

FAMILIES OF MONADS AND INSTANTONS FROM A NONCOMMUTATIVE ADHM CONSTRUCTION

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ABSTRACT. We give a θ -deformed version of the ADHM construction of $SU(2)$ instantons with arbitrary topological charge on the sphere S^4 . Classically the instanton gauge fields are constructed from suitable monad data; we show that in the deformed case the set of monads is itself a noncommutative space. We use these monads to construct noncommutative ‘families’ of $SU(2)$ instantons on the deformed sphere S^4_θ . We also compute the topological charge of each of the families. Finally we discuss what it means for such families to be gauge equivalent.

1. INTRODUCTION

The purpose of the present article is to generalise the ADHM method for constructing instantons on the four-sphere S^4 to the framework of noncommutative geometry by giving a construction of instantons on the noncommutative four-sphere S^4_θ .

Instantons arise in physics as anti-self-dual solutions of the Yang-Mills equations. Mathematically they are connections with anti-self-dual curvature on smooth G -bundles over a four-dimensional compact manifold. Since the very beginning they have been of central importance for both disciplines, an importance that has only grown over the years.

Of particular interest are instantons on $SU(2)$ -bundles over the Euclidean four-sphere S^4 . Thanks to the ADHM method of [2], the full solution to the problem of constructing such instantons on S^4 has long been known and, as a consequence, the moduli space \mathcal{M}_k of $SU(2)$ instantons with topological charge equal to k is known to be a manifold of dimension $8k - 3$. Starting with a trivial vector bundle over S^4 , the ADHM strategy is to construct an orthogonal projection to some (non-trivial) sub-bundle E in such a way that the projection of the trivial connection to E has anti-self-dual curvature.

The geometric ingredient which implements the classical ADHM construction is the Penrose twistor fibration $\mathbb{C}P^3 \rightarrow S^4$. The total space $\mathbb{C}P^3$ of the fibration is called the twistor space of S^4 and may be thought of as the bundle of projective spinors over S^4 (although it has its origins elsewhere [21]). The pull-back of an instanton bundle along this fibration is a holomorphic vector bundle over $\mathbb{C}P^3$ equipped with a set of reality conditions which identify it as such a pull-back [24]. In this way, the construction of instantons is equivalent to the construction of holomorphic bundles over twistor space.

Using powerful results from algebraic geometry, one gives an explicit description of all relevant holomorphic vector bundles over a complex projective space ([12, 4], *cf.* also [20]). Each of them arises as the cohomology of a *monad*: a suitable complex of vector

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$$0 \rightarrow \mathcal{A} \xrightarrow{\sigma} \mathcal{B} \xrightarrow{\tau} \mathcal{C} \rightarrow 0$$

such that σ is injective and τ is surjective. The ADHM construction tells us how to convert a given monad into an orthogonal projection of vector bundles as described above and guarantees that the resulting connection has anti-self-dual curvature.

Following the general strategy of the classical case, our goal is to give a deformed version of the ADHM method and hence a construction of $SU(2)$ instantons on the noncommutative four-sphere S_θ^4 . The techniques involved lend themselves rather neatly to the framework of noncommutative geometry; the construction of vector bundles and connections by orthogonal projection is particularly natural in light of the Serre-Swan theorem [11], which trades vector bundles for finitely generated projective modules.

The paper is organised as follows. Sect. 2 reviews the noncommutative spaces in question, namely the θ -deformed versions of the four-sphere S_θ^4 and its twistor space $\mathbb{C}\mathbb{P}_\theta^3$. We recall also the construction of the basic instanton and the principal bundle on which it is defined, as well as the details of the noncommutative twistor fibration. Sect. 3 recalls the construction of the quantum group $SL_\theta(2, \mathbb{H})$ of conformal transformations of S_θ^4 and the quantum subgroup $Sp_\theta(2)$ of isometries. The main purpose of these two sections is to gather together into one place the relevant contributions from [9, 14, 15, 16, 5] and to establish notation; in doing so we also make some novel improvements to previous versions. Sect. 4 presents the deformed ADHM construction itself. We show that in the deformed case the set of all monads is parameterised by a collection of noncommutative spaces $\widetilde{\mathcal{M}}_{\theta;k}$ indexed by k a positive integer. We use each of these spaces to construct a noncommutative ‘family’ of instantons whose topological charge we show to be equal to k . Finally in Sect. 5 we discuss what it means for families of instantons to be gauge equivalent. In particular, we show that the quantum symmetries of the sphere S_θ^4 generate gauge degrees of freedom, a feature which is a consequence of the noncommutativity and is not present in the classical construction.

2. THE TWISTOR FIBRATION

The use of the twistor fibration in the ADHM construction is crucial: this fibration captures in its geometry the very nature of the anti-self-duality equations, with the result that an instanton bundle is reinterpreted *via* pull-back in terms of holomorphic data on twistor space [24] (*cf.* also [1]). In particular, this means that twistor space plays the role of an ‘auxiliary space’ on which the ADHM construction takes place, before passing back down to the base space S^4 (we refer to [19] for more on the ADHM construction from a twistor perspective).

In this section we start by recalling the details of the algebra inclusion $\mathcal{A}(S_\theta^4) \hookrightarrow \mathcal{A}(S_\theta^7)$ as a noncommutative principal bundle with undeformed structure group $SU(2)$; associated to this principal bundle there is in particular a basic instanton bundle [14]. Next we give a description of the noncommutative twistor space in terms of its coordinate algebra $\mathcal{A}(\mathbb{C}\mathbb{P}_\theta^3)$, as well as a dualised description of the twistor fibration, now appearing [5] as an algebra inclusion $\mathcal{A}(S_\theta^4) \hookrightarrow \mathcal{A}(\mathbb{C}\mathbb{P}_\theta^3)$.

