SISSA preprint 17/2007/MATE

Global, finite energy, weak solutions for the NLS with rough, time-dependent magnetic potentials

Paolo Antonelli, Alessandro Michelangeli and Raffaele Scandone

Abstract. We prove the existence of weak solutions in the space of energy for a class of nonlinear Schrödinger equations in the presence of a external, rough, time-dependent magnetic potential. Under our assumptions, it is not possible to study the problem by means of usual arguments like resolvent techniques or Fourier integral operators, for example. We use a parabolic regularisation, and we solve the approximating Cauchy problem. This is achieved by obtaining suitable smoothing estimates for the dissipative evolution. The total mass and energy bounds allow to extend the solution globally in time. We then infer sufficient compactness properties in order to produce a global-in-time finite energy weak solution to our original problem.

Mathematics Subject Classification. 35D40, 35H30, 35Q41, 35Q55, 35K08.

Keywords. Nonlinear Schrödinger equation, Magnetic potentials, Viscosity regularisation, Strichartz estimates, Weak solutions.

1. Introduction and main result

In this work, we study the initial value problem associated with the nonlinear Schrödinger equation with magnetic potential

$$i\partial_t u = -(\nabla - \mathrm{i}\,A)^2 u + \mathcal{N}(u) \tag{1.1}$$

in the unknown $u \equiv u(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, where

$$\mathcal{N}(u) = \lambda_1 |u|^{\gamma-1}u + \lambda_2(|\cdot|^{-\alpha} * |u|^2)u \qquad \begin{array}{l} \gamma \in (1,5] \\ \alpha \in (0,3) \\ \lambda_1, \lambda_2 \geqslant 0 \end{array} \tag{1.2}$$

is a defocusing nonlinearity, both of local (pure power) and non-local (Hartree) type, and $A : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$ is the external time-dependent magnetic potential (The cases $\alpha = 0$ and $\gamma = 1$ would make $\mathcal{N}(u)$ a trivial linear term).

The novelty here will be the choice of A within a considerably larger class of rough potentials than what customarily considered in the literature so far—as a consequence, we will be in the condition to prove the existence of global-in-time weak solutions, without attacking for the moment the general issue of the global well-posedness.

Concerning the nonlinearity, in the regimes $\gamma \in (1,5)$ and $\alpha \in (0,3)$ we say that $\mathcal{N}(u)$ it is *energy sub-critical*, while for $\gamma = 5$ is *energy critical*. Given the defocusing character of the equation, it will not be restrictive henceforth to set $\lambda_1 = \lambda_2 = 1$, and in fact all our discussion applies also to the case when one of such couplings is set to zero.

Partially supported by the 2014–2017 MIUR-FIR Grant "Cond-Math: Condensed Matter and Mathematical Physics" code RBFR13WAET.

The relevance of Eq. (1.1) is hard to underestimate, both for the interest it deserves per se, given the variety of techniques that have been developed for its study, and for the applications in various contexts in physics. Among the latter, (1.1) is the typical effective evolution equation for the quantum dynamics of an interacting Bose gas subject to an external magnetic field, and as such, it can be derived in suitable scaling limits of infinitely many particles [31,37,39]: in this context, the $|u|^{\gamma-1}u$ term with $\gamma = 3$ (resp., $\gamma = 5$) arises as the self-interaction term due to a two-body (resp., three-body) inter-particle interaction of short scale, whereas the $(|\cdot|^{-\alpha} * |u|^2)u$ term accounts for a two-body interaction of mean field type, whence its non-local character. On the other hand, (1.1) arises also as an effective equation for the dynamics of quantum plasmas. Indeed, for densely charged plasmas, the pressure term in the degenerate (i.e. zero-temperature) electron gas is effectively given by a nonlinear function of the electron charge density [24], which in the wave-function dynamics corresponds to a power-type nonlinearity (see, for instance, [1] for more details).

In the absence of an external field ($A \equiv 0$), Eq. (1.1) has been studied extensively, and global well-posedness and scattering are well understood, both in the critical and in the sub-critical case [3,7,11,17, 20,22,29]. Such results are mainly based upon (variants of the) perturbation theory with respect to the linear dynamics, built on Strichartz estimates for the free Schrödinger propagator [21,25]. However, when $A \not\equiv 0$, the picture is much less developed.

The main mathematical difficulty is to obtain suitable dispersive and smoothing estimates for the *linear* magnetic evolution operator, in order to exploit a standard fixed point argument where the nonlinearity is treated as a perturbation.

For *smooth* magnetic potentials, local-in-time Strichartz estimates were established under suitable growth assumptions [33,43], based on the construction of the fundamental solution for the magnetic Schrödinger flow by means of the method of parametrices and time slicing a la Fujiwara [18], together with Kato's perturbation theory. If the potential has some Sobolev regularity and is sufficiently small, then Strichartz-type estimates were obtained [40] by studying the parametrix associated with the derivative Schrödinger equation $\partial_t u - i\Delta u + A \cdot \nabla u = 0$, exploiting the methods developed by Doi in [12,13]. Global well-posedness of (1.1) and stability results in the case of suitable smooth potentials are proved in [10,30,36].

As far as *non-smooth* magnetic potentials are concerned, magnetic Strichartz estimates are still available with a number of restrictions. When A is time independent, global-in-time magnetic Strichartz estimates were established by various authors under suitable spectral assumptions (absence of zero-energy resonances) on the magnetic Laplacian A [8,14,15], or alternatively under suitable smallness of the so-called non-trapping component of the magnetic field [9], up to the critical scaling $|A(x)| \sim |x|^{-1}$. Counterexamples at criticality are also known [16]. In the time-dependent case, magnetic Strichartz estimates are available only under suitable smallness condition of A [19,40].

Beyond the regime of Strichartz-controllable magnetic fields, very few is known, despite the extreme topicality of the problem in applications with potentials A that are rough, have strong singularities locally in space, and have a very mild decay at spatial infinity, virtually a L^∞-behaviour. This generic case can be actually covered, and global well-posedness for (1.1) was indeed established [32], by means of energy methods, as an alternative to the lack of magnetic Strichartz estimates. However, such an approach is only applicable to non-local nonlinearities with energy sub-critical potential (in the notation of (1.1): $\lambda_1 = 0$ and $\alpha \leqslant 2$), for it crucially relies on the fact that the nonlinearity is then locally Lipschitz in the energy space, power-type nonlinearities being instead way less regular and hence escaping this method. The same feature indeed allows to extend globally in time the well-posedness for the Maxwell–Schrödinger system in higher regularity spaces [35].

In this work, we are concerned precisely with the generic case where in (1.1) neither are the external magnetic fields Strichartz-controllable, nor can the nonlinearity be handled with energy methods.

The key idea is then to work out first the global well-posedness of an initial value problem in which an additional source of smoothing for the solution is introduced, as the one provided by the magnetic

Laplacian is not sufficient. In a recent work by the first author and collaborators [1], placed in the closely related setting of nonlinear Maxwell–Schrödinger systems, the regularisation was provided by Yosida's approximation of the identity. Here, instead, we introduce a parabolic regularisation, in the same spirit of [23] for the Maxwell–Schrödinger system. The net result is the addition of a heat kernel effect in the linear propagator, whence the desired smoothing.

At the removal of the regularisation by a compactness argument, we obtain one—not necessarily unique—global-in-time, weak solution with finite energy, which is going to be our main result (Theorem 1.2 below).

To be concrete, let us first state the conditions on the magnetic potential.

Assumption 1.1. *The magnetic potential A belongs to one of the two classes $\mathcal{A}_1$ or $\mathcal{A}_2$ defined by*

$$\mathcal{A}_1 := \widetilde{\mathcal{A}}_1 \cap \mathcal{R}$$
$$\mathcal{A}_2 := \widetilde{\mathcal{A}}_2 \cap \mathcal{R},$$

where

$$\widetilde{\mathcal{A}}_1 := \left\{ A = A(t,x) \;\middle|\; \begin{array}{c} \mathrm{div}_x A = 0 \;\; \textit{for a.e. } t \in \mathbb{R}, \\ A = A_1 + A_2 \;\textit{such that, for } j \in \{1,2\}, \\ A_j \in L^{a_j}_{\mathrm{loc}}(\mathbb{R}, L^{b_j}(\mathbb{R}^3,\mathbb{R}^3)) \\ a_j \in (4,+\infty], \quad b_j \in (3,6), \quad \frac{2}{a_j} + \frac{3}{b_j} < 1 \end{array} \right\}$$

and

$$\widetilde{\mathcal{A}}_2 := \left\{ A = A(t,x) \;\middle|\; \begin{array}{c} \mathrm{div}_x A = 0 \;\; \textit{for a.e. } t \in \mathbb{R}, \\ A = A_1 + A_2 \;\textit{such that, for } j \in \{1,2\}, \\ A_j \in L^{a_j}_{\mathrm{loc}}(\mathbb{R}, W^{1,\frac{3b_j}{3+b_j}}(\mathbb{R}^3,\mathbb{R}^3)) \\ a_j \in (2,+\infty], \quad b_j \in (3,+\infty], \quad \frac{2}{a_j} + \frac{3}{b_j} < 1 \end{array} \right\},$$

and where

$$\mathcal{R} := \big\{ A \in \widetilde{\mathcal{A}}_1 \;\textit{or}\; A \in \widetilde{\mathcal{A}}_2 \mid \partial_t A_j \in L^1_{\mathrm{loc}}(\mathbb{R}, L^{b_j}(\mathbb{R}^3,\mathbb{R}^3)),\; j = 1,2 \big\}.$$

Associated with such classes, we define

$$\|A\|_{\mathcal{A}_1} := \|A_1\|_{L^{a_1}_t L^{b_1}_x} + \|A_2\|_{L^{a_2}_t L^{b_2}_x}$$
$$\|A\|_{\mathcal{A}_2} := \|A_1\|_{L^{a_1}_t W^{1,\frac{3b_1}{3+b_1}}_x} + \|A_2\|_{L^{a_2}_t W^{1,\frac{3b_2}{3+b_2}}_x}.$$

A few observations are in order. First and foremost, both classes $\mathcal{A}_1$ and $\mathcal{A}_2$ include magnetic potentials for which in general the validity of Strichartz estimates for the magnetic Laplacian is not known.

A large part of our intermediate results, including in particular the local theory in the energy space, are found with magnetic potentials in the larger classes $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$. The mild amount of regularity in time provided by the intersection with the class $\mathcal{R}$ is needed to infer suitable a priori bounds on the solution from the estimates on the total energy. This allows one to extend globally in time the solution to the regularised problem.

Regularity in time of the external potential is not needed either when Eq. (1.1) is studied in the *mass sub-critical* regime, i.e. when $\gamma \in (1, \frac{7}{3})$ and $\alpha \in (0,2)$, and when $\max\{b_1, b_2\} \in (3,6)$. In this case, we are able to work with the more general condition $A \in \widetilde{\mathcal{A}}_1$. This is a customary fact in the context of Schrödinger equations with time-dependent potentials, as well known since [42] (compare Theorems [42, Theorem 1.1] and [42, Theorem 1.4] therein: L^a-integrability in time on the electric external potentials yields a L^p-theory in space, whereas additional L^a-integrability of the time derivative of the potential yields a H^2-theory in space). Our aim here of studying finite energy solutions to (1.1) thus requires some intermediate assumptions on the magnetic potential, determined by the class $\mathcal{R}$ above. See also Proposition 1.7 in [2] where a similar issue is considered.

The additional requirement on ∇A present in the class $\mathcal{A}_2$ is taken to accommodate slower decay at infinity for A, way slower than the behaviour $|A(x)| \sim |x|^{-1}$ (and in fact even a L^∞-behaviour) which, as mentioned before, is critical for the validity of magnetic Strichartz inequalities.

Last, it is worth remarking that the divergence-free condition, $\mathrm{div}_x A = 0$, is assumed merely for convenience: our entire analysis can be easily extended to the cases where $\mathrm{div}_x A$ belongs to suitable Lebesgue spaces and consider it as a given (electrostatic) scalar potential.

Here is finally our main result. Clearly, there is no fundamental difference in studying solutions forward or backward in time, and as customary we shall only consider henceforth the problem for $t \geqslant 0$. Our entire discussion can be repeated for the case $t \leqslant 0$.

Theorem 1.2. (Existence of global, finite energy weak solutions)
Let the magnetic potential A be such that $A \in \mathcal{A}_1$ or $A \in \mathcal{A}_2$, and take $\gamma \in (1,5]$, $\alpha \in (0,3)$. Then, for every initial datum $f \in H^1(\mathbb{R}^3)$, the Cauchy problem

$$\begin{cases} \mathrm{i}\,\partial_t u \;=\; -(\nabla - \mathrm{i}\,A)^2 u + |u|^{\gamma-1}u + (|\cdot|^{-\alpha} * |u|^2)u \\ u(0,\cdot) \;=\; f \end{cases} \qquad t \in [0,+\infty),\quad x \in \mathbb{R}^3 \tag{1.3}$$

admits a global weak H^1-solution

$$u \;\in\; L^\infty_{\mathrm{loc}}([0,+\infty), H^1(\mathbb{R}^3)) \cap W^{1,\infty}_{\mathrm{loc}}([0,+\infty), H^{-1}(\mathbb{R}^3)),$$

meaning that (1.1) is satisfied for a.e. $t \in [0,+\infty)$ as an identity in H^{-1} and $u(0,\cdot) = f$. Moreover, the energy

$$\mathcal{E}(u)(t) := \int_{\mathbb{R}^3} \Big(\tfrac{1}{2} |(\nabla - \mathrm{i}A(t))\,u|^2 + \tfrac{1}{\gamma+1}|u|^{\gamma+1} + \tfrac{1}{4}(|x|^{-\alpha} * |u|^2)|u|^2 \Big)\,\mathrm{d}x$$

is finite and bounded on compact intervals.

In the remaining part of this Introduction, let us elaborate further on the general ideas behind our proof of Theorem 1.2.

As previously mentioned, we introduce a small dissipation term in the equation

$$i\partial_t u \;=\; -(1-\mathrm{i}\,\varepsilon)(\nabla - \mathrm{i}\,A)^2 u + \mathcal{N}(u) \tag{1.4}$$

and we study the approximated problem. Similar parabolic regularisation procedures are commonly used in PDEs, see, for example, the vanishing viscosity approximation in fluid dynamics or in systems of conservation laws, and in fact this was also exploited in a similar context by Guo, Nakamitsu, and Strauss to study on the existence of finite energy weak solutions to the Maxwell–Schrödinger system [23].

By exploiting the parabolic regularisation, we can now regard $i\partial_t u + (1-\mathrm{i}\,\varepsilon)\Delta u$ as the main linear part in the equation and treat $(1-\mathrm{i}\,\varepsilon)(2\,\mathrm{i}\,A\cdot\nabla u + |A|^2 u) + \mathcal{N}(u)$ as a perturbation.

Evidently, this cannot be done in the purely Hamiltonian case $\varepsilon = 0$. Indeed, the term $A \cdot \nabla u$ is not a Kato perturbation of the free Laplacian and the whole derivative Schrödinger equation must be considered as the principal part [40].

We can instead establish the local well-posedness in the energy space for the approximated Cauchy problem

$$\begin{cases} \mathrm{i}\,\partial_t u \;=\; -(1-\mathrm{i}\,\varepsilon)(\nabla - \mathrm{i}\,A)^2 u + |u|^{\gamma-1}u + (|\cdot|^{-\alpha} * |u|^2)u \\ u(0,\cdot) \;=\; f \end{cases} \qquad t \in [0,T],\quad x \in \mathbb{R}^3. \tag{1.5}$$

We first obtain suitable Strichartz-type and smoothing estimates for the viscous magnetic evolution semi-group. This is done by exploiting the smoothing effect of the heat-Schrödinger semi-group $t \mapsto e^{(i+\varepsilon)t\Delta}$

and by inferring the same space–time bounds also for the viscous magnetic evolution, in a similar fashion as in [34,43] scalar (electrostatic) potentials are treated as perturbations of the free Schrödinger evolution.

Next, the a priori bounds on the total mass and the total energy allow us to extend the solution of the regularised problem globally in time. It is worth stressing that such global well-posedness holds in the energy critical case too: indeed, when $\gamma = 5$ the bounds deduced from the energy dissipation provide a uniform-in-time control on some Strichartz-type norms, and the argument is then completed by means of the blow-up alternative for the critical case.

The mass/energy a priori bounds turn out to be uniform in the regularising parameter $\varepsilon > 0$, which yields the needed compactness for the sequence of approximating solutions. It is then possible to remove the regularisation and to show the existence of a finite energy weak solution to our original problem (1.3), at the obvious price of loosing the uniqueness, as well as its continuous dependence on the initial data.

The material is organised as follows. In Sect. 2, we collect the preliminary notions and results we need in our analysis. In particular, we clarify the notion of weak (and strong) H^1-solution and we derive suitable space–time estimates for the heat-Schrödinger evolution. In Sect. 3, we study the smoothing property of the magnetic linear Schrödinger equation with a parabolic regularisation. In Sect. 4, we prove local existence for the regularised magnetic nonlinear Schrödinger equation (1.4). In Sect. 5, we prove mass and energy estimates for (1.4) together with certain a priori bounds. In Sect. 6, we use the energy estimates and the a priori bounds to extend the solution (forward) globally in time, both in the energy sub-critical and in critical case. In Sect. 7, using a compactness argument, we remove the regularisation, eventually proving the main theorem.

