-1)e/\rho^2, 1, \mathbb{I}, \mathbb{I}/\rho, 1/e\Big), \tag{6.8}$$

the symmetric quasilinear form is

$$\begin{bmatrix} (\gamma-1)e/\rho^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{I} & 0 & 0 \\ 0 & 0 & 0 & \mathbb{I}/\rho & 0 \\ 0 & 0 & 0 & 0 & 1/e \end{bmatrix} \begin{pmatrix} \rho \\ u \\ w \\ b \\ e \end{pmatrix}_t$$

$$+ \begin{bmatrix} (\gamma-1)eu/\rho^2 & (\gamma-1)e/\rho & 0 & 0 & 0 \\ (\gamma-1)e/\rho & u-\nu'\rho_x/\rho & 0 & b^T/\rho & (\gamma-1) \\ 0 & 0 & (u+\nu'\rho_x/\rho)\mathbb{I} & -\beta\mathbb{I}/\rho & 0 \\ 0 & b/\rho & -\beta\mathbb{I}/\rho & (u/\rho+\eta'\rho_x/\rho)\mathbb{I} & 0 \\ 0 & (\gamma-1)-\nu u_x/(\rho e) & -\nu w_x^T/(\rho e) & -\eta b_x^T/(\rho e) & u/e+(\gamma-1)\kappa'\rho_x/(R\rho e) \end{bmatrix} \begin{pmatrix} \rho \\ u \\ w \\ b \\ e \end{pmatrix}_x$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \nu/\rho & 0 & 0 & 0 \\ 0 & 0 & \nu\mathbb{I}/\rho & 0 & 0 \\ 0 & 0 & 0 & \eta\mathbb{I}/\rho^2 & 0 \\ 0 & 0 & 0 & 0 & (\gamma-1)\kappa/(R\rho e) \end{bmatrix} \begin{pmatrix} \rho \\ u \\ w \\ b \\ e \end{pmatrix}_{xx} \tag{6.9}$$

Depending on $\eta > 0$ or $\eta = 0$, we have that the dimension of the hyperbolic part is 1 or 3, respectively. We consider the two cases separately.

6.2.1. $\eta > 0$. The equation of the singular dynamics are given by

$$\left\{ \begin{array}{c} \rho_x = -\rho v_2, \quad u_x = e_x = 0, \; w_x = b_x = 0 \\ v_{2,x} = -((\gamma-1)\rho e/\nu + (\nu'\rho/\nu)v_2)v_2 \\ v_{3,x} = -((\nu'\rho/\nu)v_2)v_3 \\ v_{4,x} = -((\eta'\rho/\eta)v_2)v_4 \\ v_{5,x} = -((\kappa'\rho/\kappa)v_2)v_5 \end{array} \right. \tag{6.10}$$

It is clear that the singular surface $\{u = 0\}$ is invariant, so that condition L is verified.

The equations for the slow dynamics are

$$
(6.11)\qquad \begin{cases} \rho_x = v_1, \quad u_x = 0, \quad w_x = v_3,\ b_x = v_4,\ e_x = v_5 \\ v_1 = -(b^T/((\gamma-1)e))v_4 - (\rho/e)v_5 \\ v_{3,x} = ((\nu'/\nu)v_1)v_3 - (\beta/\nu)v_4 \\ v_{4,x} = -(\beta\rho/\eta)v_3 + ((\eta'\rho/\eta)v_1)v_4 \\ v_{5,x} = -(\nu R/((\gamma-1)\kappa))|v_3|^2 - (\eta R/((\gamma-1)\kappa))|v_4|^2 \\ \quad +((R\rho u)/((\gamma-1)\kappa) + (\kappa'/\kappa)v_1)v_5 \end{cases}
$$

Again the singular surface is invariant, so that condition M is verified.

6.2.2. $\eta = 0$. In this case the fast dynamics correspond to the system

$$
(6.12)\qquad \begin{cases} \rho_x = -\rho v_2, \quad u_x = e_x = 0,\ w_x = 0, \quad b_x = -bv_2 + \beta v_3 \\ v_{2,x} = -((\gamma-1)\rho e/\nu + (\nu'\rho/\nu)v_2 + |b|^2/\nu)v_2 + (\beta b^T/\nu)v_3 \\ v_{3,x} = (\beta b/\nu)v_2 - ((\nu'\rho/\nu)v_2 + \beta^2/\nu)v_3 \\ v_{5,x} = -((\kappa'\rho/\kappa)v_2)v_5 \end{cases}
$$

In this case, the singular dynamics is 3 dimensional, and it is clear from the above system that $\{u = 0\}$ is invariant.

The slow dynamics is given by

$$
(6.13)\qquad \begin{cases} \rho_x = v_1, \quad u_x = 0,\ w_x = b_x = 0, \quad e_x = v_5 \\ v_1 = -(\rho/e)v_5 \\ v_{5,x} = ((R\rho u)/((\gamma-1)\kappa) + (\kappa'/\kappa)v_1)v_5 \end{cases}
$$

We observe the presence of 2 more linearly degenerate characteristic fields, corresponding to $w$.

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