2.1. The noncommutative Hopf fibration. With $\lambda = \exp(2\pi i\theta)$ the deformation parameter, the coordinate algebra $\mathcal{A}(S_\theta^4)$ of the noncommutative four-sphere S_θ^4 is the

-algebra generated by a central real element x and elements $\alpha, \beta, \alpha^, \beta^*$, modulo the relations

$$(1) \quad \alpha\beta = \lambda\beta\alpha, \quad \alpha^*\beta^* = \lambda\beta^*\alpha^*, \quad \beta^*\alpha = \lambda\alpha\beta^*, \quad \beta\alpha^* = \lambda\alpha^*\beta,$$

together with the sphere relation

$$(2) \quad \alpha^*\alpha + \beta^*\beta + x^2 = 1.$$

Similarly, the coordinate algebra of the noncommutative seven-sphere $\mathcal{A}(S_\theta^7)$ is generated as a *-algebra by the elements $\{z_j, z_j^* \mid j = 1, \dots, 4\}$ and is subject to the commutation relations

$$(3) \quad z_j z_l = \eta_{jl} z_l z_j, \quad z_j z_l^* = \eta_{lj} z_l^* z_j, \quad z_j^* z_l^* = \eta_{jl} z_l^* z_j^*,$$

as well as the sphere relation

$$(4) \quad z_1^* z_1 + z_2^* z_2 + z_3^* z_3 + z_4^* z_4 = 1.$$

Compatibility with the $SU(2)$ principal bundle structure requires that the deformation matrix (η_{jk}) be given by

$$(5) \quad (\eta_{jk}) = \begin{pmatrix} 1 & 1 & \bar{\mu} & \mu \\ 1 & 1 & \mu & \bar{\mu} \\ \mu & \bar{\mu} & 1 & 1 \\ \bar{\mu} & \mu & 1 & 1 \end{pmatrix}, \quad \mu = \exp(i\pi\theta).$$

The values of the deformation parameters λ, μ are precisely those which allow an embedding of the classical group $SU(2)$ into the group $\text{Aut } \mathcal{A}(S_\theta^7)$. We denote by $\mathcal{A}(\mathbb{C}_\theta^4)$ the algebra generated by the $\{z_j, z_j^*\}$ subject to the relations (3); the quotient by the additional sphere relation yields the algebra $\mathcal{A}(S_\theta^7)$. The algebra inclusion $\mathcal{A}(S_\theta^4) \hookrightarrow \mathcal{A}(S_\theta^7)$ is given explicitly by

$$(6) \quad \alpha = 2(z_1 z_3^* + z_2^* z_4), \quad \beta = 2(z_2 z_3^* - z_1^* z_4), \quad x = z_1 z_1^* + z_2 z_2^* - z_3 z_3^* - z_4 z_4^*.$$

One easily verifies that for the right $SU(2)$ -action on $\mathcal{A}(S_\theta^7)$ given on generators by

$$(7) \quad (z_1, z_2^*, z_3, z_4^*) \mapsto (z_1, z_2^*, z_3, z_4^*) \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}, \quad w = \begin{pmatrix} w^1 & -\bar{w}^2 \\ w^2 & \bar{w}^1 \end{pmatrix} \in SU(2),$$

the invariant subalgebra is generated as expected by α, β, x and their conjugates, so one indeed has

$$\text{Inv}_{SU(2)} \mathcal{A}(S_\theta^7) = \mathcal{A}(S_\theta^4).$$

When $\theta = 0$ we recover the usual algebras of functions on the classical spheres S^4 and S^7 . The inclusion $\mathcal{A}(S^4) \hookrightarrow \mathcal{A}(S^7)$ is just a dualised description of the standard $SU(2)$ Hopf fibration $S^7 \rightarrow S^4$.

These noncommutative spheres have canonical differential calculi arising as deformations of the classical ones. Explicitly, one has a first order differential calculus $\Omega^1(S_\theta^7)$ on $\mathcal{A}(S_\theta^7)$ spanned as an $\mathcal{A}(S_\theta^7)$ -bimodule by $\{dz_j, dz_j^*, j = 1, \dots, 4\}$, subject to the relations

$$z_i dz_j = \eta_{ij} dz_j z_i, \quad z_i dz_j^* = \eta_{ji} dz_j^* z_i,$$

with η_{ij} as before. One also has relations

$$dz_i dz_j + \eta_{ij} dz_j dz_i = 0, \quad dz_i dz_j^* + \eta_{ji} dz_j^* dz_i = 0,$$

allowing one to extend the first order calculus to a differential graded algebra $\Omega(S_\theta^7)$. There is a unique differential d on $\Omega(S_\theta^7)$ such that $d : z_j \mapsto dz_j$. Furthermore, $\Omega(S_\theta^7)$ has an involution given by the graded extension of the map $z_j \mapsto z_j^*$. The story is similar for the four-sphere, in that the differential graded algebra $\Omega(S_\theta^4)$ is generated in degree one by $d\alpha, d\alpha^*, d\beta, d\beta^*, dx$, subject to relations

$$\begin{aligned} \alpha d\beta &= \lambda(d\beta)\alpha, & \beta^* d\alpha &= \lambda(d\alpha)\beta^*, \\ d\alpha d\beta + \lambda d\beta d\alpha &= 0, & d\beta^* d\alpha + \lambda d\alpha d\beta^* &= 0. \end{aligned}$$

The above are the same as the relations (1) or (3) but with d inserted. As vector spaces, the graded components $\Omega^k(S_\theta^7)$ and $\Omega^k(S_\theta^4)$ of k -forms on the noncommutative spheres are identical to their classical counterparts, although the algebra relations between forms are twisted. In particular this means that the Hodge $*$ -operator on S_θ^4 ,

$$*_\theta : \Omega^k(S_\theta^4) \rightarrow \Omega^{4-k}(S_\theta^4),$$

is defined by the same formula as it is classically. One still has that $*_\theta^2 = 1$, whence there is a direct sum decomposition of two-forms

$$\Omega^2(S_\theta^4) = \Omega_+^2(S_\theta^4) \oplus \Omega_-^2(S_\theta^4),$$

with $\Omega_\pm^2(S_\theta^4) := \{\omega \in \Omega^2(S_\theta^4) \mid *_\theta \omega = \pm \omega\}$ the spaces of self-dual and anti-self-dual two-forms.