2. Preliminaries and notation

In this section, we collect the definitions and main tools that we shall use in the rest of the work.

We begin with a few remarks on our notation. For two positive quantities P and Q, we write $P \lesssim Q$ to mean that $P \leqslant CQ$ for some constant C independent of the variables or of the parameters which P and Q depend on, unless explicitly declared; in the latter case we write, self-explanatorily, $P \lesssim_\alpha Q$, and the like. Given $p_1, \dots, p_n \in [1, +\infty]$, we define $p = p_1 * p_2 \dots * p_n$ by

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n}.$$

The same operation can be extended component-wise to vectors in $[1, +\infty]^d$, and we still denote it by $*$. Thus, for example, $(s, p) = (s_1, p_1) * (s_2, p_2)$ will mean $s^{-1} = s_1^{-1} + s_2^{-1}$ and $p^{-1} = p_1^{-1} + p_2^{-1}$. Given $p \in [1, +\infty]$, we denote by p' its Hölder dual exponent, defined by $p * p' = 1$. Henceforth, we use the symbols div, ∇ and Δ to denote derivations in the spatial variables only. When referring to the vector field $A : \mathbb{R}^3 \to \mathbb{R}^3$, conditions like $A \in L^p(\mathbb{R}^3)$ are to be understood as $A \in L^p(\mathbb{R}^3, \mathbb{R}^3)$. As customary, in a self-explanatory manner we will frequently make only the dependence on t explicit in symbols such as $A(t)$, $u(t)$ and $\mathcal{N}(u(t))$, $(\nabla - \mathrm{i}\, A(t))u$, instead of writing $A(t, x)$, $u(t, x)$, $(\mathcal{N}(u))(t, x)$, $((\nabla - \mathrm{i}\, A)u)(t, x)$, etc. The short-cut "NLS" refers as usual to nonlinear Schrödinger equation, in the sense that will be specified in the following. For sequences and convergence of sequences, we write $(u_n)_n$ and $u_n \to u$ for $(u_n)_{n \in \mathbb{N}}$ and $u_n \to u$ as $n \to +\infty$.

2.1. Magnetic Laplacian and magnetic Sobolev space

We clarify now the meaning of the symbol $(\nabla - \mathrm{i}A(t))^2$. As mentioned already in Introduction, formally

$$(\nabla - \mathrm{i}A(t))^2 = \Delta - 2\,\mathrm{i}\, A(t) \cdot \nabla - \mathrm{i}\, \mathrm{div} A(t) - |A(t)|^2.$$

In our setting of divergence-free magnetic potentials, this becomes

$$(\nabla - \mathrm{i}A(t))^2 = \Delta - 2\,\mathrm{i}\,A(t)\cdot\nabla - |A(t)|^2.$$

If $A(t) \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ for almost every $t \in \mathbb{R}$, which will always be our case, then we define the *magnetic Laplacian* $(\nabla - \mathrm{i}A(t))^2$ as a (time-dependent) distributional operator, according to the following straightforward lemma.

Lemma 2.1. (Distributional meaning of the magnetic Laplacian) *Assume that, for almost every $t \in \mathbb{R}$, $A(t) \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ with $\mathrm{div}A(t) = 0$. Then for almost every $t \in \mathbb{R}$, $(\nabla - \mathrm{i}A(t))^2$ is a map from $L^1_{\mathrm{loc}}(\mathbb{R}^3)$ to $\mathcal{D}'(\mathbb{R}^3)$, which acts on a generic $f \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ as*

$$(\nabla - \mathrm{i}A(t))^2 f \;=\; \Delta f - 2\,\mathrm{i}\,A(t)\cdot\nabla f - |A(t)|^2 f.$$

In order to qualify such a distribution as an element of a suitable functional space, it is natural to deal with the magnetic Sobolev space defined as follows (Here, with respect to our general setting, A is meant to be a magnetic vector potential at a fixed time).

Definition 2.2. Let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$. We define *magnetic Sobolev space*

$$H^1_A(\mathbb{R}^3) := \{f \in L^2(\mathbb{R}^3)\,|\,(\nabla - \mathrm{i}\,A)f \in L^2(\mathbb{R}^d)\}$$

equipped with the norm

$$\|f\|^2_{H^1_A(\mathbb{R}^3)} := \|f\|^2_{L^2(\mathbb{R}^d)} + \|(\nabla - \mathrm{i}\,A)f\|^2_{L^2(\mathbb{R}^d)},$$

which makes $H^1_A(\mathbb{R}^3)$ a Banach space.

We recall [27, Theorem 7.21] that, when $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$, any $f \in H^1_A(\mathbb{R}^3)$ satisfies the *diamagnetic inequality*

$$|(\nabla|f|)(x)| \leq |((\nabla - \mathrm{i}\,A)f)(x)| \quad \text{for a.e. } x \in \mathbb{R}^3. \tag{2.1}$$

The following two lemmas express useful magnetic estimates in our regime for A.

Lemma 2.3. *Assume that $A \in \mathcal{A}_1$ or $A \in \mathcal{A}_2$. Then, for almost every $t \in \mathbb{R}$,*

$$\|2\,\mathrm{i}\,A(t)\cdot\nabla f + |A(t)|^2 f\|_{H^{-1}(\mathbb{R}^3)} \;\lesssim\; C_A(t)\|f\|_{H^1(\mathbb{R}^3)}, \tag{2.2}$$

where

$$C_A(t) := 1 + \|A_1(t)\|^2_{L^{b_1}(\mathbb{R}^3)} + \|A_2(t)\|^2_{L^{b_2}(\mathbb{R}^3)}.$$

In particular, for almost every $t \in \mathbb{R}$, $(\nabla - \mathrm{i}A(t))^2$ is a continuous map from $H^1(\mathbb{R}^3)$ to $H^{-1}(\mathbb{R}^3)$.

Proof. The proof is based on a straightforward application of Sobolev's embedding and Hölder's inequality. □

Lemma 2.4. *Let $A \in L^b(\mathbb{R}^3)$ with $b \in [3, +\infty]$.*

(i) *One has*

$$\|f\|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{H^1_A(\mathbb{R}^3)}, \quad q \in [2, 6] \tag{2.3}$$

with the constant in (2.3) independent of A, hence the embedding $H^1_A(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ for $q \in [2, 6]$.

(ii) *One has*

$$\begin{aligned}(1 + \|A\|_{L^b(\mathbb{R}^3)})^{-1}\|f\|_{H^1(\mathbb{R}^3)} \;&\lesssim\; \|f\|_{H^1_A(\mathbb{R}^3)} \\ &\lesssim\; (1 + \|A\|_{L^b(\mathbb{R}^3)})\|f\|_{H^1(\mathbb{R}^3)},\end{aligned} \tag{2.4}$$

whence $H^1_A(\mathbb{R}^3) \cong H^1(\mathbb{R}^3)$ as an isomorphism between Banach spaces

Proof. The proof is based on a straightforward application of Sobolev's embedding, Hölder's inequality and the diamagnetic inequality. □

Remark 2.5. As an immediate consequence of Lemma 2.4, given a potential $A \in \widetilde{\mathcal{A}}_1$ or $A \in \widetilde{\mathcal{A}}_2$, for almost every $t > 0$ the magnetic Sobolev spaces $H^1_{A(t)}(\mathbb{R}^3)$ are all equivalent to the ordinary Sobolev space $H^1(\mathbb{R}^3)$.

2.2. Notion of solutions

We give now the precise notion of strong and weak solutions to the Cauchy problem (1.3) and its regularised version (1.5).

For the sake of a comprehensive discussion, let us consider the general Cauchy problem

$$\begin{cases} \mathrm{i}\,\partial_t u \;=\; c\,(\Delta u - 2\,\mathrm{i}\,A(t)\cdot\nabla u - |A(t)|^2 u) + \mathcal{N}(u) \\ u(0,\cdot) := f \\ \qquad t \in I{:=}[0,T), \quad x \in \mathbb{R}^3, \end{cases} \tag{2.5}$$

for some $T > 0$ and $c \in \mathbb{C}$ with $\mathfrak{Im}\, c \geqslant 0$. Here the choices $c = -1$ and $c = -1+\mathrm{i}\varepsilon$ correspond, respectively, to (1.3) and (1.5).

Definition 2.6. Let $I{:=}[0,T)$ for some $T > 0$. Given an initial datum $f \in H^1(\mathbb{R}^3)$, we say that

(i) a local strong H^1-solution u to (2.5) on I is a function

$$u \;\in\; \mathcal{C}(I, H^1(\mathbb{R}^3)) \cap \mathcal{C}^1(I; H^{-1}(\mathbb{R}^3))$$

such that $\mathrm{i}\,\partial_t u = c\,(\Delta u - 2\,\mathrm{i}\,A(t)\cdot\nabla u - |A(t)|^2 u) + \mathcal{N}(u)$ in $H^{-1}(\mathbb{R}^3)$ for all $t \in I$ and $u(0) = f$;

(ii) a local weak H^1-solution u to (2.5) on I is a function

$$u \;\in\; L^\infty(I, H^1(\mathbb{R}^3)) \cap W^{1,\infty}(I; H^{-1}(\mathbb{R}^3))$$

such that $\mathrm{i}\,\partial_t u = c\,(\Delta u - 2\,\mathrm{i}\,A(t)\cdot\nabla u - |A(t)|^2 u) + \mathcal{N}(u)$ in $H^{-1}(\mathbb{R}^3)$ for a.e. $t \in I$ and $u(0) = f$.

Moreover, a function $u \in L^\infty_{\mathrm{loc}}([0,+\infty), H^1(\mathbb{R}^3))$ is called

(iii) a global strong H^1-solution u to (2.5) if it is a local strong solution for every interval $I = [0,T)$;

(iv) a global weak H^1-solution u to (2.5) if it is a local weak solution for every interval $I = [0,T)$.

Next, we recall the notion of local and global well-posedness [3, Section 3.1].

Definition 2.7. We say that equation

$$\mathrm{i}\,\partial_t u \;=\; c\,(\Delta u - 2\,\mathrm{i}\,A(t)\cdot\nabla u - |A(t)|^2 u) + \mathcal{N}(u)$$

is locally well-posed in $H^1(\mathbb{R}^3)$ if the following conditions hold:

(i) For any initial datum $f \in H^1(\mathbb{R}^3)$, the Cauchy problem (2.5) admits a unique local strong H^1-solution, defined on a maximal interval $[0, T_{\max})$, with $T_{\max} = T_{\max}(f) \in (0,+\infty]$.

(ii) One has continuous dependence on the initial data, i.e. if $f_n \to f$ in $H^1(\mathbb{R}^3)$ and $0 \ni I \subset [0, T_{\max})$ is a closed interval, then the maximal strong H^1-solution to (2.5) with initial datum f_n is defined on I for n large enough and satisfies $u_n \to u$ in $\mathcal{C}(I, H^1(\mathbb{R}^3))$.

(iii) In the energy sub-critical case, one has the blow-up alternative: if $T_{\max} < +\infty$, then

$$\lim_{t \uparrow T_{\max}} \|u(t,\cdot)\|_{H^1(\mathbb{R}^3)} \;=\; +\infty.$$

We say that the same equation is globally well-posed in $H^1(\mathbb{R}^3)$ if it is locally well-posed, and if for any initial datum $f \in H^1(\mathbb{R}^3)$, the Cauchy problem (2.5) admits a global strong H^1-solution.

2.3. Smoothing estimates for the heat-Schrödinger flow

Let us now analyse the smoothing properties of the heat and the Schrödinger flows generated by the free Laplacian.

We begin by recalling the well-known dispersive estimates for the Schrödinger equation

$$\|e^{\mathrm{i}t\Delta} f\|_{L^p(\mathbb{R}^3)} \;\lesssim\; |t|^{-\frac{3}{2}\left(\frac{1}{p'}-\frac{1}{p}\right)}\|f\|_{L^{p'}(\mathbb{R}^3)}, \quad p\in[2,+\infty],\;\; t\neq 0, \tag{2.6}$$

and the L^p–L^r estimates for the heat flow

$$\|e^{t\Delta} f\|_{L^r(\mathbb{R}^3)} \lesssim t^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\|f\|_{L^p(\mathbb{R}^3)} \tag{2.7}$$

$$1\leqslant p\leqslant r\leqslant +\infty,\; t>0.$$

$$\|\nabla e^{t\Delta} f\|_{L^r(\mathbb{R}^3)}\lesssim t^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{2}}\|f\|_{L^p(\mathbb{R}^3)} \tag{2.8}$$

We also recall the definition of admissible pairs for the Schrödinger flow in three dimensions.

Definition 2.8. A pair (q,r) is called admissible if

$$\frac{2}{q}+\frac{3}{r}=\frac{3}{2}, \qquad r\in[2,6].$$

The pair $(2,6)$ is called endpoint, while the others are called non-endpoint. The pair (s,p) is called dual admissible if $(s,p)=(q',r')$ for some admissible pair (q,r), namely

$$\frac{2}{s}+\frac{3}{p}=\frac{7}{2}, \qquad p\in[\tfrac{6}{5},2].$$

The dispersive estimate (2.6) yields a whole class of space–time estimates for the Schrödinger flow [22, 25, 42].

Proposition 2.9. (Strichartz estimates)

(i) *For any admissible pair (q,r), the following homogeneous estimate holds:*

$$\|e^{\mathrm{i}t\Delta} f\|_{L^q(\mathbb{R};L^r(\mathbb{R}^3))} \;\lesssim\; \|f\|_{L^2(\mathbb{R}^3)}. \tag{2.9}$$

(ii) *Let I be an interval of $\mathbb{R}$ (bounded or not), and $\tau,t\in\overline{I}$. For any admissible pair (q,r) and any dual admissible pair (s,p), the following inhomogeneous estimate holds:*

$$\left\|\int_\tau^t e^{\mathrm{i}(t-\sigma)\Delta}F(\sigma)\,\mathrm{d}\sigma\right\|_{L^q(I;L^r(\mathbb{R}^3))} \;\lesssim\; \|F\|_{L^s(I;L^p(\mathbb{R}^3))}. \tag{2.10}$$

Similarly (see e.g. [41, Section 2.2.2]), by means of (2.7) and (2.8) one infers an analogous class of space–time estimates for the heat propagator.

Proposition 2.10. (Space–time estimates for $e^{t\Delta}$)

(i) *For any admissible pair (q,r), the following homogeneous estimate holds:*

$$\|e^{t\Delta} f\|_{L^q([0,+\infty),L^r(\mathbb{R}^3))} \;\lesssim\; \|f\|_{L^2(\mathbb{R}^3)}. \tag{2.11}$$

(ii) *Let $I\subseteq\mathbb{R}$ be an interval of the form $[\tau,T)$, with $T\in(\tau,+\infty]$. For any admissible pair (q,r) and any dual admissible pair (s,p), the following inhomogeneous estimate holds:*

$$\left\|\int_\tau^t e^{(t-\sigma)\Delta}F(\sigma)\,\mathrm{d}\sigma\right\|_{L^q(I;L^r(\mathbb{R}^3))} \;\lesssim\; \|F\|_{L^s(I;L^p(\mathbb{R}^3))}. \tag{2.12}$$

We can also combine the previous results in order to infer L^p–L^r estimates (Proposition 2.11) and space–time estimates (Proposition 2.12) for the heat-Schrödinger propagator.