2.2. The basic instanton. Amongst the nice properties of the classical Hopf fibration is that its canonical connection is an anti-instanton: its curvature is a self-dual two-form with values in the Lie algebra $\mathfrak{su}(2)$ of the structure group. This property holds also in the noncommutative case, giving a simple example of a noncommutative instanton. It has an elegant description [14] in terms of the function algebras $\mathcal{A}(S_\theta^7), \mathcal{A}(S_\theta^4)$ as follows. One considers a pair of elements of the right $\mathcal{A}(S_\theta^7)$ -module $\mathcal{A}(S_\theta^7)^4 := \mathbb{C}^4 \otimes \mathcal{A}(S_\theta^7)$ given by

$$|\psi_1\rangle = (z_1 \ z_2 \ z_3 \ z_4)^t, \quad |\psi_2\rangle = (-z_2^* \ z_1^* \ -z_4^* \ z_3^*)^t.$$

With the natural Hermitian structure on $\mathcal{A}(S_\theta^7)^4$ given by $\langle \xi | \eta \rangle = \sum_i \xi_i^* \eta_i$, one easily sees that $\langle \psi_j | \psi_l \rangle = \delta_{jl}$. It is convenient to introduce the matrix-valued function Ψ on S_θ^7 given by

$$(8) \quad \Psi = (|\psi_1\rangle \ |\psi_2\rangle) = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ -z_2^* & z_1^* & -z_4^* & z_3^* \end{pmatrix}^t.$$

From orthonormality of the columns one has that $\Psi^* \Psi = 1$ and hence the matrix

$$(9) \quad p := \Psi \Psi^* = \frac{1}{2} \begin{pmatrix} 1+x & 0 & \alpha & -\bar{\mu} \beta^* \\ 0 & 1+x & \beta & \mu \alpha^* \\ \alpha^* & \beta^* & 1-x & 0 \\ -\mu \beta & \bar{\mu} \alpha & 0 & 1-x \end{pmatrix}$$

is a self-adjoint idempotent of rank two, *i.e.* $p^* = p = p^2$ and $\text{Tr } p = 2$. The action (7) of $\text{SU}(2)$ on $\mathcal{A}(S_\theta^7)$ now takes the form

$$\Psi \mapsto \Psi w, \quad w \in \text{SU}(2),$$

from which the $SU(2)$ -invariance of the entries of p is immediately deduced. We may also write the commutation relations of $\mathcal{A}(S_\theta^7)$ in the useful form

$$(10) \quad \Psi_{ia}\Psi_{jb} = \eta_{ij}\Psi_{jb}\Psi_{ia}, \quad a, b = 1, 2 \quad i, j = 1, 2, 3, 4.$$

If ρ is the natural representation of $SU(2)$ on \mathbb{C}^2 , the finitely generated projective right $\mathcal{A}(S_\theta^4)$ -module $\mathcal{E} := p\mathcal{A}(S_\theta^4)^4$ is isomorphic to the module of equivariant maps from $\mathcal{A}(S_\theta^7)$ to \mathbb{C}^2 ,

$$\mathcal{E} \cong \{\phi \in \mathcal{A}(S_\theta^7) \otimes \mathbb{C}^2 \mid (w \otimes \text{id})\phi = (\text{id} \otimes \rho(w^{-1}))\phi \text{ for all } w \in SU(2)\}.$$

The module \mathcal{E} has the role of the space of sections of the ‘associated bundle’ $E = S_\theta^7 \times_{SU(2)} \mathbb{C}^2$. With the projection $p = \Psi\Psi^*$ there comes the canonical Grassmann connection defined on the module \mathcal{E} by

$$\nabla := p \circ d : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}(S_\theta^4)} \Omega^1(S_\theta^4).$$

The curvature of ∇ is $\nabla^2 = p(dp)^2$, which may be shown to be self-dual with respect to the Hodge operator,

$$*_\theta(p(dp)^2) = p(dp)^2.$$

The complementary projector $1 - p$ yields a connection whose curvature is anti-self-dual and hence an instanton on the noncommutative four-sphere, which we call the *basic instanton*. Noncommutative index theory computes its ‘topological charge’ to be equal to -1 .

Using the standard basis (e_1, e_2) of \mathbb{C}^2 , equivariant maps are written as $\phi = \sum_a \phi_a \otimes e_a$. On them, one has explicitly that

$$\nabla(\phi_a) = d\phi_a + \sum_b \omega_{ab}\phi_b,$$

where the connection one-form $\omega = \omega_{ab}$ is found to be

$$(11) \quad \omega_{ab} = \frac{1}{2} \sum_j ((\Psi^*)_{aj} d\Psi_{jb} - d(\Psi^*)_{aj} \Psi_{jb}).$$

From this it is easy to see that $\omega_{ab} = -(\omega^*)_{ba}$ and $\sum_a \omega_{aa} = 0$, so that ω is an element of $\Omega^1(S_\theta^7) \otimes \mathfrak{su}(2)$.