Proposition 2.11. (Pointwise-in-time estimates for the heat-Schrödinger flow)
For any $t>0$, $p\in[1,2]$, *and* $r\in[2,+\infty]$,

$$\|e^{(\mathrm{i}+\varepsilon)t\Delta}f\|_{L^r(\mathbb{R}^3)} \lesssim \varepsilon^{-\frac{3}{2}|\frac{1}{p'}-\frac{1}{r}|}t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})}\|f\|_{L^p(\mathbb{R}^3)} \tag{2.13}$$

$$\|\nabla e^{(\mathrm{i}+\varepsilon)t\Delta}f\|_{L^r(\mathbb{R}^3)} \lesssim \varepsilon^{-\frac{3}{2}|\frac{1}{p'}-\frac{1}{r}|-\frac{1}{2}}t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}}\|f\|_{L^p(\mathbb{R}^3)}. \tag{2.14}$$

Proof. The proof is straightforward and follows by combining the decay estimates of both the heat and the Schrödinger propagators, see formulas (2.6)–(2.8) above. In fact, similar decay estimates follow also by simply ignoring the hyperbolic part given by the Schrödinger evolution, however with a worse control in terms of ε. □

Proposition 2.12. (Space–time estimates for the heat-Schrödinger flow)
Let $\varepsilon>0$ *and let* (q,r) *be an admissible pair.*

(i) *One has (homogeneous Strichartz estimate)*

$$\|e^{(\mathrm{i}+\varepsilon)t\Delta}f\|_{L^q([0,T],L^r(\mathbb{R}^3))} \lesssim \|f\|_{L^2(\mathbb{R}^3)}. \tag{2.15}$$

(ii) *Let* $T>0$ *and let the pair* (s,p) *satisfy*

$$\frac{2}{s}+\frac{3}{p}=\frac{7}{2}, \qquad \begin{cases} \frac{1}{2}\leqslant\frac{1}{p}\leqslant 1 & 2\leqslant r<3 \\ \frac{1}{2}\leqslant\frac{1}{p}<\frac{1}{r}+\frac{2}{3} & 3\leqslant r\leqslant 6. \end{cases} \tag{2.16}$$

Then (inhomogeneous retarded Strichartz estimate)

$$\left\|\int_0^t e^{(\mathrm{i}+\varepsilon)(t-\tau)\Delta}F(\tau)\,\mathrm{d}\tau\right\|_{L^q([0,T],L^r(\mathbb{R}^3))} \lesssim_\varepsilon \|F\|_{L^s([0,T],L^p(\mathbb{R}^3))}. \tag{2.17}$$

(iii) *Assume in addition that* (q,r) *is non-endpoint. Let* $T>0$ *and let the pair* (s,p) *satisfy*

$$\frac{2}{s}+\frac{3}{p}=\frac{5}{2}, \qquad \frac{1}{2}\leqslant\frac{1}{p}<\frac{1}{r}+\frac{1}{3}. \tag{2.18}$$

Then (inhomogeneous retarded Strichartz estimate)

$$\left\|\nabla\int_0^t e^{(\mathrm{i}+\varepsilon)(t-\tau)\Delta}F(\tau)\,\mathrm{d}\tau\right\|_{L^q([0,T],L^r(\mathbb{R}^3))} \lesssim_\varepsilon \|F\|_{L^s([0,T],L^p(\mathbb{R}^3))}. \tag{2.19}$$

Remark 2.13. In (2.16), the range of admissible pairs (s,p) is *larger* as compared to the case of the Schrödinger equation. In fact, dispersive equations, even if hyperbolic, have the remarkable property of enjoying a class of smoothing estimates. More specifically, for the Schrödinger equation it can be proved that the inhomogeneous part in the Duhamel formula enjoys the gain of regularity by one derivative in space, see Theorem 4.4 in [28]. However, it is straightforward to check that estimate (2.19) for the heat-Schrödinger semi-group is stronger than estimate (4.26) in [28] and it is better suited to study our problem.

Proof of Proposition 2.12. We begin with the proof of part (i). Combining the homogeneous Strichartz estimate (2.9) for the Schrödinger flow with the estimate $\|e^{\varepsilon t\Delta}f\|_{L^r(\mathbb{R}^3)} \lesssim \|f\|_{L^r(\mathbb{R}^3)}$, which follows by (2.7), we get

$$\begin{aligned}\|e^{(\mathrm{i}+\varepsilon)t\Delta}f\|_{L^q([0,+\infty),L^r(\mathbb{R}^3))} &= \|e^{\varepsilon t\Delta}e^{\mathrm{i}t\Delta}f\|_{L^q([0,+\infty),L^r(\mathbb{R}^3))} \\ &\lesssim \|e^{\mathrm{i}t\Delta}f\|_{L^q([0,+\infty),L^r(\mathbb{R}^3))} \lesssim \|f\|_{L^2(\mathbb{R}^3)},\end{aligned}$$

which proves (2.15). Next we prove part (ii). In the special case $(q,r)=(+\infty,2)$ and $(s,p)=(1,2)$, the dispersive estimate (2.13) yields

$$\begin{aligned}&\left\| \int_0^{+\infty} e^{(\mathrm{i}+\varepsilon)|t-\tau|\Delta} F(\tau)\,\mathrm{d}\tau \right\|_{L^\infty([0,T],L^2(\mathbb{R}^3))} \\ &\quad\lesssim \int_0^{+\infty} \|F(\tau)\|\,\mathrm{d}\tau \;=\; \|F\|_{L^1([0,T],L^2(\mathbb{R}^3))}.\end{aligned} \tag{2.20}$$

For the generic case, namely $(p,r)\neq(2,2)$, owing to (2.13) one obtains

$$\begin{aligned}&\left\| \int_0^{+\infty} e^{(\mathrm{i}+\varepsilon)|t-\tau|\Delta} F(\tau)\,\mathrm{d}\tau \right\|_{L^q([0,T],L^r(\mathbb{R}^3))} \\ &\quad\lesssim_\varepsilon \left\| \int_0^{+\infty} |t-\tau|^{-\gamma} \|F(\tau)\|_{L^p(\mathbb{R}^3)}\,\mathrm{d}\tau \right\|_{L^q[0,T]},\end{aligned}$$

where $\gamma := \frac{3}{2}(\frac{1}{p}-\frac{1}{r})\in(0,1)$ by the assumptions on p,r. The Hardy–Littlewood–Sobolev inequality in time yields then

$$\left\| \int_0^{+\infty} e^{(\mathrm{i}+\varepsilon)|t-\tau|\Delta} F(\tau)\,\mathrm{d}\tau \right\|_{L^q([0,T],L^r(\mathbb{R}^3))} \lesssim_\varepsilon \|F\|_{L^s([0,T],L^p(\mathbb{R}^3))} \tag{2.21}$$

with $\frac{1}{s}=1+\frac{1}{q}-\gamma$, namely $\frac{2}{s}+\frac{3}{p}=\frac{7}{2}$. Now, using estimates (2.20) and (2.21) and the Christ–Kiselev lemma [6], we deduce the "retarded estimate" (2.17). The proof of part (iii) proceeds similarly as for part (ii). Indeed, owing to the dispersive estimate with gradient (2.14) and the Hardy–Littlewood–Sobolev inequality in time,

$$\begin{aligned}&\left\| \nabla \int_0^{+\infty} e^{(\mathrm{i}+\varepsilon)|t-\tau|\Delta} F(\tau)\,\mathrm{d}\tau \right\|_{L^q([0,T],L^r(\mathbb{R}^3))} \\ &\quad\lesssim_\varepsilon \left\| \int_0^{+\infty} |t-\tau|^{-\gamma} \|F(\tau)\|_{L^p(\mathbb{R}^3)}\,\mathrm{d}\tau \right\|_{L^q(0,+\infty)} \lesssim_\varepsilon \|F\|_{L^s([0,T],L^p(\mathbb{R}^3))},\end{aligned}$$

where now $\gamma=\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{2}\in(0,1)$ and the exponent s is given by $\frac{2}{s}+\frac{3}{p}=\frac{5}{2}$; this, and again the result by Christ–Kieselev, then imply (2.19). □

For our analysis, it will be necessary to apply the above Strichartz estimates for the heat-Schrödinger flow in a regime of indices that guarantees also to control the smallness of the constant in each such inequalities in terms of the smallness of T. This leads us to introduce the following admissibility condition.

Definition 2.14. Let (q,r) be a admissible pair.

(i) A pair (s,p) is called a (q,r)-admissible pair if

$$\frac{2}{s}+\frac{3}{p}<\frac{7}{2}, \qquad \begin{cases} \frac{1}{2}\leqslant\frac{1}{p}\leqslant 1 & 2\leqslant r<3 \\ \frac{1}{2}\leqslant\frac{1}{p}<\frac{1}{r}+\frac{2}{3} & 3\leqslant r\leqslant 6. \end{cases} \tag{2.22}$$

(ii) A pair (s,p) is a called (q,r)-grad-admissible pair if

$$\frac{2}{s}+\frac{3}{p}<\frac{5}{2}, \quad \frac{1}{2}\leqslant\frac{1}{p}<\frac{1}{r}+\frac{1}{3}. \tag{2.23}$$

Remark 2.15. If (s,p) is a (q,r)-grad-admissible pair, then it is also (q,r)-admissible. Moreover, if (s,p) is a (q,r)-admissible pair (resp. (q,r)-grad-admissible), and (q_1,r_1) is another admissible pair with $r_1<r$, then (s,p) is also a (q_1,r_1)-admissible pair (resp. (q_1,r_1)-grad-admissible) pair.

We can state now a useful Corollary to Proposition 2.12.

Corollary 2.16. *Let $\varepsilon>0$ and $T>0$, and let (q,r) be a admissible pair.*

(i) *For any (q,r)-admissible pair (s,p),*

$$\begin{aligned}\left\|\int_0^t e^{(\mathrm{i}+\varepsilon)(t-\tau)\Delta}F(\tau)\,\mathrm{d}\tau\right\|_{L^q([0,T],L^r(\mathbb{R}^3))} &\lesssim_\varepsilon T^\theta\|F\|_{L^s([0,T],L^p(\mathbb{R}^3))} \\ \theta &:= \tfrac{7}{4}-\tfrac{1}{s}-\tfrac{3}{2p}.\end{aligned} \tag{2.24}$$

(ii) *Assume in addition that (q,r) is non-endpoint. For any (q,r)-grad-admissible pair,*

$$\begin{aligned}\left\|\nabla\int_0^t e^{(\mathrm{i}+\varepsilon)(t-\tau)\Delta}F(\tau)\,\mathrm{d}\tau\right\|_{L^q([0,T],L^r(\mathbb{R}^3))} &\lesssim_\varepsilon T^\theta\|F\|_{L^s([0,T],L^p(\mathbb{R}^3))} \\ \theta &:= \tfrac{5}{4}-\tfrac{1}{s}-\tfrac{3}{2p}.\end{aligned} \tag{2.25}$$

In either case, it follows by the assumptions that $\theta>0$.

2.4. Further technical lemmas

We conclude this section by collecting a few technical lemmas that will be useful for setting up the fixed point argument (Sect. 3).

Let us first introduce the following.

Definition 2.17. Given $T>0$, we define

$$X^{(4,3)}[0,T]:=L^\infty([0,T],H^1(\mathbb{R}^3))\cap L^4([0,T],W^{1,3}(\mathbb{R}^3))$$

equipped with the Banach norm

$$\|\cdot\|_{X^{(4,3)}[0,T]}:=\|\cdot\|_{L^\infty([0,T],H^1(\mathbb{R}^3))}+\|\cdot\|_{L^4([0,T],W^{1,3}(\mathbb{R}^3))}.$$

Remark 2.18. By interpolation we have that, for every admissible pair (q,r) with $r\in[2,3]$.

$$\|u\|_{L^q([0,T],W^{1,r}(\mathbb{R}^3))}\lesssim\|u\|_{X^{(4,3)}[0,T]}. \tag{2.26}$$

Furthermore, Sobolev embedding also yields

$$\|u\|_{L^q([0,T],L^{\frac{3r}{3-r}}(\mathbb{R}^3))}\lesssim\|u\|_{X^{(4,3)}[0,T]} \tag{2.27}$$

for any admissible pair (q,r) with $r\in[2,3]$.

Lemma 2.19.

(i) *Let* $A \in \widetilde{\mathcal{A}}_1$ *or* $A \in \widetilde{\mathcal{A}}_2$. *There exist* $(4,3)$*-grad-admissible pairs* (s_1,p_1), (s_2,p_2) *such that, for any* $u \in X^{(4,3)}[0,T]$,

$$A_i \cdot \nabla u \ \in \ L^{s_i}([0,T], L^{p_i}(\mathbb{R}^3)), \qquad i \in \{1,2\},$$

and

$$\|A_i \cdot \nabla u\|_{L^{s_i}([0,T],L^{p_i}(\mathbb{R}^3))} \ \lesssim \ \|A\|_{L^{a_i}([0,T],L^{b_i}(\mathbb{R}^3))} \|u\|_{X^{(4,3)}[0,T]}. \tag{2.28}$$

(ii) *Let* $A \in \widetilde{\mathcal{A}}_1$. *There exist four* $(4,3)$*-grad-admissible pairs* (s_{ij},p_{ij}), $i,j \in \{1,2\}$, *such that, for any* $u \in X^{(4,3)}[0,T]$,

$$A_i \cdot A_j \, u \ \in \ L^{s_{ij}}([0,T], L^{p_{ij}}(\mathbb{R}^3))$$

and

$$\begin{aligned} &\|A_i \cdot A_j \, u\|_{L^{s_{ij}}([0,T],L^{p_{ij}}(\mathbb{R}^3))} \\ &\quad \lesssim \|A_i\|_{L^{a_i}([0,T],L^{b_i}(\mathbb{R}^3))} \, \|A_j\|_{L^{a_j}([0,T],L^{b_j}(\mathbb{R}^3))} \, \|u\|_{X^{(4,3)}[0,T]}. \end{aligned} \tag{2.29}$$

(iii) *Let* $A \in \widetilde{\mathcal{A}}_2$. *There exist four* $(4,3)$*-admissible pairs* (s_{ij},p_{ij}), $i,j \in \{1,2\}$, *such that, for any* $u \in X^{(4,3)}[0,T]$,

$$A_i \cdot A_j \, u \ \in \ L^{s_{ij}}([0,T], W^{1,p_{ij}}(\mathbb{R}^3))$$

and

$$\begin{aligned} &\|A_i \cdot A_j \, u\|_{L^{s_{ij}}([0,T],W^{1,p_{ij}}(\mathbb{R}^3))} \\ &\quad \lesssim \big(\|A_i\|_{L^{a_i}([0,T],L^{b_i}(\mathbb{R}^3))} + \|\nabla A_i\|_{L^{a_i}([0,T],L^{3b_i/(3+b_i)}(\mathbb{R}^3))}\big) \\ &\qquad \times \big(\|A_j\|_{L^{a_j}([0,T],L^{b_j}(\mathbb{R}^3))} + \|\nabla A_j\|_{L^{a_j}([0,T],L^{3b_j/(3+b_j)}(\mathbb{R}^3))}\big) \\ &\qquad \times \|u\|_{X^{(4,3)}[0,T]}. \end{aligned} \tag{2.30}$$

Proof. The proof consists in repeatedly applying Hölder's inequality and the Sobolev embedding; we omit the standard details. □

Lemma 2.20. *Let* $A \in \widetilde{\mathcal{A}}_1$ *or* $A \in \widetilde{\mathcal{A}}_2$, *and let* $\varepsilon > 0$. *There exists a constant* $\theta_A > 0$ *such that, for every* $T \in (0,1]$,

$$\bigg\| \int_0^t e^{(\mathrm{i}+\varepsilon)(t-\sigma)\Delta} A(\sigma) \cdot \nabla u(\sigma) \, \mathrm{d}\sigma \bigg\|_{X^{(4,3)}[0,T]} \ \lesssim_{\varepsilon,A} \ T^{\theta_A} \|u\|_{X^{(4,3)}[0,T]}.$$

Proof. Because of Lemma 2.19(i),

$$A_i \cdot \nabla u \in L^{s_i}([0,T], L^{p_i}(\mathbb{R}^3)), \qquad i \in \{1,2\},$$

for some (s_1,p_1), (s_2,p_2) which are $(4,3)$-grad-admissible pairs. Applying Corollary 2.16(ii) and Lemma 2.19(i) to $A_i \cdot \nabla u$ and setting

$$\theta_A := \min\left\{ \frac{5}{4} - \frac{1}{s_1} - \frac{3}{2p_1}, \frac{5}{4} - \frac{1}{s_2} - \frac{3}{2p_2} \right\}$$

the thesis follows. □

Lemma 2.21. *Let* $A \in \widetilde{\mathcal{A}}_1$ *or* $A \in \widetilde{\mathcal{A}}_2$, *and let* $\varepsilon > 0$. *There exists a constant* $\theta_A > 0$ *such that, for every* $T \in (0,1]$,

$$\bigg\| \int_0^t e^{(\mathrm{i}+\varepsilon)(t-\sigma)\Delta} |A(\sigma)|^2 u(\sigma) \, \mathrm{d}\sigma \bigg\|_{X^{(4,3)}[0,T]} \ \lesssim_{\varepsilon,A} \ T^{\theta_A} \|u\|_{X^{(4,3)}[0,T]}.$$

Proof. The proof is similar to the previous one. For example, in the case $A \in \widetilde{\mathcal{A}}_1$, by Lemma 2.19(ii) we have

$$A_i \cdot A_j u \in L^{s_{ij}}([0,T], L^{p_{ij}}(\mathbb{R}^3)), \qquad i,j \in \{1,2\}.$$

Then, we apply Corollary 2.16(i). □

3. The regularised magnetic Laplacian

We discuss now the existence of the linear magnetic viscous propagator, and we prove that, with our assumptions on the magnetic potential, the propagator enjoys the same Strichartz-type estimates for the heat-Schrödinger flow obtained already in Sect. 2.3.

The main result of this section is the following.

Theorem 3.1. *Assume that $A \in \widetilde{\mathcal{A}}_1$ or $A \in \widetilde{\mathcal{A}}_2$. For given $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $f \in H^1(\mathbb{R}^3)$ consider the inhomogeneous Cauchy problem*

$$\begin{cases} \mathrm{i}\,\partial_t u \;=\; -\,(1-\mathrm{i}\,\varepsilon)(\Delta u - 2\,\mathrm{i}\,A\cdot\nabla u - |A|^2 u) + F + G \\ u(\tau,\cdot) \;= f \end{cases} \tag{3.1}$$

and the associated integral equation

$$\begin{aligned} u(t,\cdot) &= e^{(\mathrm{i}+\varepsilon)(t-\tau)\Delta} f \\ &\quad - \mathrm{i}\int_\tau^t e^{(\mathrm{i}+\varepsilon)(t-\sigma)\Delta}\big((1-\mathrm{i}\,\varepsilon)(2\,\mathrm{i}\,A\cdot\nabla u + |A|^2 u)(\sigma) + F(\sigma) + G(\sigma)\big)\,\mathrm{d}\sigma, \end{aligned} \tag{3.2}$$

where

- *$F \in L^{\widetilde{s}}(\mathbb{R}, W^{1,\widetilde{p}}(\mathbb{R}^3))$ for some pair $(\widetilde{s},\widetilde{p})$ that is $(4,3)$-admissible pair or satisfies (2.16) with $(q,r) = (4,3)$, namely $\frac{2}{\widetilde{s}} + \frac{3}{\widetilde{p}} \leqslant \frac{7}{2}$, $\frac{1}{2} \leqslant \frac{1}{\widetilde{p}} < 1$;*
- *$G \in L^s(\mathbb{R}, L^p(\mathbb{R}^3))$, for some pair (s,p) that is $(4,3)$-grad-admissible or satisfies (2.18) with $(q,r) = (4,3)$, namely $\frac{2}{s} + \frac{3}{p} \leqslant \frac{5}{2}$, $\frac{1}{2} \leqslant \frac{1}{p} < \frac{2}{3}$.*

Then, there exists a unique solution $u \in \mathcal{C}([\tau,+\infty), H^1(\mathbb{R}^3))$ to (3.2). Moreover, for any $T > \tau$ and for any Strichartz pair (q,r), with $r \in [2,3]$,

$$\begin{aligned} \|u\|_{L^q([\tau,T],W^{1,r}(\mathbb{R}^3))} &\lesssim_{\varepsilon,A,T} \\ &\lesssim_{\varepsilon,A,T} \|f\|_{H^1(\mathbb{R}^3)} + \|F\|_{L^s(\mathbb{R},L^p(\mathbb{R}^3))} + \|G\|_{L^{\widetilde{s}}(\mathbb{R},W^{1,\widetilde{p}}(\mathbb{R}^3))}. \end{aligned} \tag{3.3}$$

Theorem 3.1 shows the existence of a unique solution u to the integral equation (3.2). From the assumptions on the magnetic potential and the source terms F, G and by using standard arguments in the theory of evolution equations (see, for example, [4]), we may also infer that u satisfies (3.1) for almost every $t \in \mathbb{R}$ in the sense of distributions. In the case when $F = G = 0$, the solution u to (3.1) defines an evolution operator, namely for any $f \in H^1(\mathbb{R}^3)$ the *magnetic viscous evolution* is defined by $\mathcal{U}_{\varepsilon,A}(t,\tau) f = u(t)$ where u is the solution to (3.1) with $F = G = 0$. As a consequence of Theorem 3.1, we have that $\mathcal{U}_{\varepsilon,A}(t,\tau)$ enjoys a class of Strichartz-type estimates.

Proposition 3.2. *The family $\{\mathcal{U}_{\varepsilon,A}(t,\tau)\}_{t,\tau}$ of operators on $H^1(\mathbb{R}^3)$ satisfies the following properties:*

- *$\mathcal{U}_{\varepsilon,A}(t,s)\,\mathcal{U}_{\varepsilon,A}(s,\tau) = \mathcal{U}_{\varepsilon,A}(t,\tau)$ for any $\tau < s < t$;*
- *$\mathcal{U}_{\varepsilon,A}(t,t) = \mathbb{1}$;*
- *the map $(t,\tau) \mapsto \mathcal{U}_{\varepsilon,A}(t,\tau)$ is strongly continuous in $H^1(\mathbb{R}^3)$;*

- *for any admissible pair (q,r) with $r \in [2,3]$, and for any F, G satisfying the same assumptions as in Theorem 3.1, one has*

$$\|\mathcal{U}_{\varepsilon,A}(t,\tau)f\|_{L^q([\tau,T],W^{1,r}(\mathbb{R}^3))} \lesssim_{\varepsilon,A,T} \|f\|_{H^1(\mathbb{R}^3)} \tag{3.4}$$

$$\left\| \int_\tau^t \mathcal{U}_{\varepsilon,A}(t,\sigma)\, F(\sigma)\,\mathrm{d}\sigma \right\|_{L^q([\tau,T],W^{1,r}(\mathbb{R}^3))} \lesssim_{\varepsilon,A,T} \|F\|_{L^{\tilde{s}}([\tau,T],W^{1,\tilde{p}}(\mathbb{R}^3))} \tag{3.5}$$

$$\left\| \int_\tau^t \mathcal{U}_{\varepsilon,A}(t,\sigma)\, G(\sigma)\,\mathrm{d}\sigma \right\|_{L^q([\tau,T],W^{1,r}(\mathbb{R}^3))} \lesssim_{\varepsilon,A,T} \|G\|_{L^{s}([\tau,T],L^{p}(\mathbb{R}^3))}. \tag{3.6}$$

Once we defined the magnetic viscous evolution operator $\mathcal{U}_{\varepsilon,A}(t,\tau)$, we see that we can write the integral formulation for (3.1) in the following way

$$u(t) \;=\; \mathcal{U}_{\varepsilon,A}(t,\tau)f - \mathrm{i}\int_\tau^t \mathcal{U}_{\varepsilon,A}(t,\sigma)\big(F(\sigma)+G(\sigma)\big)\,\mathrm{d}\sigma. \tag{3.7}$$

We will use formula (3.7) and Strichartz-type estimates (3.4)–(3.6) in order to set up a fixed point argument and show the existence of solutions to the nonlinear problem (1.5).

Let us now proceed with proving Theorem 3.1. As already mentioned, the proof is based upon a contraction argument in the space introduced in Definition 2.17 and requires the magnetic estimates established Lemmas 2.19, 2.20, and 2.21.

Proof of Theorem 3.1. It is clearly not restrictive to set the initial time $\tau = 0$. For given $T \in (0,1]$ and $M > 0$, we consider the ball of radius M in $X^{(4,3)}[0,T]$, i.e.

$$\mathcal{X}_{T,M} := \{u \in X^{(4,3)}[0,T] \mid \|u\|_{X^{(4,3)}[0,T]} \leqslant M\}.$$

Moreover, we define the solution map $u \mapsto \varPhi u$ where, for $t \in [0,T]$,

$$\begin{aligned}(\varPhi u)(t) &:= e^{(\mathrm{i}+\varepsilon)t\Delta}f - (\mathrm{i}+\varepsilon)\\ &\quad\times\int_0^t e^{(\mathrm{i}+\varepsilon)(t-\sigma)\Delta}\big((2\,\mathrm{i}\,A(\sigma)\cdot\nabla + |A(\sigma)|^2)u(\sigma) + F(\sigma) + G(\sigma)\big)\,\mathrm{d}\sigma.\end{aligned} \tag{3.8}$$

Thus, finding a solution to the integral Eq. (3.2), with $\tau = 0$, is equivalent to finding a fixed point for the map $\varPhi$. We shall then prove Theorem 3.1 by showing that, for suitable T and M, the map $\varPhi$ is a contraction on $\mathcal{X}_{T,M}$. To this aim, let us consider a generic $u \in \mathcal{X}_{T,M}$: owing to Strichartz estimates (2.15) and (2.19) and to Lemmas 2.20 and 2.21, there exist positive constants $C \equiv C_{\varepsilon,A}$ and $\theta \equiv \theta_A$ such that, for $T \in (0,1]$,

$$\begin{aligned}\|\varPhi u\|_{X^{(4,3)}[0,T]} \;\leqslant\; C\Bigg(&\|f\|_{H^1(\mathbb{R}^3)} + \sum_{i=1}^N \|F_i\|_{L^{s_i}([0,T],L^{p_i}(\mathbb{R}^3)}\\ &+ \sum_{i=1}^N \|G_i\|_{L^{\tilde{s}_i}([0,T],L^{\tilde{p}_i}(\mathbb{R}^3)} + T^\theta\|u\|_{X^{(4,3)}[0,T]}\Bigg).\end{aligned} \tag{3.9}$$

It is possible to restrict further M and T such that

$$M \;>\; 2C\left(\|f\|_{H^1(\mathbb{R}^3)} + \sum_{i=1}^N \|F_i\|_{L^{s_i}([0,T],L^{p_i}(\mathbb{R}^3)} + \sum_{i=1}^N \|G_i\|_{L^{\tilde{s}_i}([0,T],L^{\tilde{p}_i}(\mathbb{R}^3)}\right)$$

and $2CT^\theta < 1$, in which case (3.9) yields

$$\|\Phi u\|_{X^{(4,3)}[0,T]} \leqslant M(\tfrac{1}{2} + CT^\theta) < M.$$

This proves that Φ maps indeed $\mathcal{X}_{T,M}$ into itself. Next, for generic $u, v \in \mathcal{X}_{T,M}$, and with the above choice of M and T, (3.9) also yields

$$\begin{aligned}\|\Phi u - \Phi v\|_{X^{(4,3)}[0,T]} &= \|\Phi(u-v)\|_{X^{(4,3)}[0,T]} \leqslant CT^\theta \|u-v\|_{X^{(4,3)}[0,T]} \\ &< \frac{1}{2}\|u-v\|_{X^{(4,3)}[0,T]},\end{aligned}$$

which proves that Φ is indeed a contraction on $\mathcal{X}_{T,M}$. By Banach's fixed point theorem, we conclude that the integral equation $u = \Phi u$ has a unique solution in $\mathcal{X}_{T,M}$. Furthermore, $\Phi u \in \mathcal{C}([0,T], H^1(\mathbb{R}^3))$. Hence, we have found a local solution $u \in \mathcal{C}([0,T], H^1(\mathbb{R}^3))$ to the integral equation (3.2), which satisfies (3.3). Moreover, since the local existence time T does not depend on the initial data, this solution can be extended globally in time, and (3.3) is satisfied for any $T > 0$. □

As the last result of this section, we show the propagator $\mathcal{U}_{\varepsilon,A}(t,\tau)$ is stable under small perturbations of the magnetic potential and of the initial datum.

Proposition 3.3. (Stability) *Let $\tau \in \mathbb{R}, T > \tau$, and let us assume that $A^{(1)}, A^{(2)} \in \widetilde{\mathcal{A}}_1$, with $\|A^{(1)} - A^{(2)}\|_{\mathcal{A}_1} < \delta$ or $A^{(1)}, A^{(2)} \in \widetilde{\mathcal{A}}_2$, with $\|A^{(1)} - A^{(2)}\|_{\mathcal{A}_2} < \delta$, where $\delta > 0$ is sufficiently small. Let $u_1, u_2 \in \mathcal{C}([\tau,T); H^1(\mathbb{R}^3))$ be the solutions to*

$$\begin{cases} \mathrm{i}\,\partial_t u_j = -(1-\mathrm{i}\,\varepsilon)(\nabla - \mathrm{i}A^{(j)})^2 u_j + F_j \\ u(\tau,\cdot) = f_j \end{cases} \tag{3.10}$$

for given $f_1, f_2 \in H^1(\mathbb{R}^3)$ and given $F_1, F_2 \in L^s([\tau,T], W^{1,p}(\mathbb{R}^3))$, where (s,p) is dual admissible. Then, for any admissible pair (q,r) with $r \in [2,3]$, we have

$$\|u_1 - u_2\|_{L^q([\tau,T],W^{1,r}(\mathbb{R}^3))} \lesssim \delta + \|f_1 - f_2\|_{H^1} + \|F_1 - F_2\|_{L^s([\tau,T],W^{1,p}(\mathbb{R}^3))}.$$

Proof. We prove the Proposition under the assumptions $A^{(1)}, A^{(2)} \in \mathcal{A}_1$ and $\|A^{(1)} - A^{(2)}\|_{\mathcal{A}_1} < \delta$, the other case being completely analogous. From (3.10), we infer that the function $\widetilde{u}{:=}u_1 - u_2$ satisfies

$$\begin{cases} \mathrm{i}\,\partial_t \widetilde{u} = -(1-\mathrm{i}\,\varepsilon)(\nabla - iA^{(1)})^2 \widetilde{u} + 2\,\mathrm{i}\widetilde{A}\cdot\nabla u_2 + \widetilde{A}\cdot(A^{(1)} + A^{(2)})u_2 + \widetilde{F} \\ \widetilde{u}(\tau,\cdot) = \widetilde{f} \end{cases}$$

or equivalently

$$\begin{aligned}\widetilde{u}(t) &= \mathcal{U}_{\varepsilon,A^{(1)}}(t,0)\widetilde{f} \\ &\quad - \mathrm{i}\int_\tau^t \mathcal{U}_{\varepsilon,A^{(1)}}(t,\sigma)\big(2\,\mathrm{i}\widetilde{A}\cdot\nabla u_2 + \widetilde{A}\cdot(A^{(1)} + A^{(2)})u_2 + \widetilde{F}\big)(\sigma)\,\mathrm{d}\sigma,\end{aligned} \tag{3.11}$$

where $\widetilde{f}{:=}f_1 - f_2$, $\widetilde{A}{:=}A_1 - A_2$, and $\widetilde{F}{:=}F_1 - F_2$. Since u_1 and u_2 solve (3.10) on the time interval $[\tau,T]$, estimate (3.3) yields

$$\begin{aligned}\|u_j\|_{L^q([\tau,T];W^{1,r}(\mathbb{R}^3))} &\leqslant C(\|f_j\|_{H^1}, \|A^{(j)}\|_{\mathcal{A}_1}, \|F_j\|_{L^s([\tau,T],W^{1,p}(\mathbb{R}^3))}) \\ & j \in \{1,2\}\end{aligned}$$

for any admissible pair (q,r) with $r \in [2,3]$. By applying the Strichartz-type estimates stated in Proposition 3.2 and the estimates of Lemma 2.19 to Eq. (3.11), we have

$$\begin{aligned}\|\tilde{u}\|_{L^q([\tau,T],W^{1,r}(\mathbb{R}^3))} &\lesssim \|\widetilde{f}\|_{H^1} + \|\widetilde{A}\|_{\mathcal{A}_1}\|u_2\|_{X^{(4,3)}[\tau,T]} \\ &+ \|\widetilde{A}\|_{\mathcal{A}_1}\Big(\|A^{(1)}\|_{\mathcal{A}_1} + \|A^{(2)}\|_{\mathcal{A}_1}\Big)\|u_2\|_{X^{(4,3)}[\tau,T]} + \|\tilde{F}\|_{L^s([\tau,T],W^{1,p}(\mathbb{R}^3))},\end{aligned}$$

from which the result follows. □

4. Local well-posedness for the regularised magnetic NLS

In this section, we turn our attention to the nonlinear problem (1.5). Using the existence result and the Strichartz-type estimates established, respectively, in Theorem 3.1 and Proposition 3.2, we set up our fixed point argument associated with the integral equation

$$u(t) \;=\; \mathcal{U}_{\varepsilon,A}(t,0)f - \mathrm{i}\int_0^t \mathcal{U}_{\varepsilon,A}(t,\sigma)\,\mathcal{N}(u)(\sigma)\,\mathrm{d}\sigma. \tag{4.1}$$

We first focus on the case of energy sub-critical nonlinearities.

Proposition 4.1. (Local well-posedness, energy sub-critical case) *Let $\varepsilon > 0$. Assume that $A \in \widetilde{\mathcal{A}}_1$ or $A \in \widetilde{\mathcal{A}}_2$ and that the exponents in the nonlinearity (1.2) are in the regime $\gamma \in (1,5)$ and $\alpha \in (0,3)$. Then, for any $f \in H^1(\mathbb{R}^3)$, there exists a unique solution $u \in \mathcal{C}([0,T_{max}),H^1(\mathbb{R}^3))$ to (4.1) on a maximal interval $[0,T_{max})$ such that the following blow-up alternative holds: if $T_{max} < +\infty$, then $\lim_{t\uparrow T_{max}}\|u(t)\|_{H^1} = +\infty$.*

Proof. Since the linear propagator $\mathcal{U}_{\varepsilon,A}(t,\tau)$ satisfies the same Strichartz-type estimates as the heat-Schrödinger flow, and since the nonlinearities considered here are sub-critical perturbation of the linear flow, a customary contraction argument in the space

$$\mathcal{C}([0,T],H^1(\mathbb{R}^3)) \cap L^{q(\gamma)}([0,T],W^{1,r(\gamma)}(\mathbb{R}^3)) \cap L^{q(\alpha)}([0,T],W^{1,r(\alpha)}(\mathbb{R}^3)), \tag{4.2}$$

where

$$(q(\gamma),r(\gamma)) := \Big(\tfrac{4(\gamma+1)}{\gamma-1},\tfrac{3(\gamma+1)}{\gamma+2}\Big) \tag{4.3}$$

(see e.g. [29, Theorems 2.1 and 3.1]) and

$$(q(\alpha),r(\alpha)) := \begin{cases} (+\infty,2) & \alpha \in (0,2] \\ \Big(\tfrac{6}{\alpha-2},\tfrac{18}{13-2\alpha}\Big) & \alpha \in (2,3) \end{cases} \tag{4.4}$$

(see e.g. [28, Section 5.2]), guarantees the existence of a unique local solution for sufficiently small T. We observe, in particular, that with the above choice one has $r(\gamma), r(\alpha) \in [2,3)$. Furthermore, by a customary continuation argument we can extend such a solution over a maximal interval for which the blow-up alternative holds true. We omit the standard details; they are part of the well-established theory of semi-linear equations. □

In the presence of a energy critical nonlinearity ($\gamma = 5$), the above arguments cannot be applied. Indeed, when $\gamma = 5$ we cannot apply Corollary 2.16 with that nonlinearity, in order to obtain the factor T^θ, $\theta > 0$ and apply the standard contraction argument. However, it is possible to exploit a similar idea as in [5] to infer a local well-posedness result when $\gamma = 5$.