2.3. Noncommutative twistor space. It is well-known that, as a real six-dimensional manifold, the space \mathbb{CP}^3 may be identified with the set of all 4×4 Hermitian projector matrices of rank one: this is because each such matrix uniquely determines and is uniquely determined by a one-dimensional subspace of \mathbb{C}^4 . Thus the coordinate algebra $\mathcal{A}(\mathbb{CP}^3)$ of \mathbb{CP}^3 has a defining matrix of generators

$$(12) \quad Q = \begin{pmatrix} t_1 & x_1 & x_2 & x_3 \\ x_1^* & t_2 & y_3 & y_2 \\ x_2^* & y_3^* & t_3 & y_1 \\ x_3^* & y_2^* & y_1^* & t_4 \end{pmatrix},$$

with $t_j^* = t_j$, $j = 1, \dots, 4$ and $\text{Tr } Q = \sum_j t_j = 1$, as well as the relations corresponding to the fact that $Q^2 = Q$, that is to say $\sum_j Q_{kj} Q_{jl} = Q_{kl}$. The noncommutative twistor algebra $\mathcal{A}(\mathbb{CP}_\theta^3)$ is obtained by deforming these relations: with deformation parameter $\lambda = \exp(2\pi i\theta)$, one has that t_1, \dots, t_4 are central, that

$$x_1 x_3 = \bar{\lambda} x_3 x_1, \quad x_2 x_1 = \bar{\lambda} x_1 x_2, \quad x_2 x_3 = \bar{\lambda} x_3 x_2$$

as well as the auxiliary relations

$$y_1 y_2 = \bar{\lambda} y_2 y_1, \quad y_1 y_3 = \bar{\lambda} y_3 y_1, \quad y_2 y_3 = \bar{\lambda} y_3 y_2, \quad x_1(y_1, y_2, y_3) = (\bar{\lambda}^2 y_1, \bar{\lambda} y_2, \lambda y_3) x_1, \\ x_2(y_1, y_2, y_3) = (\bar{\lambda} y_1, y_2, \lambda y_3) x_2, \quad x_3(y_1, y_2, y_3) = (\bar{\lambda} y_1, \bar{\lambda} y_2, y_3) x_3,$$

and similar relations obtained by taking the adjoint under $*$ of those above (we refer to [5] for further details). To proceed further it is useful to note that classically $\mathbb{C}\mathbb{P}^3$ is the quotient of the sphere S^7 by the action of the diagonal $U(1)$ subgroup of $SU(2)$. This remains true in the noncommutative case and one identifies the generators of $\mathcal{A}(\mathbb{C}\mathbb{P}_\theta^3)$ as

$$(13) \quad Q_{jl} = z_j z_l^*,$$

via the generators $\{z_j, z_j^*\}$ of $\mathcal{A}(S_\theta^7)$. Indeed, from equation (13) one could infer the relations on the generators of $\mathcal{A}(\mathbb{C}\mathbb{P}_\theta^3)$ from those on the generators of $\mathcal{A}(S_\theta^7)$. By its very definition $\mathcal{A}(\mathbb{C}\mathbb{P}_\theta^3)$ is the invariant subalgebra of $\mathcal{A}(S_\theta^7)$ under this $U(1)$ -action and equation (13) defines an inclusion of algebras

$$\mathcal{A}(\mathbb{C}\mathbb{P}_\theta^3) \hookrightarrow \mathcal{A}(S_\theta^7),$$

giving a noncommutative principal bundle with structure group $U(1)$. We thus have algebra inclusions

$$(14) \quad \mathcal{A}(S_\theta^4) \hookrightarrow \mathcal{A}(\mathbb{C}\mathbb{P}_\theta^3) \hookrightarrow \mathcal{A}(S_\theta^7),$$

with the left-hand arrow still to be determined. As in the classical case, this inclusion is not a principal fibration (the typical fibre is a copy of $\mathbb{C}\mathbb{P}^1$) but we may nevertheless express the generators of $\mathcal{A}(\mathbb{C}\mathbb{P}_\theta^3)$ in terms of the generators of $\mathcal{A}(S_\theta^4)$. For this we need the non-degenerate map on $\mathcal{A}(\mathbb{C}_\theta^4)$ given on generators by

$$(15) \quad J(z_1, z_2, z_3, z_4) := (-z_2^*, z_1^*, -z_4^*, z_3^*)$$

and extended as an anti-algebra map. Classically, in doing so we would be identifying the set of quaternions \mathbb{H} with the set of 2×2 matrices over \mathbb{C} of the form

$$q = c_1 + c_2 j \in \mathbb{H} \mapsto \begin{pmatrix} c_1 & -\bar{c}_2 \\ c_2 & \bar{c}_1 \end{pmatrix} \in M_2(\mathbb{C}),$$

and the map J corresponds to right multiplication by the quaternion j . In the deformed case, this very same identification defines the algebra $\mathcal{A}(\mathbb{H}_\theta^2)$ to be equal to the algebra $\mathcal{A}(\mathbb{C}_\theta^4)$ equipped with the map J [16].