Proposition 4.2. (Local existence and uniqueness, energy critical case) *Let $A \in \widetilde{\mathcal{A}}_1$ or $A \in \widetilde{\mathcal{A}}_2$ and let the exponents in the nonlinearity (1.2) be in the regime $\gamma = 5$ and $\alpha \in (0,3)$. Let $\varepsilon > 0$ and $f \in H^1(\mathbb{R}^3)$. There exists $\eta_0 > 0$ such that, if*

$$\|\nabla e^{\mathrm{i}t\Delta}f\|_{L^6([0,T],L^{\frac{18}{7}}(\mathbb{R}^3))} \leqslant \eta \tag{4.5}$$

for some (small enough) $T > 0$ *and some* $\eta < \eta_0$*, then there exists a unique solution* $u \in \mathcal{C}([0,T], H^1(\mathbb{R}^3))$ *to* (4.1)*. Moreover, this solution can be extended on a maximal interval* $[0, T_{max})$ *such that the following blow-up alternative holds true:* $T_{max} < \infty$ *if and only if* $\|u\|_{L^6([0,T_{max}),L^{18}(R^3))} = \infty$*.*

Proof. A direct application of a well-known argument by Cazenave and Weissler [5] (we refer to [26, Section 3] for a more recent discussion). In particular, having established Strichartz estimates for $\mathcal{U}_{\varepsilon,A}(t,\tau)$ relative to the pair $(q,r) = (6, \frac{18}{7})$, we proceed exactly as in the proof of [26, Theorem 3.4 and Corollary 3.5], so as to find a unique solution u to the integral equation (4.1) in the space

$$\mathcal{C}([0,T], H^1(\mathbb{R}^3)) \cap L^6([0,T], W^{1,\frac{18}{7}}(\mathbb{R}^3)) \cap L^{q(\alpha)}([0,T], W^{1,r(\alpha)}(\mathbb{R}^3)) \tag{4.6}$$

with $(q(\alpha), r(\alpha))$ given by (4.4), together with the $L^6_t L^{18}_x$-blow-up alternative. □

We conclude this section by stating the analogous stability property of Proposition 3.3 also for the nonlinear problem

Proposition 4.3. *Let* $\tau \geq 0$*,* $T \in (\tau, \infty)$ *and let us assume that* $A^{(1)}, A^{(2)} \in \widetilde{\mathcal{A}}_1$ *with* $\|A^{(1)} - A^{(2)}\|_{\mathcal{A}_1} < \delta$ *or that* $A^{(1)}, A^{(2)} \in \widetilde{\mathcal{A}}_2$ *with* $\|A^{(1)} - A^{(2)}\|_{\mathcal{A}_2} < \delta$*, for some* $\delta > 0$ *small enough. Let us consider* $u_1, u_2 \in \mathcal{C}([\tau, T]; H^1(\mathbb{R}^3))$ *solutions to*

$$\begin{cases} \mathrm{i}\,\partial_t u_j \;=\; -\,(1 - \mathrm{i}\,\varepsilon)(\nabla - iA^{(j)})^2 u_j + \mathcal{N}(u_j) \\ u(\tau, \cdot) \;= f_j, \end{cases}$$

where $j \in \{1,2\}$*,* $f_1, f_2 \in H^1$*,* $\mathcal{N}(u)$ *is given by* (1.2) *with* $\gamma \in (1,5], \alpha \in (0,3)$*. Then, for any admissible pair* (q,r) *with* $r \in [2,3]$*, we have*

$$\|u_1 - u_2\|_{L^q([\tau,T],W^{1,r}(\mathbb{R}^3))} \lesssim \delta + \|f_1 - f_2\|_{H^1}.$$

5. Mass and energy estimates

In this section, we establish some a priori estimates which will be needed in order to extend the local approximating solution obtained in Sect. 4 over arbitrary time intervals. In particular, we will show that the total mass and energy are uniformly bounded. Furthermore, by exploiting the dissipative regularisation, we will infer some a priori space–time bounds which will allow to extend globally the solution also in the energy critical case.

The two quantities of interest are defined as follows.

Definition 5.1. Let $T > 0$. For each $u \in L^\infty([0,T), H^1(\mathbb{R}^3))$ and $t \in [0,T)$, mass and energy of u are defined, at almost every time $t \in [0,T)$, as

$$(\mathcal{M}(u))(t) := \int_{\mathbb{R}^3} |u(t,x)|^2 \,\mathrm{d}x$$

$$(\mathcal{E}(u))(t) := \int_{\mathbb{R}^3} \Big(\tfrac{1}{2} |(\nabla - \mathrm{i}A(t))\, u|^2 + \tfrac{1}{\gamma+1} |u|^{\gamma+1} + \tfrac{1}{4} (|x|^{-\alpha} * |u|^2)|u|^2 \Big) \,\mathrm{d}x.$$

In what follows, we will consider potentials $A \in \mathcal{A}_1$ or $A \in \mathcal{A}_2$, so to have the time regularity needed in order to study the energy functional.

Proposition 5.2. *Assume that* $A \in \mathcal{A}_1$ *or* $A \in \mathcal{A}_2$ *and that the exponents in nonlinearity* (1.2) *are in the whole regime* $\gamma \in (1,5]$ *and* $\alpha \in (0,3)$*. For fixed* $\varepsilon > 0$*, let* $u_\varepsilon \in \mathcal{C}([0,T), H^1(\mathbb{R}^3))$ *be the local solution to* *the regularised equation (1.4) for some $T > 0$. Then, the mass, the energy and the H^1-norm of u_ε are bounded in time over $[0, T)$, uniformly in $\varepsilon > 0$, that is,*

$$\sup_{t\in[0,T]} \mathcal{M}(u_\varepsilon) \;\lesssim\; 1 \tag{5.1}$$

$$\sup_{t\in[0,T]} \mathcal{E}(u_\varepsilon) \;\lesssim_{A,T}\; 1 \tag{5.2}$$

$$\|u_\varepsilon\|_{L^\infty([0,T),H^1(\mathbb{R}^3))} \;\lesssim_{A,T}\; 1, \tag{5.3}$$

and moreover, one has the a priori bounds

$$\int_0^T\!\!\int_{\mathbb{R}^3} \Big(|(\nabla - \mathrm{i}A(t))u_\varepsilon|^2\big(|u_\varepsilon|^{\gamma-1} + (|x|^{-\alpha} * |u_\varepsilon|^2)\big) + (\gamma-1)|u_\varepsilon|^{\gamma-1}|\nabla|u_\varepsilon||^2 + (|x|^{-\alpha} * \nabla|u_\varepsilon|^2)\nabla|u_\varepsilon|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \;\lesssim_{A,T}\; \varepsilon^{-1}. \tag{5.4}$$

Remark 5.3. *At fixed $\varepsilon > 0$*, the finiteness of $\mathcal{M}(u_\varepsilon)(t)$ and of $\mathcal{E}(u_\varepsilon)(t)$ for all $t \in [0, T)$ is obvious for the mass, since by assumption $u_\varepsilon(t) \in L^2(\mathbb{R}^3)$ for every $t \in [0, T)$, and it is also straightforward for the energy, since the property that $((\nabla - \mathrm{i}A)u_\varepsilon)(t) \in L^2(\mathbb{R}^3)$ for every $t \in [0, T)$ is also part of the assumption, and moreover, it is a standard property (see e.g. [3, Section 3.2]) that both $\int_{\mathbb{R}^3} |u_\varepsilon|^{\gamma+1}\,\mathrm{d}x$ and $\int_{\mathbb{R}^3}(|x|^{-\alpha} * |u_\varepsilon|^2)|u_\varepsilon|^2\,\mathrm{d}x$ are finite for every $t \in [0, T)$, and both in the energy sub-critical and in critical regime. The virtue of Proposition 5.2 is thus to produce the bounds (5.1)–(5.3) that are *uniform* in ε. The non-uniformity in T of (5.2) and (5.3) is due to the fact that the magnetic potential is only $\mathrm{AC}_{\mathrm{loc}}$ in time: for AC-potentials such bounds would be uniform in T as well.

Proof of Proposition 5.2. We recall that u_ε satisfies

$$\mathrm{i}\,\partial_t u_\varepsilon \;=\; -(1-\mathrm{i}\,\varepsilon)(\nabla - \mathrm{i}\,A)^2 u_\varepsilon + \mathcal{N}(u_\varepsilon)$$

as an identity at every t between H^{-1}-functions in space.

Let us first prove the thesis in a regular case, and later work out a density argument for the general case.

It is straightforward to see, by means of a customary contraction argument in $L^\infty([0, T], H^s(\mathbb{R}^3))$ for arbitrary $s > 0$, that if $f \in \mathcal{S}(\mathbb{R}^3)$ and $A \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}, \mathcal{S}(\mathbb{R}^3))$, then the solution u_ε to the local Cauchy problem (1.5) is smooth in space, whence in particular $u_\varepsilon \in \mathcal{C}^1([0, T), H^1(\mathbb{R}^3))$, a fact that justifies the time derivations in the computations that follow.

From

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{M}(u_\varepsilon))(t) &= -\,2\,\mathfrak{Re}\int_{\mathbb{R}^3} \overline{u_\varepsilon}\,\big((\mathrm{i}+\varepsilon)(\nabla - \mathrm{i}\,A)^2 u_\varepsilon - \mathrm{i}\,|u_\varepsilon|^{\gamma-1}u_\varepsilon - \mathrm{i}\,(|\cdot|^{-\alpha} * |u_\varepsilon|^2)\,u_\varepsilon\big)\,\mathrm{d}x \\ &= -\,2\varepsilon\int_{\mathbb{R}^3} |(\nabla - \mathrm{i}\,A)u_\varepsilon|^2 dx \;\leqslant\; 0,\end{aligned}$$

one deduces $(\mathcal{M}(u_\varepsilon))(t) \leqslant (\mathcal{M}(u_\varepsilon))(0)$, whence (5.1).

Next, we compute

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E}(u_\varepsilon))(t) = \mathfrak{Re}\int_{\mathbb{R}^3} \Big(& \big((\nabla - \mathrm{i}\, A)\partial_t u_\varepsilon - \mathrm{i}\,(\partial_t A)u_\varepsilon\big)\cdot \overline{(\nabla - \mathrm{i}\, A)u_\varepsilon} \\
& + \big(|u_\varepsilon|^{\gamma-1} + (|x|^{-\alpha} * |u_\varepsilon|^2)\big)\, \overline{u_\varepsilon}\, \partial_t u_\varepsilon \Big)\,\mathrm{d}x \\
= {} & \mathfrak{Re}\int_{\mathbb{R}^3} (\partial_t u_\varepsilon)\big(-\overline{(\nabla - \mathrm{i}\, A)^2 u_\varepsilon} + |u_\varepsilon|^{\gamma-1}\,\overline{u_\varepsilon} + (|x|^{-\alpha} * |u_\varepsilon|^2)\,\overline{u_\varepsilon}\,\big)\,\mathrm{d}x \\
& + \int_{\mathbb{R}^3} A\cdot(\partial_t A)|u_\varepsilon|^2 + (\partial_t A)\cdot \mathfrak{Im}\,(u_\varepsilon \overline{\nabla u_\varepsilon}\,)\,\mathrm{d}x \\
= {} & \varepsilon \int_{\mathbb{R}^3} \big(-|(\nabla - \mathrm{i}\, A)^2 u_\varepsilon|^2 + (\,|u_\varepsilon|^{\gamma-1} + |x|^{-\alpha} * |u_\varepsilon|^2)\,\mathfrak{Re}\,(\overline{u_\varepsilon}(\nabla - \mathrm{i}\, A)^2 u_\varepsilon)\big)\,\mathrm{d}x \\
& + \int_{\mathbb{R}^3} \big(A\cdot(\partial_t A)|u_\varepsilon|^2 + (\partial_t A)\cdot \mathfrak{Im}\,(u_\varepsilon \overline{\nabla u_\varepsilon}\,)\,\mathrm{d}x \\
= {} & -\varepsilon \int_{\mathbb{R}^3} |(\nabla - \mathrm{i}\, A)^2 u_\varepsilon|^2\,\mathrm{d}x - \varepsilon\,\mathcal{R}(u_\varepsilon)(t) + \mathcal{S}(u_\varepsilon)(t),
\end{aligned} \tag{5.5}
$$

where

$$
\begin{aligned}
\mathcal{R}(u_\varepsilon)(t) &:= -\int_{\mathbb{R}^3} (\,|u_\varepsilon|^{\gamma-1} + |x|^{-\alpha} * |u_\varepsilon|^2)\,\mathfrak{Re}\,(\overline{u_\varepsilon}(\nabla - \mathrm{i}\, A)^2 u_\varepsilon)\,\mathrm{d}x \\
\mathcal{S}(u_\varepsilon)(t) &:= \int_{\mathbb{R}^3} \big(A\cdot(\partial_t A)|u_\varepsilon|^2 + (\partial_t A)\cdot \mathfrak{Im}\,(u_\varepsilon \overline{\nabla u_\varepsilon}\,)\,\big)\,\mathrm{d}x.
\end{aligned}
$$

From

$$
\begin{aligned}
\mathcal{R}(u_\varepsilon)(t) = & -\int_{\mathbb{R}^3} (\,|u_\varepsilon|^{\gamma-1} + |x|^{-\alpha} * |u_\varepsilon|^2)\big(-|(\nabla - \mathrm{i}\, A)u_\varepsilon|^2 + \tfrac{1}{2}\Delta|u_\varepsilon|^2\big)\,\mathrm{d}x \\
= & + \int_{\mathbb{R}^3} |u_\varepsilon|^{\gamma-1}|(\nabla - \mathrm{i}\, A)u_\varepsilon|^2\,\mathrm{d}x + (\gamma-1)\int_{\mathbb{R}^3} |u_\varepsilon|^{\gamma-1}|\nabla|u_\varepsilon||^2\,\mathrm{d}x \\
& + \int_{\mathbb{R}^3} (\,|x|^{-\alpha} * |u_\varepsilon|^2)\,|(\nabla - \mathrm{i}\, A)u_\varepsilon|^2\,\mathrm{d}x \\
& + \tfrac{1}{2}\int_{\mathbb{R}^3} (\,|x|^{-\alpha} * \nabla|u_\varepsilon|^2)\,\nabla|u_\varepsilon|^2\,\mathrm{d}x
\end{aligned} \tag{5.6}
$$

we see that

$$
\mathcal{R}(u_\varepsilon)(t) \;\geqslant\; 0. \tag{5.7}
$$

This is obvious for the first three summands in the r.h.s. of (5.6), whereas for the last one, setting $\phi{:=}\nabla|u_\varepsilon|^2$, Plancherel's formula gives

$$
\int_{\mathbb{R}^3} (\,|x|^{-\alpha} * \phi)\,\overline{\phi}\,\mathrm{d}x \;=\; (2\pi)^{\frac{3}{2}} \int_{\mathbb{R}^3} (\widehat{|\cdot|^{-\alpha}})(\xi)\,|\widehat{\phi}(\xi)|^2\,\mathrm{d}\xi\,,
$$

and since $\widehat{|\cdot|^{-\alpha}}$ is positive, the fourth summand too is positive. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E}(u_\varepsilon))(t) \;\leqslant\; \mathcal{S}(u_\varepsilon)(t). \tag{5.8}$$

In order to estimate $\mathcal{S}(u_\varepsilon)(t)$, it is checked by direct inspection that there are $M_1, M_2 \in [2,6]$ such that

$$b_1 * 2 * M_1 \;=\; b_2 * 2 * M_2 \;=\; 1,$$

whence, for every $t \in [0,T)$ and $j \in \{1,2\}$,

$$\begin{aligned}\|u_\varepsilon(t)\|_{L^M_j(\mathbb{R}^3)} \;&\lesssim\; \|u_\varepsilon(t)\|_{H^1(\mathbb{R}^3)} \\ &\lesssim\; \big(1 + \|A_1(t)\|_{L^{b_1}(\mathbb{R}^3)} + \|A_2(t)\|_{L^{b_2}(\mathbb{R}^3)}\big)\,\|u_\varepsilon\|_{H^1_{A(t)}}\end{aligned}$$

(Sobolev's embedding and norm equivalence (2.4)). Thus, by Hölder's inequality,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^3}(\partial_t A(t))\cdot\mathfrak{Im}\,(u_\varepsilon(t)\,\overline{\nabla u_\varepsilon(t)})\,\mathrm{d}x\,\Big| \\ &\quad\lesssim\; \big(\|\partial_t A_1(t)\|_{L^{b_1}(\mathbb{R}^3)} + \|\partial_t A_2(t)\|_{L^{b_2}(\mathbb{R}^3)}\big) \\ &\qquad\times\big(1 + \|A_1(t)\|_{L^{b_1}(\mathbb{R}^3)} + \|A_2(t)\|_{L^{b_2}(\mathbb{R}^3)}\big)^2\,\|u_\varepsilon(t)\|^2_{H^1_{A(t)}(\mathbb{R}^3)} \\ &\quad\leqslant\; \big(\|\partial_t A_1(t)\|_{L^{b_1}(\mathbb{R}^3)} + \|\partial_t A_2(t)\|_{L^{b_2}(\mathbb{R}^3)}\big) \\ &\qquad\times\big(1 + \|A_1(t)\|_{L^{b_1}(\mathbb{R}^3)} + \|A_2(t)\|_{L^{b_2}(\mathbb{R}^3)}\big)^2\,\big(1 + (\mathcal{E}(u_\varepsilon)(t)\big),\end{aligned} \tag{5.9}$$

the last step following from

$$\|u_\varepsilon(t)\|^2_{H^1_{A(t)}} \;\leqslant\; (\mathcal{M}(u_\varepsilon))(t) + (\mathcal{E}(u_\varepsilon))(t) \tag{5.10}$$

and from $(\mathcal{M}(u_\varepsilon))(t) \lesssim 1$. Analogously, now with Hölder exponents $M_{ij} \in [2,6]$ such that

$$b_i * b_j * \tfrac{1}{2}M_{ij} \;=\; 1 \qquad i,j \in \{1,2\},$$

we find

$$\begin{aligned}&\Big|\int_{\mathbb{R}^3} A\cdot(\partial_t A)\,|u_\varepsilon|^2\,\mathrm{d}x\,\Big| \\ &\quad\lesssim\; \big(\|\partial_t A_1(t)\|_{L^{b_1}(\mathbb{R}^3)} + \|\partial_t A_2(t)\|_{L^{b_2}(\mathbb{R}^3)}\big) \\ &\qquad\times\big(\|A_1(t)\|_{L^{b_1}(\mathbb{R}^3)} + \|A_2(t)\|_{L^{b_2}(\mathbb{R}^3)}\big)\,\|u_\varepsilon(t)\|^2_{H^1_{A(t)}(\mathbb{R}^3)} \\ &\quad\leqslant\; \big(\|\partial_t A_1(t)\|_{L^{b_1}(\mathbb{R}^3)} + \|\partial_t A_2(t)\|_{L^{b_2}(\mathbb{R}^3)}\big) \\ &\qquad\times\big(1 + \|A_1(t)\|_{L^{b_1}(\mathbb{R}^3)} + \|A_2(t)\|_{L^{b_2}(\mathbb{R}^3)}\big)\,\big(1 + (\mathcal{E}(u_\varepsilon)(t)\big).\end{aligned} \tag{5.11}$$

Combining (5.8), (5.9) and (5.11) together yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E}(u_\varepsilon))(t) \;&\lesssim\; |\mathcal{S}(u_\varepsilon)(t)| \;\lesssim\; \Lambda(t)\,\big(1 + (\mathcal{E}(u_\varepsilon)(t)\big) \\ \Lambda(t) &:= \big(\|\partial_t A_1(t)\|_{L^{b_1}(\mathbb{R}^3)} + \|\partial_t A_2(t)\|_{L^{b_2}(\mathbb{R}^3)}\big) \\ &\qquad\times\;\big(1 + \|A_1(t)\|_{L^{b_1}(\mathbb{R}^3)} + \|A_2(t)\|_{L^{b_2}(\mathbb{R}^3)}\big).\end{aligned} \tag{5.12}$$

Owing to the assumptions on A, $\Lambda \in L^1_{\mathrm{loc}}(\mathbb{R},\mathrm{d}t)$, therefore Grönwall's lemma is applicable to (5.12) and we deduce

$$(\mathcal{E}(u_\varepsilon))(t) \;\leqslant\; e^{\int_0^t \Lambda(s)\,\mathrm{d}s}\Big((\mathcal{E}(u_\varepsilon))(0) + \int_0^t \Lambda(s)\,\mathrm{d}s\Big) \;\lesssim_{A,T}\; 1,$$

which proves (5.2). Based on (5.10) and on the norm equivalence (2.4), the bounds (5.1) and (5.2) then imply also (5.3).

Let us prove now the a priori bound (5.4). Integrating (5.5) in $t \in [0,T)$ yields

$$\begin{aligned}(\mathcal{E}(u_\varepsilon))(T) &- (\mathcal{E}(u_\varepsilon))(0)\\ &= -\varepsilon\int_0^T\bigg(\int_{\mathbb{R}^3}\big(\,|(\nabla - \mathrm{i}\,A)^2 u_\varepsilon|^2\,\mathrm{d}x + \mathcal{R}(u_\varepsilon)(t)\bigg)\,\mathrm{d}t + \int_0^T \mathcal{S}(u_\varepsilon)(t)\,\mathrm{d}t,\end{aligned}$$

whence

$$\int_0^T \mathcal{R}(u_\varepsilon)(t)\,\mathrm{d}t \;\leqslant\; \frac{1}{\varepsilon}\bigg(\,|(\mathcal{E}(u_\varepsilon))(T) - (\mathcal{E}(u_\varepsilon))(0)| + \int_0^T |\mathcal{S}(u_\varepsilon)(t)|\,\mathrm{d}t\bigg).$$

The bound (5.12) for $|\mathcal{S}(u_\varepsilon)(t)|$ and bound (5.2) for $\mathcal{E}(u_\varepsilon)(t)$, together with the fact that $\Lambda \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathrm{d}t)$, then give

$$\int_0^T \mathcal{R}(u_\varepsilon)(t)\,\mathrm{d}t \;\lesssim_{A,T}\; \varepsilon^{-1}. \tag{5.13}$$

It is clear from (5.6) that the l.h.s. of the a priori bound (5.4) is controlled by $\int_0^T \mathcal{R}(u_\varepsilon)(t)\mathrm{d}t$; therefore, (5.13) implies (5.4).

This completes the proof under the additional assumption that $f \in \mathcal{S}(\mathbb{R}^3)$ and $A \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}, \mathcal{S}(\mathbb{R}^3))$. The proof in the general case of non-smooth potentials and non-smooth initial data follows by a density argument. We consider a sequence of regular potentials A_n and regular initial data f_n such that $f_n \to f$ in $H^1(\mathbb{R}^3)$ and $\|A_n - A\|_{\mathcal{A}_1} \to 0$ when $A \in \mathcal{A}_1$, or $\|A_n - A\|_{\mathcal{A}_2} \to 0$ when $A \in \mathcal{A}_2$, and we denote by $u_{\varepsilon,n}$ the solution to local Cauchy problem (1.5) with initial datum f_n and magnetic potential A_n.

Having already established Proposition 5.2 for such regular initial data and potentials, the bounds

$$\|u_{\varepsilon,n}\|_{L^\infty([0,T);L^2(\mathbb{R}^3))} \;\lesssim\; 1 \tag{5.14}$$

$$\|u_{\varepsilon,n}\|_{L^\infty([0,T);H^1(\mathbb{R}^3))} \;\lesssim_{A,T}\; 1 \tag{5.15}$$

hold for every n uniformly in $\varepsilon > 0$. The latter fact, together with the stability property

$$\|u_{n,\varepsilon} - u_\varepsilon\|_{L^\infty[0,T),H^1(\mathbb{R}^3))} \to 0 \qquad \text{uniformly in } \varepsilon \tag{5.16}$$

given by Proposition 4.3, then implies (5.1) and (5.3) also in the general case. Analogously, since for fixed t the mass $\mathcal{M}(u)(t)$ and the energy $\mathcal{E}(u)(t)$ depend continuously on the H^1-norm of $u(t)$, (5.16) also implies (5.1) and (5.2) in the general case.

We are left to prove the energy a priori bound (5.4). We first collect some useful facts, valid for a generic Strichartz pair (q,r), with $r \in [2,3)$. The starting point is the stability result proved in Proposition 4.3, which in this case reads

$$u_{n,\varepsilon} \;\longrightarrow\; u_\varepsilon \quad \text{in } L^q([0,T), W^{1,r}(\mathbb{R}^3)). \tag{5.17}$$

In particular,

$$u_{n,\varepsilon} \;\longrightarrow\; u_\varepsilon \quad \text{in } L^q([0,T), L^{\frac{Mr}{M-r}}(\mathbb{R}^3)), \quad M \in [3,+\infty], \tag{5.18}$$

$$\nabla u_{n,\varepsilon} \;\longrightarrow\; \nabla u_\varepsilon \;\; \text{in } L^q([0,T), L^r(\mathbb{R}^3)). \tag{5.19}$$

Moreover, the following identity is trivially satisfied (recall that $b_i > 3$):

$$(+\infty, b_i) * \left(q, \frac{b_i r}{b_i - r}\right) = (q, r), \quad i \in \{1, 2\}. \tag{5.20}$$

Now, (5.18) and Hölder's inequality yield

$$Au_{n,\varepsilon} \longrightarrow Au_\varepsilon \quad \text{in } L^q([0,T), L^r(\mathbb{R}^3)), \tag{5.21}$$

and (5.19) and (5.21) yield

$$|(\nabla - \mathrm{i}A)u_{n,\varepsilon}|^2 \longrightarrow |(\nabla - \mathrm{i}A)u_\varepsilon|^2 \quad \text{in } L^{\frac{q}{2}}([0,T), L^{\frac{r}{2}}(\mathbb{R}^3)). \tag{5.22}$$

We show now how to prove estimate (5.4) in the general case. Having already established Proposition 5.2 for regular initial data and potentials, we have in particular

$$\big\| u_{n,\varepsilon}^{\gamma-1} |(\nabla - \mathrm{i}A)u_{n,\varepsilon}|^2 \big\|_{L^1([0,T),L^1(\mathbb{R}^3))} \lesssim_{A,T} \varepsilon^{-1}, \tag{5.23}$$

$$\big\| (|x|^{-\alpha} * |u_{n,\varepsilon}|^2) |(\nabla - \mathrm{i}A)u_{n,\varepsilon}|^2 \big\|_{L^1([0,T),L^1(\mathbb{R}^3))} \lesssim_{A,T} \varepsilon^{-1}, \tag{5.24}$$

$$\big\| (|x|^{-\alpha} * \nabla|u_{n,\varepsilon}|^2) \nabla|u_{n,\varepsilon}|^2 \big\|_{L^1([0,T),L^1(\mathbb{R}^3))} \lesssim_{A,T} \varepsilon^{-1}. \tag{5.25}$$

For any $\gamma \in (1, 5]$, we can find Strichartz pairs (q_1, r_1) and (q_2, r_2), with $r_1, r_2 \in [2, 3)$, such that

$$\left(\tfrac{q_1}{\gamma-1}, \tfrac{3r_1}{(3-r_1)(\gamma-1)}\right) * \left(\tfrac{q_2}{2}, \tfrac{r_2}{2}\right) = (1, 1).$$

Then, (5.18), (5.22) and Hölder's inequality yield

$$u_{n,\varepsilon}^{\gamma-1} |(\nabla - \mathrm{i}A)u_{n,\varepsilon}|^2 \longrightarrow u_\varepsilon^{\gamma-1} |(\nabla - \mathrm{i}A)u_\varepsilon|^2 \quad \text{in } L^1([0,T), L^1(\mathbb{R}^3)), \tag{5.26}$$

which together with bound (5.23) implies

$$\big\| u_\varepsilon^{\gamma-1} |(\nabla - \mathrm{i}A)u_\varepsilon|^2 \big\|_{L^1([0,T),L^1(\mathbb{R}^3))} \lesssim_{A,T} \varepsilon^{-1}. \tag{5.27}$$

In turn, the diamagnetic inequality $|\nabla|g|| \leqslant |(\nabla - \mathrm{i}A)g|$ and (5.27) give also

$$\big\| u_\varepsilon^{\gamma-1} |\nabla|u_\varepsilon||^2 \big\|_{L^1([0,T),L^1(\mathbb{R}^3))} \lesssim_{A,T} \varepsilon^{-1}. \tag{5.28}$$

Concerning the convolution terms, for any $\alpha \in (0, 3)$ we can find Strichartz pairs $(\widetilde{q}_1, \widetilde{r}_1)$ and $(\widetilde{q}_2, \widetilde{r}_2)$, with $\widetilde{r}_1, \widetilde{r}_2 \in [2, 3)$, such that

$$\begin{aligned}
&\int_0^T \!\! \int_{\mathbb{R}^3} (|x|^{-\alpha} * |u_{n,\varepsilon}|^2) |(\nabla - \mathrm{i}A)u_{n,\varepsilon}|^2 \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad \lesssim \|u\|_{L^{\frac{2}{q}}([0,T),L^{\frac{3\widetilde{r}_1}{2(3-\widetilde{r}_1)}}(\mathbb{R}^3))} \|(\nabla - \mathrm{i}A)u_{n,\varepsilon}\|_{L^{\frac{\widetilde{q}_2}{2}}([0,T),L^{\frac{\widetilde{r}_2}{2}}(\mathbb{R}^3))},
\end{aligned}$$

which is obtained by the Hardy–Littlewood–Sobolev and Hölder's inequality. Therefore,

$$\begin{aligned}
(|x|^{-\alpha} * |u_{n,\varepsilon}|^2) |(\nabla - \mathrm{i}A)u_{n,\varepsilon}|^2 &\longrightarrow (|x|^{-\alpha} * |u_\varepsilon|^2) |(\nabla - \mathrm{i}A)u_\varepsilon|^2 \\
&\quad \text{in } L^1([0,T) L^1(\mathbb{R}^3)),
\end{aligned} \tag{5.29}$$

which together with bound (5.24) implies

$$\big\| (|x|^{-\alpha} * |u_\varepsilon|^2) |(\nabla - \mathrm{i}A)u_\varepsilon|^2 \big\|_{L^1([0,T),L^1(\mathbb{R}^3))} \lesssim_{A,T} \varepsilon^{-1}. \tag{5.30}$$

In analogous manner, using Hardy–Littlewood–Sobolev and Hölder's inequality, from (5.18) and (5.19) we get

$$\begin{aligned}
(|x|^{-\alpha} * \nabla|u_{n,\varepsilon}|^2) \nabla|u_{n,\varepsilon}|^2 &\longrightarrow (|x|^{-\alpha} * \nabla|u_\varepsilon|^2) \nabla|u_\varepsilon|^2 \\
&\quad \text{in } L^1([0,T) L^1(\mathbb{R}^3)),
\end{aligned} \tag{5.31}$$

which together with bound (5.25) implies

$$\big\|(|x|^{-\alpha} * \nabla|u_\varepsilon|^2)\nabla|u_\varepsilon|^2\big\|_{L^1([0,T),L^1(\mathbb{R}^3))} \lesssim_{A,T} \varepsilon^{-1}. \tag{5.32}$$

A priori abound (5.4) in the general case follows by combining (5.27), (5.28), (5.30) and (5.32). □

Remark 5.4. Inequality (5.10), namely

$$\|u_\varepsilon(t)\|^2_{H^1_{A(t)}} \leqslant (\mathcal{M}(u_\varepsilon))(t) + (\mathcal{E}(u_\varepsilon))(t), \qquad t \in [0,T), \tag{5.33}$$

reflects the *defocusing* structure of regularised magnetic NLS (1.4).

6. Global existence for the regularised equation

In this section, we exploit the a priori estimates for mass and energy so as to prove that the local solution to the regularised Cauchy problem (1.5), constructed in Sect. 4, can be actually extended globally in time.

We discuss first the result in the energy sub-critical case.

Theorem 6.1. (Global well-posedness, energy sub-critical case) *Assume that $A \in \mathcal{A}_1$ or $A \in \mathcal{A}_2$ and that the exponents in nonlinearity (1.2) are in the regime $\gamma \in (1,5)$ and $\alpha \in (0,3)$. Let $\varepsilon > 0$. Then, the regularised nonlinear magnetic Schrödinger Eq. (1.4) is globally well-posed in $H^1(\mathbb{R}^3)$, in the sense of Definitions 2.6 and 2.7. Moreover, the solution u_ε to (1.4) with given initial datum $f \in H^1(\mathbb{R}^3)$ satisfies the bound*

$$\|u_\varepsilon\|_{L^\infty[0,T],H^1(\mathbb{R}^3)} \lesssim_T 1 \qquad \forall T \in (0,+\infty), \tag{6.1}$$

uniformly in $\varepsilon > 0$.

Proof. The local well-posedness is proved in Proposition 4.1. Because of (5.3), the H^1-norm of u_ε is bounded on finite intervals of time. Therefore, by the blow-up alternative, the solution is necessarily global, and in particular, it satisfies the bound (6.1). □

We discuss now the analogous result in the energy critical case.