Using the identification of generators (13) the map J extends to an automorphism of $\mathcal{A}(\mathbb{C}\mathbb{P}_\theta^3)$, given in terms of the matrix generators in equation (12) by

$$J(t_1) = t_2, \quad J(t_2) = t_1, \quad J(t_3) = t_4, \quad J(t_4) = t_3, \\ J(x_1) = -x_1, \quad J(y_1) = -y_1, \quad J(x_1^*) = -x_1^*, \quad J(y_1^*) = -y_1^*, \\ J(x_2) = \mu y_2^*, \quad J(x_3) = -y_3^*, \quad J(x_2^*) = \bar{\mu} y_2, \quad J(x_3^*) = -y_3, \\ J(y_2) = \bar{\mu} x_2^*, \quad J(y_3) = -x_3^*, \quad J(y_2^*) = \mu x_2, \quad J(y_3^*) = -x_3,$$

as required for J to respect the algebra relations of $\mathcal{A}(\mathbb{C}\mathbb{P}_\theta^3)$. The subalgebra fixed by this map is precisely $\mathcal{A}(S_\theta^4)$; in fact one has an inclusion of algebras $\mathcal{A}(S_\theta^4) \hookrightarrow \mathcal{A}(\mathbb{C}\mathbb{P}_\theta^3)$ given on generators by

$$(16) \quad x \mapsto 2(t_1 + t_2 - 1), \quad \alpha \mapsto 2(x_2 + \mu y_2^*), \quad \beta \mapsto 2(-x_3^* + y_3),$$

with $\mu = \sqrt{\lambda} = \exp(\pi i \theta)$. In the notation of equation (8) we have that $Q = |\psi_1\rangle\langle\psi_1|$, and we note also that $|\psi_2\rangle = |J\psi_1\rangle$, so that equation (16) is just the statement that

$$p = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| = |\psi_1\rangle\langle\psi_1| + |J\psi_1\rangle\langle J\psi_1| = Q + J(Q).$$

This gives us the promised algebraic description of the twistor fibration (14): the generators of $\mathcal{A}(S_\theta^4)$ are identified with the degree one elements of $\mathcal{A}(\mathbb{C}\mathbb{P}_\theta^3)$ of the form $Z + J(Z)$.

3. THE QUANTUM CONFORMAL GROUP

Next, we briefly review the construction of the quantum groups which describe the symmetries of the spheres S_θ^4 and S_θ^7 (and the symmetries of the Hopf fibration defined in Sect. 2.1).

3.1. The quantum groups $\mathbf{SL}_\theta(2, \mathbb{H})$ and $\mathbf{Sp}_\theta(2)$. To begin, we need a noncommutative analogue of the set of all linear transformations of the quaternionic vector space \mathbb{H}_θ^2 defined above. To this end, we define a transformation bialgebra for the algebra $\mathcal{A}(\mathbb{H}_\theta^2)$ to be a bialgebra \mathcal{B} such that there is a $*$ -algebra map $\Delta_L : \mathcal{A}(\mathbb{C}_\theta^4) \rightarrow \mathcal{B} \otimes \mathcal{A}(\mathbb{C}_\theta^4)$ commuting with the map J of equation (15). The set of all transformation bialgebras for $\mathcal{A}(\mathbb{H}_\theta^2)$ forms a category in the natural way; we define the bialgebra $\mathcal{A}(\mathbf{M}_\theta(2, \mathbb{H}))$ as the universal initial object in the category, meaning that whenever \mathcal{B} is a transformation bialgebra for $\mathcal{A}(\mathbb{H}_\theta^2)$ there is a morphism of transformation bialgebras $\mathcal{A}(\mathbf{M}_\theta(2, \mathbb{H})) \rightarrow \mathcal{B}$ [16]. Using the universality property, one finds that $\mathcal{A}(\mathbf{M}_\theta(2, \mathbb{H}))$ is the associative algebra generated by the entries of the following 4×4 matrix:

$$(17) \quad A = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} = \begin{pmatrix} a_1 & -a_2^* & b_1 & -b_2^* \\ a_2 & a_1^* & b_2 & b_1^* \\ c_1 & -c_2^* & d_1 & -d_2^* \\ c_2 & c_1^* & d_2 & d_1^* \end{pmatrix}.$$

With our earlier notation, we think of this matrix as generated by four quaternion-valued functions, writing

$$a = (a_{ij}) = \begin{pmatrix} a_1 & -a_2^* \\ a_2 & a_1^* \end{pmatrix}$$

and similarly for the other entries b, c, d . The coalgebra structure on $\mathcal{A}(\mathbf{M}_\theta(2, \mathbb{H}))$ is given by

$$\Delta(A_{ij}) = \sum_l A_{il} \otimes A_{lj}, \quad \epsilon(A_{ij}) = \delta_{ij}$$

for $i, j = 1, \dots, 4$, and its $*$ -structure is evident from the matrix (17). The coaction Δ_L is determined to be

$$(18) \quad \Delta_L : \mathcal{A}(\mathbb{C}_\theta^4) \rightarrow \mathcal{A}(\mathbf{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(\mathbb{C}_\theta^4), \quad \Delta_L(\Psi_{ia}) = \sum_j A_{ij} \otimes \Psi_{ja},$$

where Ψ is the matrix in equation (8) (although here we do not assume the sphere relation and instead think of the entries of Ψ as generators of the algebra $\mathcal{A}(\mathbb{C}_\theta^4)$). The relations between the generators of $\mathcal{A}(\mathbf{M}_\theta(2, \mathbb{H}))$ are found from the requirement that Δ_L make $\mathcal{A}(\mathbb{C}_\theta^4)$ into an $\mathcal{A}(\mathbf{M}_\theta(2, \mathbb{H}))$ -comodule algebra. One computes

$$(19) \quad \Delta_L(\Psi_{ia}\Psi_{jb}) = \sum_{km} (A_{im}A_{jl} - \eta_{ij}\eta_{lm}A_{jl}A_{im}) \otimes \Psi_{ma}\Psi_{lb}$$

and, since the products $\Psi_{ma}\Psi_{lb}$ may be taken to be all independent as k, l, a, b vary, we must have that

$$(20) \quad A_{im}A_{jl} = \eta_{ij}\eta_{lm}A_{jl}A_{im}$$

for $i, j, l, m = 1, \dots, 4$. It is not difficult to see that the algebra generated by the a_{ij} is commutative, as are the algebras generated by the b_{ij}, c_{ij}, d_{ij} , although overall the algebra is noncommutative due to some not trivial relations among components in different blocks.