Theorem 6.2. (Global existence and uniqueness, energy critical case) *Assume that $A \in \mathcal{A}_1$ or $A \in \mathcal{A}_2$ and that the exponents in nonlinearity (1.2) are in the regime $\gamma = 5$ and $\alpha \in (0,3)$. Let $\varepsilon > 0$ and $f \in H^1(\mathbb{R}^3)$. The Cauchy problem (1.5) has a unique global strong H^1-solution u_ε, in the sense of Definition 2.6. Moreover, u satisfies the bound*

$$\|u_\varepsilon\|_{L^\infty[0,T],H^1(\mathbb{R}^3)} \lesssim_T 1 \qquad \forall T \in (0,+\infty), \tag{6.2}$$

uniformly in $\varepsilon > 0$.

Proof. The existence of a unique local solution u_ε is proved in Proposition 4.2. The a priori bound (5.4) implies that

$$\int_0^T\!\!\int_{\mathbb{R}^3} \big(|u_\varepsilon|^2\,\nabla|u_\varepsilon|\big)^2\,\mathrm{d}x\,\mathrm{d}t \lesssim \varepsilon^{-1},$$

which, together with Sobolev's embedding, yields

$$\begin{aligned}\|u_\varepsilon\|^6_{L^6([0,T],L^{18}(\mathbb{R}^3))} &= \|u_\varepsilon^3\|^2_{L^2([0,T],L^6(\mathbb{R}^3))} \lesssim \int_0^T\int_{\mathbb{R}^3} |\nabla|u_\varepsilon|^3|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\lesssim \int_0^T\int_{\mathbb{R}^3} |u_\varepsilon|^4\,|\nabla|u_\varepsilon||^2\,\mathrm{d}x\,\mathrm{d}t \lesssim \varepsilon^{-1} < +\infty.\end{aligned} \tag{6.3}$$

Owing to (6.3) and to the blow-up alternative proved in Proposition 4.2, we conclude that the solution u can be extended globally, and moreover, using again (5.3), it satisfies the bound (6.2). □

Remark 6.3. As anticipated in Introduction, right after stating the assumptions on the magnetic potential, let us comment here about the fact that in the *mass sub-critical* regime ($\gamma \in (1, \frac{7}{3})$ and $\alpha \in (0,2)$) we can work with the larger class $\widetilde{\mathcal{A}}_1$ instead of $\mathcal{A}_1$ and still prove the extension of the local solution globally in time with finite H^1-norm on arbitrary finite time interval. This is due to the fact that, for a potential $u \in \widetilde{\mathcal{A}}_1$ and in the mass sub-critical regime, in order to extend the solution globally neither need we estimate (5.3) as in the proof of Theorem 6.1, nor need we estimate (5.4) as in the proof of Theorem 6.2. Indeed, we can first prove local well-posedness in $L^2(\mathbb{R}^3)$ for the regularised magnetic NLS (1.4), using a fixed point argument based on the space–time estimates for the heat-Schrödinger flow, in the very same spirit of the proof of Theorem 3.1. Then, we can extend such a solution globally in time using only the mass a priori bound (5.1), for proving such a bound does not require any time-regularity assumption on the magnetic potential. Moreover, since the nonlinearities are mass sub-critical and since we can prove convenient estimates on the commutator $[\nabla, (\nabla - \mathrm{i}\,A)^2]$ when $\max\{b_1, b_2\} \in (3,6)$, we can show that the global L^2-solution exhibits *persistence of* H^1*-regularity* in the sense that it stays in $H^1(\mathbb{R}^3)$ for every positive time provided that the initial datum belongs already to $H^1(\mathbb{R}^3)$. This way, we obtain existence and uniqueness of one global strong H^1-solution.

7. Removing the regularisation

In this section, we prove our main Theorem 1.2. The proof is based on a compactness argument that we develop in Sect. 7.1, so as to remove the ε-regularisation, and leads to a *local* weak H^1-solution to (1.3).

The reason why by compactness we can only produce local solutions is merely due to the local-in-time regularity of magnetic potentials belonging to the class $\mathcal{A}_1$ or $\mathcal{A}_2$—*globally*-in-time regular potentials, say, $AC(\mathbb{R})$-potentials, would instead allow for a direct removal of the regularisation globally in time.

In order to circumvent this simple obstruction, in Sect. 7.2 we work out a straightforward “gluing” argument, eventually proving Theorem 1.2.

7.1. Local weak solutions

The main result of this subsection is the following.

Proposition 7.1. *Assume that $A \in \mathcal{A}_1$ or $A \in \mathcal{A}_2$ and that the exponents in the nonlinearity (1.2) are in the whole regime $\gamma \in (1,5]$ and $\alpha \in (0,3)$. Let $T > 0$, and $f \in H^1(\mathbb{R}^3)$. For any sequence $(\varepsilon_n)_n$ of positive numbers with $\varepsilon_n \downarrow 0$, let u_n be the unique global strong H^1-solution to Cauchy problem (1.5) with viscosity parameter $\varepsilon = \varepsilon_n$ and with initial datum f, as provided by Theorem 6.1 in the energy sub-critical case and by Theorem 6.2 in the energy critical case. Then, up to a subsequence, u_n converges weakly-$*$ in $L^\infty([0,T], H^1(\mathbb{R}^3))$ to a local weak H^1-solution u to the magnetic NLS (1.1) in the time interval $[0,T]$ and with initial datum f.*

In order to set up the compactness argument that proves Proposition 7.1, we need a few auxiliary results, as follows.

Lemma 7.2. *The sequence* $(u_n)_n$ *in the assumption of Proposition 7.1 is bounded in* $L^\infty([0,T],H^1(\mathbb{R}^3))$, *i.e.*

$$\|u_n\|_{L^\infty([0,T],H^1(\mathbb{R}^3))} \lesssim_{A,T} 1, \tag{7.1}$$

and hence, up to a subsequence, $(u_n)_n$ *admits a weak-$*$ limit* u *in* $L^\infty([0,T],H^1(\mathbb{R}^3))$.

Proof. An immediate consequence of uniform-in-ε bounds (6.1) and (6.2) and the Banach–Alaoglu theorem. □

Lemma 7.3. *For the sequence* $(u_n)_n$ *in the assumption of Proposition 7.1 there exist indices* $p_i, p_{ij} \in [\frac{6}{5}, 2]$, $i,j \in \{1,2\}$, *such that*

$$(A_i \cdot \nabla u_n)_n \text{ is a bounded sequence in } L^\infty([0,T],L^{p_i}(\mathbb{R}^3)), \tag{7.2}$$

$$(A_i \cdot A_j u_n)_n \text{ is a bounded sequence in } L^\infty([0,T],L^{p_{ij}}(\mathbb{R}^3)). \tag{7.3}$$

Proof. For $p_i{:=}b_i * 2 \in [\frac{6}{5}, 2]$, $i \in \{1,2\}$, the bound (7.1) and Hölder's inequality give

$$\begin{aligned}\|A_i \cdot \nabla u_n\|_{L^\infty([0,T],L^{p_i}(\mathbb{R}^3))} &\lesssim \|A_i\|_{L^\infty([0,T],L^{b_i}(\mathbb{R}^3))}\|\nabla u_n\|_{L^\infty([0,T],L^2(\mathbb{R}^3))}\\ &\lesssim_{A,T} 1,\end{aligned}$$

which proves (7.2). Moreover, there exist $M_{ij} \in [2,6]$, $i,j \in \{1,2\}$, such that $p_{ij}{:=}b_i * b_j * M_{ij} \in [\frac{6}{5}, 2]$; therefore, the bound (7.1), Hölder's inequality and Sobolev's embedding give

$$\begin{aligned}&\|A_i \cdot A_j u_n\|_{L^\infty([0,T],L^{p_{ij}}(\mathbb{R}^3))}\\ &\lesssim \|A_i\|_{L^\infty([0,T],L^{b_i}(\mathbb{R}^3))}\|A_j\|_{L^\infty([0,T],L^{b_j}(\mathbb{R}^3))}\|u_n\|_{L^\infty([0,T],L^{M_{ij}}(\mathbb{R}^3))} \lesssim_{A,T} 1,\end{aligned}$$

which proves (7.3). □

Lemma 7.4. *For the sequence* $(u_n)_n$ *in the assumption of Proposition 7.1, and for every* $\gamma \in (1,5]$ *and* $\alpha \in (1,3)$, *there exist indices* $p(\gamma), \widetilde{p}(\alpha) \in [\frac{6}{5}, 2]$ *such that*

$$\big(|u_n|^{\gamma-1}u_n\big)_n \text{ is a bounded sequence in } L^\infty([0,T],L^{p(\gamma)}(\mathbb{R}^3)), \tag{7.4}$$

$$\big((\,|\cdot|^{-\alpha} * |u_n|^2)u_n\big)_n \text{ is a bounded sequence in } L^\infty([0,T],L^{\widetilde{p}(\alpha)}(\mathbb{R}^3)). \tag{7.5}$$

Proof. For any $\gamma \in (1,5]$, there exists $M{:=}M(\gamma) \in [2,6]$ such that $M/\gamma \in [\frac{6}{5}, 2]$, whence

$$\begin{aligned}\||u_n|^{\gamma-1}u\|_{L^\infty([0,T],L^{M/\gamma}(\mathbb{R}^3))} &\leqslant \|u_n\|^\gamma_{L^\infty([0,T],L^M(\mathbb{R}^3))}\\ &\lesssim \|u_n\|^\gamma_{L^\infty([0,T],H^1(\mathbb{R}^3))} \lesssim_{A,T} 1,\end{aligned}$$

based on the bound (7.1) and Sobolev's embedding, which proves (7.4), with $p(\gamma){:=}M/\gamma$. Next, let us use the Hardy–Littlewood–Sobolev inequality, for $m(\alpha) \in (1, \frac{3}{3-\alpha})$ and $g \in L^{m(\alpha)}(\mathbb{R}^3)$,

$$\big\|\,|\cdot|^{-\alpha} * g\,\big\|_{L^{q(m(\alpha))}(\mathbb{R}^3)} \lesssim \|g\|_{L^{m(\alpha)}(\mathbb{R}^3)}, \qquad q(m){:=}\tfrac{3m(\alpha)}{3-(3-\alpha)m(\alpha)}.$$

Taking

$$\begin{aligned}m(\alpha) &\in (1, \tfrac{3}{3-\alpha}) &&\text{if } \alpha \in (0,2]\\ m(\alpha) &\in (1, 3] &&\text{if } \alpha \in (2,3),\end{aligned} \tag{7.6}$$

the Hardy–Littlewood–Sobolev inequality above and Sobolev's embedding yield

$$\begin{aligned}\big\|\,|\cdot|^{-\alpha} * |u|^2\big\|_{L^\infty([0,T],L^{q(m(\alpha))}(\mathbb{R}^3))} &\lesssim \|u^2\|_{L^\infty([0,T],L^{m(\alpha)}(\mathbb{R}^3))}\\ &\lesssim \|u\|^2_{L^\infty([0,T],H^1(\mathbb{R}^3))}.\end{aligned} \tag{7.7}$$

Since $\frac{3}{4-\alpha} < 1$ for $\alpha \in (0,3)$, we can find $m(\alpha)$ that satisfies (7.6) as well as $q(m(\alpha)) * 2 \in [\frac{6}{5}, 2]$, namely

$$m(\alpha) \in \left(\frac{3}{4-\alpha}, \frac{3}{3-\alpha}\right). \tag{7.8}$$

As a consequence, for $\widetilde{p}(\alpha):=q(m(\alpha))*2\in[\frac{6}{5},2]$ one has

$$\begin{aligned}&\big\|(\,|\cdot|^{-\alpha}*|u_n|^2)u_n\big\|_{L^\infty([0,T],L^{\widetilde{p}(\alpha)}(\mathbb{R}^3))}\\&\lesssim\;\big\|\,|\cdot|^{-\alpha}*|u_n|^2\big\|_{L^\infty([0,T],L^{q(m(\alpha))}(\mathbb{R}^3))}\,\|u_n\|_{L^\infty([0,T],L^2(\mathbb{R}^3))}\\&\lesssim_{A,T}\;\|u_n\|_{L^\infty([0,T],H^1(\mathbb{R}^3))}\;\lesssim_{A,T}\;1,\end{aligned}$$

based on Hölder's inequality (first step), bound (7.7) (second step), and Sobolev's embedding (third step), which proves (7.5). □

Corollary 7.5. *For the sequence $(u_n)_n$ in the assumption of Proposition 7.1 there exist indices p_i, p_{ij}, $p(\gamma)$, and $\widetilde{p}(\alpha)$ in $[\frac{6}{5},2]$, and there exists functions $X_i\in L^\infty([0,T],L^{p_i}(\mathbb{R}^3))$, $Y_{ij}\in L^\infty([0,T],L^{p_{ij}}(\mathbb{R}^3))$, $N_1\in L^\infty([0,T],L^{p(\gamma)}(\mathbb{R}^3))$, and $N_2\in L^\infty([0,T],L^{\widetilde{p}(\alpha)}(\mathbb{R}^3))$ such that*

$$A_i\cdot\nabla u_n\rightharpoonup X_i\qquad\textit{weakly-}*\textit{ in }L^\infty([0,T],L^{p_i}(\mathbb{R}^3))\tag{7.9}$$

$$A_i\cdot A_j u_n\rightharpoonup Y_{ij}\qquad\textit{weakly-}*\textit{ in }L^\infty([0,T],L^{p_{ij}}(\mathbb{R}^3))\tag{7.10}$$

$$|u_n|^{\gamma-1}u_n\rightharpoonup N_1\qquad\textit{weakly-}*\textit{ in }L^\infty([0,T],L^{p(\gamma)}(\mathbb{R}^3))\tag{7.11}$$

$$(|\cdot|^{-\alpha}*|u_n|^2)u_n\rightharpoonup N_2\qquad\textit{weakly-}*\textit{ in }L^\infty([0,T],L^{\widetilde{p}(\alpha)}(\mathbb{R}^3)).\tag{7.12}$$

Proof. An immediate consequence of Lemmas 7.3 and 7.4, using the Banach–Alaoglu theorem. □

Lemma 7.6. *For the sequence $(u_n)_n$ in the assumption of Proposition 7.1, for the corresponding weak limit u identified in Lemma 7.2, and for the exponents p_i, $i\in\{1,2\}$ identified in Corollary 7.5, one has*

$$A_i\cdot\nabla u_n\;\rightarrow\;A_i\cdot\nabla u\qquad\textit{weakly in }L^2([0,T],L^{p_i}(\mathbb{R}^3)).\tag{7.13}$$

Proof. Because of the bound (7.1), up to a subsequence

$$\nabla u_n\rightarrow\nabla u\qquad\text{weakly in }L^2([0,T],L^2(\mathbb{R}^3)).$$

Now, since $p_i=b_i*2$ and hence $p_i'*b_i=2$, and since $A_i\in L^\infty([0,T]L^{b_i}(\mathbb{R}^3))$, one has $A_i\eta\in L^2([0,T],L^2(\mathbb{R}^3))$ for any $\eta\in L^2([0,T],L^{p_i'}(\mathbb{R}^3))$. Then,

$$\int_0^T\!\!\int_{\mathbb{R}^3}A_i\cdot(\nabla u_n-\nabla u)\overline{\eta}\,\mathrm{d}x\,\mathrm{d}t\;=\;\int_0^T\!\!\int_{\mathbb{R}^3}(\nabla u_n-\nabla u)A_i\overline{\eta}\,\mathrm{d}x\,\mathrm{d}t\;\rightarrow\;0,$$

thus concluding the proof. □

Lemma 7.7. *Let Ω be an open, bounded subset of $\mathbb{R}^3$ and let $M\in[1,+\infty]$. For the sequence $(u_n)_n$ in the assumption of Proposition 7.1, and for the corresponding weak limit u identified in Lemma 7.2,*

$$u_n|_\Omega\rightarrow u|_\Omega\qquad\textit{strongly in }L^M([0,T],L^4(\Omega)).\tag{7.14}$$

Proof. Because of (7.1), $(u_n|)_n$ is a bounded sequence in $L^M([0,T],H^1(\mathbb{R}^3))$ for any $M\in[1,+\infty]$. Moreover, for every time $t\in[0,T]$ u_n satisfies

$$\mathrm{i}\,\partial_t u_n\;=\;-(1-\mathrm{i}\,\varepsilon)(\Delta u_n-2\,\mathrm{i}\,A\cdot\nabla u_n-|A|^2u_n)+\mathcal{N}(u_n)$$

as an identity between H^{-1}-functions. Hence, owing to estimate (2.2) and to the boundedness of the map $\mathcal{N}(u):H^1(\mathbb{R}^3)\rightarrow H^{-1}(\mathbb{R}^3)$,

$$\begin{aligned}&\|\partial_t u_n\|_{L^\infty([0,T],H^{-1}(\mathbb{R}^3))}\\&\lesssim\;\|(\nabla-\mathrm{i}\,A)^2u_n\|_{L^\infty([0,T],H^{-1}(\mathbb{R}^3))}+\|\mathcal{N}(u_n)\|_{L^\infty([0,T],H^{-1}(\mathbb{R}^3))}\\&\lesssim_A\;\|u_n\|_{L^\infty([0,T],H^1(\mathbb{R}^3))}\;\lesssim_{A,T}\;1.\end{aligned}\tag{7.15}$$

In particular,

$$\|\partial_t u_n|_\Omega\|_{L^1([0,T],H^{-1}(\Omega))} \lesssim_{A,T} 1. \tag{7.16}$$

Therefore, (7.14) follows as an application of Aubin–Lions compactness lemma (see e.g. [38, Section 7.3]) to bound (7.16) and with respect to the compact inclusion $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and the continuous inclusion $L^4(\Omega) \hookrightarrow H^{-1}(\Omega)$. □

Lemma 7.8. *For the limit function u identified in Lemma 7.2 and for the limit functions X_i, Y_{ij} and N_i identified in Corollary 7.5, one has the pointwise identities for $t \in [0,T]$ and a.e. $x \in \mathbb{R}^3$:*

$$A_i \cdot \nabla u = X_i \tag{7.17}$$
$$A_i \cdot A_j u = Y_{ij} \tag{7.18}$$
$$|u|^{\gamma-1} u = N_1 \tag{7.19}$$
$$\big(\,|\cdot|^{-\alpha} * |u|^2\big) u = N_2. \tag{7.20}$$

Proof. For the sequence $(u_n)_n$ in the assumption of Proposition 7.1, and for the exponents p_i, $i \in \{1,2\}$ identified in Corollary 7.5, one has

$$A_i \cdot \nabla u_n \;\to\; A_i \cdot \nabla u \qquad \text{weakly in } L^2([0,T], L^{p_i}(\mathbb{R}^3)). \tag{7.21}$$

Indeed, because of bound (7.1), up to a subsequence

$$\nabla u_n \to \nabla u \qquad \text{weakly in } L^2([0,T], L^2(\mathbb{R}^3))\,;$$

therefore, since $p_i = b_i * 2$ and hence $p_i' * b_i = 2$, and since $A_i \in L^\infty([0,T]L^{b_i}(\mathbb{R}^3))$, one has $A_i\eta \in L^2([0,T], L^2(\mathbb{R}^3))$ for any $\eta \in L^2([0,T], L^{p_i'}(\mathbb{R}^3))$,

$$\int_0^T\!\!\int_{\mathbb{R}^3} A_i \cdot (\nabla u_n - \nabla u)\overline{\eta}\,\mathrm{d}x\,\mathrm{d}t \;=\; \int_0^T\!\!\int_{\mathbb{R}^3} (\nabla u_n - \nabla u)A_i\overline{\eta}\,\mathrm{d}x\,\mathrm{d}t \;\to\; 0.$$

Limits (7.9) and (7.13) imply

$$\int_0^T\!\!\int_{\mathbb{R}^3} \big(A_i \cdot \nabla u_n - A_i \cdot \nabla u\big)\,\varphi\,\mathrm{d}x\,\mathrm{d}t \;\to\; 0$$
$$\int_0^T\!\!\int_{\mathbb{R}^3} \big(A_i \cdot \nabla u_n - X_i\big)\,\varphi\,\mathrm{d}x\,\mathrm{d}t \;\to\; 0$$

for arbitrary $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^3)$, whence the pointwise identity (7.17). Let now Ω be an open and bounded subset of $\mathbb{R}^3$, and let $M \in [1,+\infty]$. Since, as seen in (7.14), $u_n|_\Omega$ converges to $u|_\Omega$ in $L^M([0,T], L^4(\Omega))$, then up to a subsequence one has also pointwise convergence, whence

$$A_i \cdot A_j u_n|_\Omega \to A_i \cdot A_j u|_\Omega \tag{7.22}$$
$$|u_n|^{\gamma-1} u_n|_\Omega \to |u|^{\gamma-1} u|_\Omega \tag{7.23}$$
$$\big(\,|\cdot|^{-\alpha} * |u_n|^2\big) u_n|_\Omega \to \big(\,|\cdot|^{-\alpha} * |u|^2\big) u|_\Omega \tag{7.24}$$

pointwise for $t \in [0,T]$ and a.e. $x \in \Omega$. Therefore, (7.18), (7.19) and (7.20) follow by the uniqueness of the pointwise limit and the arbitrariness of Ω, combining, respectively, (7.10), (7.11) and (7.12) with, respectively, (7.22), (7.23) and (7.24). □

With the material collected so far, we can complete the argument for the removal of the parabolic regularisation, locally in time.

Proof of Proposition 7.1. We want to show that the function u identified in Lemma 7.2 is actually a local weak H^1-solution, in the sense of Definition 2.6 to the magnetic NLS (1.1) with initial datum f in the time interval $[0,T]$. All the exponents p_i, p_{ij}, $p(\gamma)$ and $\widetilde{p}(\alpha)$ identified in Corollary 7.5 belong to the interval $\left[\frac{6}{5},2\right]$, and then by Sobolev's embedding the functions $X_i = A_i \cdot \nabla u$, $Y_{ij} = A_i \cdot A_j u$, $N_1 = |u|^{\gamma-1}u$ and $N_2 = (|\cdot|^{-\alpha} * u^2)u$ discussed in Corollary 7.5 and Lemma 7.8 all belong to $H^{-1}(\mathbb{R}^3)$, and so too does Δu, obviously. Therefore, (1.1) is satisfied by u as an identity between H^{-1}-functions. As a consequence, one can repeat the argument used to derive estimate (7.15), whence $\partial_t u \in L^\infty([0,T], H^{-1}(\mathbb{R}^3))$. Thus, $u \in W^{1,\infty}([0,T], H^{-1}(\mathbb{R}^3))$. On the other hand, $u_n \in C^1([0,T], H^{-1}(\mathbb{R}^3))$, and Lemma 7.2 implies

$$\int_0^T\!\!\int_{\mathbb{R}^3} \eta(t,x)\big(u_n(t,x) - u(t,x)\big)\,\mathrm{d}x\,\mathrm{d}t \;\to\; 0 \qquad \forall \eta \in L^1([0,T], H^{-1}(\mathbb{R}^3).$$

For $\eta(t,x) = \delta(t-t_0,x)\varphi(x)$, where t_0 is arbitrary in $[0,T]$ and φ is arbitrary in $L^2(\mathbb{R}^3)$, the limit above reads $u_n(t_0,\cdot) \to u(t_0,\cdot)$ weakly in $L^2(\mathbb{R}^3)$, whence $u(0,\cdot) = f(\cdot)$. □

7.2. Proof of the main theorem

It is already evident at this stage that had we assumed the magnetic potential to be an AC function for all times, then the proof of the existence of a global weak solution with finite energy would be completed with the proof of Proposition 7.1 above, in full analogy with the scheme of the work [23] we mentioned in Introduction.

Our potential being in general only $\mathrm{AC}_{\mathrm{loc}}$ in time, we cannot appeal to bounds that are uniform in time (indeed, our (6.1) and (6.2) are T-dependent), and the following straightforward strategy must be added in order to complete the proof of our main result.

Proof of Theorem 1.2. We set $T = 1$ and we choose an arbitrary sequence $(\varepsilon_n)_n$ of positive numbers with $\varepsilon_n \downarrow 0$. Let u_n be the unique local strong H^1-solution to regularised magnetic NLS (1.4) with viscosity parameter $\varepsilon = \varepsilon_n$ and with initial datum $f \in H^1(\mathbb{R}^3)$. By Proposition 7.1, there exists a subsequence $(\varepsilon_{n'})_{n'}$ of $(\varepsilon_n)_n$ such that $u_{n'} \to u_1$ weakly-$*$ in $L^\infty([0,1], H^1(\mathbb{R}^3))$, where u_1 is a local weak H^1-solution to the magnetic NLS (1.1) with $u_1(0) = f$. If we take instead $T = 2$ and repeat the argument, we find a subsequence $(\varepsilon_{n''})_{n''}$ of $(\varepsilon_{n'})_{n'}$ such that $u_{n''} \to u_2$ weakly-$*$ in $L^\infty([0,2], H^1(\mathbb{R}^3))$, where u_2 is a local weak H^1-solution to (1.1) with $u_2(0) = f$, now in the time interval $[0,2]$. Moreover, having refined the $u_{n'}$'s in order to obtain the $u_{n''}$'s, necessarily $u_2(t) = u_1(t)$ for $t \in [0,1]$. Iterating this process, we construct for any $N \in \mathbb{N}$ a function u_N which is a local weak H^1-solution to (1.1) in the time interval $[0,N]$, with $u_N(0) = f$ and $u_N(t) = u_{N-1}(t)$ for $t \in [0,N-1]$. It remains to define

$$u(t,x) := u_N(t,x) \qquad x \in \mathbb{R}^3, \quad t \in [0,+\infty) \quad N = [t].$$

Since $u_N \in L^\infty([0,N], H^1(\mathbb{R}^3)) \cap W^{1,\infty}([0,N], H^{-1}(\mathbb{R}^3))$ for every $N \in N$, such u turns out to be a global weak H^1-solution to (1.3) with finite energy for a.e. $t \in \mathbb{R}$, uniformly on compact time intervals. □

References

[1] Antonelli, P., d'Amico, M., Marcati, P.: Nonlinear Maxwell-Schrödinger system and quantum magneto-hydrodynamics in 3-D. Commun. Math. Sci. **15**, 451–479 (2017)

[2] Carles, R.: Nonlinear Schrödinger equation with time dependent potential. Commun. Math. Sci. **9**(4), 937–964 (2011)

[3] Cazenave, T.: Semilinear Schrödinger Equations. Courant Lecture Notes in Mathematics, vol. 10. New York University Courant Institute of Mathematical Sciences, New York (2003)

[4] Cazenave, T., Haraux, A.: An introduction to semilinear evolution equations, translated from the 1990 French original by Y. Martel and revised by the authors. Oxford Lecture Series in Mathematics and Applications, 13. Clarendon, Oxford University Press, New York (1998)
[5] Cazenave, T., Weissler, F.B.: The Cauchy problem for the critical nonlinear Schrödinger equation in H^s. Nonlinear Anal. **14**, 807–836 (1990)
[6] Christ, M., Kiselev, A.: Maximal functions associated to filtrations. J. Funct. Anal. **179**, 409–425 (2001)
[7] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T.: Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$. Ann. Math. **167**(2), 767–865 (2008)
[8] D'Ancona, P., Fanelli, L.: Strichartz and smoothing estimates of dispersive equations with magnetic potentials. Commun. Partial Differential Equations **33**, 1082–1112 (2008)
[9] D'Ancona, P., Fanelli, L., Vega, L., Visciglia, N.: Endpoint Strichartz estimates for the magnetic Schrödinger equation. J. Funct. Anal. **258**, 3227–3240 (2010)
[10] De Bouard, A.: Nonlinear Schrodinger equations with magnetic fields. Differ. Integral Equ. **4**, 73–88 (1991)
[11] Dodson, B.: Global well-posedness and scattering for the defocusing, cubic nonlinear Schrödinger equation when $n = 3$ via a linear-nonlinear decomposition. Discrete Cont. Dyn. Syst. **33**, 1905–1926 (2013)
[12] Doi, S.-I.: On the Cauchy problem for Schrödinger type equations and the regularity of solutions. J. Math. Kyoto Univ. **34**, 319–328 (1994)
[13] Doi, S.-I.: Remarks on the Cauchy problem for Schrödinger-type equations. Commun. Partial Differ. Equ. **21**, 163–178 (1996)
[14] Erdoğan, M.B., Goldberg, M., Schlag, W.: Strichartz and smoothing estimates for Schrödinger operators with large magnetic potentials in $\mathbb{R}^3$. J. Eur. Math. Soc. (JEMS) **10**, 507–531 (2008)
[15] Erdoğan, M.B., Goldberg, M., Schlag, W.: Strichartz and smoothing estimates for Schrödinger operators with almost critical magnetic potentials in three and higher dimensions. Forum Math. **21**, 687–722 (2009)
[16] Fanelli, L., Garcia, A.: Counterexamples to Strichartz estimates for the magnetic Schrödinger equation. Commun. Contemp. Math. **13**, 213–234 (2011)
[17] Fang, D., Han, Z., Dai, J.: The nonlinear Schrödinger equations with combined nonlinearities of power-type and Hartree-type. Chin. Ann. Math. Ser. B **32**, 435–474 (2011)
[18] Fujiwara, D.: A construction of the fundamental solution for the Schrödinger equation. J. Anal. Math. **35**, 41–96 (1979)
[19] Georgiev, V., Stefanov, A., Tarulli, M.: Smoothing-Strichartz estimates for the Schrödinger equation with small magnetic potential. Discrete Cont. Dyn. Syst. **17**, 771–786 (2007)
[20] Ginibre, J., Velo, G., Scattering theory in the energy space for a class of nonlinear Schrödinger equations. In: Semigroups, Theory and Applications, Vol. I (Trieste, vol. 141 of Pitman Res. Notes Math. Ser., Longman Sci. Tech. Harlow 1986, pp. 110–120 (1984)
[21] Ginibre, J., Velo, G.: Smoothing properties and retarded estimates for some dispersive evolution equations. Commun. Math. Phys. **144**, 163–188 (1992)
[22] Ginibre, J., Velo, G.: Scattering theory in the energy space for a class of Hartree equations. In: Nonlinear Wave Equations (Providence, RI, vol. 263 of Contemp. Math., Amer. Math. Soc., Providence, RI 2000, pp. 29–60 (1998)
[23] Guo, Y., Nakamitsu, K., Strauss, W.: Global finite-energy solutions of the Maxwell-Schrödinger system. Commun. Math. Phys. **170**, 181–196 (1995)
[24] Haas, F.: Quantum plasmas, vol. 65 of Springer Series on Atomic, Optical, and Plasma Physics, Springer, New York. An hydrodynamic approach (2011)
[25] Keel, M., Tao, T.: Endpoint Strichartz estimates. Am. J. Math. **120**, 955–980 (1998)
[26] Killip, R., Vişan, M.: Nonlinear Schrödinger equations at critical regularity. In: Evolution Equations, vol. 17 of Clay Math. Proc., , pp. 325–437. Amer. Math. Soc., Providence (2013)
[27] Lieb, E.H., Loss, M.: Analysis. Graduate Studies in Mathematics, vol. 14, 2nd edn. American Mathematical Society, Providence (2001)
[28] Linares, F., Ponce, G.: Introduction to Nonlinear Dispersive Equations, Universitext, 2nd edn. Springer, New York (2015)
[29] Miao, C., Xu, G., Zhao, L.: The Cauchy problem of the Hartree equation. J. Partial Differ. Equ. **21**, 22–44 (2008)
[30] Michel, L.: Remarks on non-linear Schrödinger equation with magnetic fields. Commun. Partial Differ. Equ. **33**, 1198–1215 (2008)
[31] Michelangeli, A.: Role of scaling limits in the rigorous analysis of Bose-Einstein condensation. J. Math. Phys. **48**, 102102 (2007)
[32] Michelangeli, A.: Global well-posedness of the magnetic Hartree equation with non-Strichartz external fields. Nonlinearity **28**, 2743 (2015)
[33] Mizutani, H.: Strichartz estimates for Schrödinger equations with variable coefficients and unbounded potentials II. Superquadratic potentials. Commun. Pure Appl. Anal. **13**, 2177–2210 (2014)
[34] Naibo, V., Stefanov, A.: On some Schrödinger and wave equations with time dependent potentials. Math. Ann. **334**, 325–338 (2006)

[35] Nakamura, M., Wada, T.: Global existence and uniqueness of solutions to the Maxwell-Schrödinger equations. Commun. Math. Phys. **276**, 315–339 (2007)
[36] Nakamura, Y., Shimomura, A.: Local well-posedness and smoothing effects of strong solutions for nonlinear Schrödinger equations with potentials and magnetic fields. Hokkaido Math. J. **34**, 37–63 (2005)
[37] Olgiati, A.: Remarks on the derivation of Gross–Pitaevskii equation with magnetic Laplacian. In: Dell'Antonio, G., Michelangeli, A. (eds.) Advances in Quantum Mechanics: Contemporary Trends and Open Problems, vol. 18 of Springer INdAM Series, pp. 257–266. Springer International Publishing (2017)
[38] Roubíček, T.S.: Nonlinear partial differential equations with applications, vol. 153 of International Series of Numerical Mathematics, 2nd edn. Birkhäuser/Springer Basel AG, Basel (2013)
[39] Schlein, B.: Derivation of effective evolution equations from microscopic quantum dynamics (2008). arXiv:0807.4307
[40] Stefanov, A.: Strichartz estimates for the magnetic Schrödinger equation. Adv. Math. **210**, 246–303 (2007)
[41] Wang, B., Huo, Z., Hao, C., Guo, Z.: Harmonic Analysis Method for Nonlinear Evolution Equations. I. World Scientific Publishing Co. Pte. Ltd., Hackensack (2011)
[42] Yajima, K.: Existence of solutions for Schrödinger evolution equations. Commun. Math. Phys. **110**, 415–426 (1987)
[43] Yajima, K.: Schrödinger evolution equations with magnetic fields. J. Anal. Math. **56**, 29–76 (1991)

Paolo Antonelli
Gran Sasso Science Institute – GSSI
Via Crispi 7
67100 L'Aquila
Italy
e-mail: paolo.antonelli@gssi.it

Alessandro Michelangeli and Raffaele Scandone
SISSA – International School for Advanced Studies
Via Bonomea 265
34136 Trieste
Italy
e-mail: alemiche@sissa.it

Raffaele Scandone
e-mail: rscandone@sissa.it