Of course, $\mathcal{A}(M_\theta(2, \mathbb{H}))$ is not quite a Hopf algebra since it does not have an antipode. We obtain a Hopf algebra by passing to the quotient of $\mathcal{A}(M_\theta(2, \mathbb{H}))$ by the Hopf $*$ -ideal generated by the element $D - 1$, where $D = \det A$ is the formal determinant of the matrix A in (17). We denote the quotient by $\mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))$, the coordinate algebra on the quantum group $\text{SL}_\theta(2, \mathbb{H})$ of matrices in $M_\theta(2, \mathbb{H})$ with determinant one, and continue to write the generators of the quotient as A_{ij} . The algebra $\mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))$ inherits a $*$ -bialgebra structure from that of $\mathcal{A}(M_\theta(2, \mathbb{H}))$ and as in [16] we use the determinant to define an antipode $S : \mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \rightarrow \mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))$. The datum $(\mathcal{A}(\text{SL}_\theta(2, \mathbb{H})), \Delta, \epsilon, S)$ constitutes a Hopf $*$ -algebra.

The Hopf algebra $\mathcal{A}(\text{Sp}_\theta(2))$ is the quotient of $\mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))$ by the two-sided $*$ -Hopf ideal generated by

$$\sum_l (A^*)_{li}A_{lj} - \delta_{ij}, \quad i, j = 1, \dots, 4.$$

In this algebra we have the relations $A^*A = AA^* = 1$, or equivalently that $S(A) = A^*$. This Hopf algebra is the coordinate algebra on the quantum group $\text{Sp}_\theta(2)$, the subgroup of $\text{SL}_\theta(2, \mathbb{H})$ of unitary matrices.

Finally there is an inclusion of algebras $\mathcal{A}(S_\theta^7) \hookrightarrow \mathcal{A}(\text{Sp}_\theta(2))$ given on generators by the $*$ -algebra map

$$(21) \quad z_1 \mapsto a_1, \quad z_2 \mapsto a_2, \quad z_3 \mapsto c_1, \quad z_4 \mapsto c_2.$$

This means that we may identify the first two columns of the matrix A with the matrix Ψ of equation (8). Similarly there is an algebra inclusion $\mathcal{A}(S_\theta^4) \hookrightarrow \mathcal{A}(\text{Sp}_\theta(2))$ given by

$$(22) \quad x \mapsto a_1a_1^* - a_2a_2^* + c_1c_1^* - c_2c_2^*, \quad \alpha \mapsto a_1c_1^* - a_2^*c_2, \quad \beta \mapsto -a_1^*c_2 + a_2c_1^*.$$

These inclusions yield algebra isomorphisms of $\mathcal{A}(S_\theta^7)$ and $\mathcal{A}(S_\theta^4)$ with certain subalgebras of $\mathcal{A}(\text{Sp}_\theta(2))$ of coinvariants under coactions by appropriate sub-Hopf algebras, thus realising the noncommutative spheres as quantum homogeneous spaces for $\text{Sp}_\theta(2)$. We refer to [16] for details of these constructions.

3.2. Quantum conformal transformations. We now review how the quantum groups obtained in the previous section (co)act on the spheres S_θ^7 and S_θ^4 as ‘quantum symmetries’. The coaction

$$(23) \quad \Delta_L : \mathcal{A}(\mathbb{C}_\theta^4) \rightarrow \mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(\mathbb{C}_\theta^4), \quad \Delta_L(\Psi_{ia}) = \sum_j A_{ij} \otimes \Psi_{ja},$$

is by construction a $*$ -algebra map and so, if we assume that the quantity

$$r^2 := \sum_j z_j^* z_j$$

is invertible with inverse r^{-2} , then we may also define an inverse for the quantity

$$\rho^2 := \Delta_L \left(\sum_j z_j^* z_j \right)$$

by $\rho^{-2} := \Delta_L(r^{-2})$. Inverting r^2 corresponds to deleting the origin in \mathbb{C}_θ^4 and we define the coordinate algebra of the corresponding subset of \mathbb{C}_θ^4 by

$$\mathcal{A}_0(\mathbb{C}_\theta^4) := \mathcal{A}(\mathbb{C}_\theta^4)[r^{-2}],$$

the algebra $\mathcal{A}(\mathbb{C}_\theta^4)$ with r^{-2} adjoined. Extending Δ_L as a $*$ -algebra map gives a well-defined coaction

$$\Delta_L : \mathcal{A}_0(\mathbb{C}_\theta^4) \rightarrow \mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}_0(\mathbb{C}_\theta^4)$$

for which $\mathcal{A}_0(\mathbb{C}_\theta^4)$ is an $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H}))$ -comodule algebra. Writing $\mathcal{A}_0(\tilde{\mathbb{C}}_\theta^4) := \Delta_L(\mathcal{A}_0(\mathbb{C}_\theta^4))$ for the image of $\mathcal{A}_0(\mathbb{C}_\theta^4)$ under Δ_L , both ρ^2 and ρ^{-2} are central in the algebra $\mathcal{A}_0(\tilde{\mathbb{C}}_\theta^4)$, since r^2 and r^{-2} are central in $\mathcal{A}_0(\mathbb{C}_\theta^4)$.

Now the coaction Δ_L descends to a coaction of the Hopf algebra $\mathcal{A}(\mathrm{Sp}_\theta(2))$,

$$(24) \quad \Delta_L : \mathcal{A}_0(\mathbb{C}_\theta^4) \rightarrow \mathcal{A}(\mathrm{Sp}_\theta(2)) \otimes \mathcal{A}_0(\mathbb{C}_\theta^4),$$

by the same formula (18) now viewed for the quotient $\mathcal{A}(\mathrm{Sp}_\theta(2))$. In particular, for this coaction one has

$$(\Psi^* \Psi)_{ab} \mapsto \sum_{ijl} (A^*)_{li} A_{ij} \otimes (\Psi^*)_{al} \Psi_{jb} = \sum_{jl} \delta_{lj} \otimes (\Psi^*)_{al} \Psi_{jb} = 1 \otimes (\Psi^* \Psi)_{ab},$$

since the generators A_{ij} satisfy the relations $\sum_i (A^*)_{li} A_{ij} = \delta_{lj}$ in the algebra $\mathcal{A}(\mathrm{Sp}_\theta(2))$. Then both $\mathcal{A}(S_\theta^4)$ and $\mathcal{A}(S_\theta^7)$ are $\mathcal{A}(\mathrm{Sp}_\theta(2))$ -comodule algebras, since this coaction preserves the sphere relations (2) and (4).

In contrast, the spheres S_θ^7 and S_θ^4 are not preserved under the coaction of the larger quantum group $\mathrm{SL}_\theta(2, \mathbb{H})$. Although defined on the algebra $\mathcal{A}_0(\mathbb{C}_\theta^4)$, the coaction Δ_L of $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H}))$ is not well-defined on the seven-sphere $\mathcal{A}(S_\theta^7)$ since it does not preserve the sphere relation $r^2 = 1$ of equation (4). By definition, we have instead that $\Delta_L(r^2) = \rho^2$, meaning that the coaction of $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H}))$ ‘inflates’ the sphere $\mathcal{A}(S_\theta^7)$ [16]. Since r^2 is a central element of $\mathcal{A}_0(\mathbb{C}_\theta^4)$, we may evaluate it as a positive real number. The result is the coordinate algebra of a noncommutative sphere $S_{\theta,r}^7$ of radius r ; as this radius varies in $\mathcal{A}_0(\mathbb{C}_\theta^4)$, it sweeps out a family of seven-spheres. Similarly, evaluation of the central element ρ^2 in $\mathcal{A}_0(\tilde{\mathbb{C}}_\theta^4)$ yields the coordinate algebra of a noncommutative sphere $\tilde{S}_{\theta,\rho}^7$ of radius ρ and, as the value of ρ varies in $\mathcal{A}_0(\tilde{\mathbb{C}}_\theta^4)$, it sweeps out another family of seven-spheres. The coaction Δ_L of $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H}))$ on $\mathcal{A}_0(\mathbb{C}_\theta^4)$ serves to map the family parameterised by r^2 onto the family parameterised by ρ^2 .

A similar fact is found for the generators α, β, x of the four-sphere algebra $\mathcal{A}(S_\theta^4)$. The coaction of $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H}))$ does not preserve the sphere relation but gives instead that

$$\Delta_L(\alpha^* \alpha + \beta^* \beta + x^2) = \rho^4,$$

and the four-sphere S_θ^4 is also inflated. Let us write $\mathcal{A}(Q_\theta)$ for the subalgebra of $\mathcal{A}_0(\mathbb{C}_\theta^4)$ generated by α, β, x and their conjugates. Then as r^4 varies in $\mathcal{A}(Q_\theta)$, we get a family of noncommutative four-spheres. Similarly, we define $\tilde{\alpha} := \Delta_L(\alpha)$, $\tilde{\beta} := \Delta_L(\beta)$, $\tilde{x} := \Delta_L(x)$ and so forth, and write $\mathcal{A}(\tilde{Q}_\theta)$ for the subalgebra of $\mathcal{A}_0(\tilde{\mathbb{C}}_\theta^4)$ that they generate. It is precisely the $\mathrm{SU}(2)$ -invariant subalgebra of $\mathcal{A}_0(\tilde{\mathbb{C}}_\theta^4)$, and as ρ^4 varies in $\mathcal{A}(\tilde{Q}_\theta)$ we get another family of noncommutative four-spheres. The coaction of the quantum group $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H}))$ maps the family parameterised by r^4 onto the family parameterised by ρ^4 .

Thus there is a family of $\mathrm{SU}(2)$ -principal fibrations given by the algebra inclusion $\mathcal{A}(Q_\theta) \hookrightarrow \mathcal{A}_0(\mathbb{C}_\theta^4)$, the family being parameterised by the function r^2 . For a fixed value

of r^2 we get an $SU(2)$ principal bundle $S_{\theta,r}^7 \rightarrow S_{\theta,r^2}^4$. Similarly, the algebra inclusion $\mathcal{A}(\tilde{Q}_\theta) \hookrightarrow \mathcal{A}_0(\tilde{\mathbb{C}}_\theta^4)$ defines a family of $SU(2)$ -principal fibrations parameterised by the function ρ^2 . The above construction shows that the coaction of the quantum group $\mathcal{A}(SL_\theta(2, \mathbb{H}))$ carries the former family of principal fibrations onto the latter.

All of this means that, as things stand, we cannot use the presentations of $\mathcal{A}(S_\theta^4)$ and $\mathcal{A}(S_\theta^7)$ of Sect. 2.1 to give a well-defined coaction of $\mathcal{A}(SL_\theta(2, \mathbb{H}))$, since the sphere relations we use to define them are not preserved by the coaction. Rather we should work with the families of spheres all at once (this is the price we have to pay for working with the coaction of a Hopf algebra rather than the action of a group). To do this, we note that the algebra $\mathcal{A}(S_\theta^4)$ may be identified with the subalgebra of $\mathcal{A}_0(\mathbb{C}_\theta^4)$ generated by $r^{-2}\alpha$, $r^{-2}\beta$, $r^{-2}x$, together with their conjugates, since the sphere relation

$$(25) \quad (r^{-2}\alpha)(r^{-2}\alpha)^* + (r^{-2}\beta)(r^{-2}\beta)^* + (r^{-2}x)^2 = 1$$

is automatically satisfied in $\mathcal{A}_0(\mathbb{C}_\theta^4)$. The result of doing so is that we have a well-defined coaction,

$$\Delta_L : \mathcal{A}(S_\theta^4) \rightarrow \mathcal{A}(SL_\theta(2, \mathbb{H})) \otimes \mathcal{A}(S_\theta^4),$$

defined on the generators $r^{-2}\alpha$, $r^{-2}\beta$, $r^{-2}x$ and their conjugates, since the sphere relation (25) is now preserved by Δ_L . In this way, we think of $SL_\theta(2, \mathbb{H})$ as the quantum group of conformal transformations of S_θ^4 .

In these new terms, the construction of the defining projector for $\mathcal{A}(S_\theta^4)$ needs to be modified only slightly. We now take the normalised matrix

$$(26) \quad \Psi = r^{-1} \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ -z_2^* & z_1^* & -z_4^* & z_3^* \end{pmatrix}^t,$$

at the price of including the generator r^{-1} as well (not a problem in the smooth closure [16]). Thanks to the relation (25), we still have $\Psi^*\Psi = 1$ so that the required projector is

$$(27) \quad p := \Psi\Psi^* = \frac{1}{2}r^{-2} \begin{pmatrix} r^2 + x & 0 & \alpha & -\bar{\mu}\beta^* \\ 0 & r^2 + x & \beta & \mu\alpha^* \\ \alpha^* & \beta^* & r^2 - x & 0 \\ -\mu\beta & \bar{\mu}\alpha & 0 & r^2 - x \end{pmatrix}.$$

By the above discussion, the coaction Δ_L of $\mathcal{A}(SL_\theta(2, \mathbb{H}))$ is now well-defined on the algebra generated by the entries of this matrix. Writing $\tilde{\Psi}_{ia} := \Delta_L(\Psi_{ia})$, the image of p under Δ_L is computed to be

$$(28) \quad P := \tilde{\Psi}\tilde{\Psi}^* = \frac{1}{2}\rho^{-2} \begin{pmatrix} \rho^2 + \tilde{x} & 0 & \tilde{\alpha} & -\bar{\mu}\tilde{\beta}^* \\ 0 & \rho^2 + x & \tilde{\beta} & \mu\tilde{\alpha}^* \\ \tilde{\alpha}^* & \tilde{\beta}^* & \rho^2 - x & 0 \\ -\mu\tilde{\beta} & \bar{\mu}\tilde{\alpha} & 0 & \rho^2 - \tilde{x} \end{pmatrix}.$$

The entries of these projectors generate respectively subalgebras of $\mathcal{A}_0(\mathbb{C}_\theta^4)$ and $\mathcal{A}_0(\tilde{\mathbb{C}}_\theta^4)$, each parameterising the families of noncommutative four-spheres discussed above.

Finally, we observe that similar statements may be made about the $U(1)$ -principal fibration $S_\theta^7 \rightarrow \mathbb{C}\mathbb{P}_\theta^3$. We do not need a sphere relation in order to define the coordinate algebra $\mathcal{A}(\mathbb{C}\mathbb{P}_\theta^3)$: in Sect. 2.3 it was merely convenient to do so. Instead, we may identify

$\mathcal{A}(\mathbb{CP}_\theta^3)$ as the $U(1)$ -invariant subalgebra of $\mathcal{A}_0(\mathbb{C}^4)$ generated by elements $t_1 = r^{-2}z_1z_2^*$, $x_1 = r^{-2}z_1z_2^*$, $x_2 = r^{-2}z_1z_3^*$, $x_3 = r^{-2}z_1z_4^*$ and so forth.

4. A NONCOMMUTATIVE ADHM CONSTRUCTION

There is a well-known solution to the problem of constructing instantons on the classical four-sphere S^4 which goes under the name of ADHM construction. It uses techniques of linear algebra to construct vector bundles over twistor space \mathbb{CP}^3 , which are in turn put together to construct a vector bundle over S^4 equipped with an instanton connection. Moreover, it is known that all such connections are obtained in this way [2, 3].

Our goal here is to generalise the ADHM method to a deformed version which constructs instantons on the noncommutative sphere S_θ^4 . The classical construction may be obtained from our deformed version by setting $\theta = 0$. As usual our approach stems from writing the classical construction in a dualised language which does not depend on the commutativity of the available function algebras, although here the situation is not as straightforward as one might first expect. The deformed construction is rather more subtle than it is in the commutative case and produces noncommutative ‘families’ of instantons.

4.1. A noncommutative space of monads. The algebra $\mathcal{A}(\mathbb{C}_\theta^4)$ has a natural \mathbb{Z} -grading given by assigning to its generators the degrees

$$\deg(z_j) = 1, \quad \deg(z_j^*) = -1, \quad j = 1, \dots, 4,$$

which results in a decomposition $\mathcal{A}(\mathbb{C}_\theta^4) = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n$. Then for each $r \in \mathbb{Z}$ there is a ‘degree shift’ map from $\mathcal{A}(\mathbb{C}_\theta^4)$ to itself whose image we denote $\mathcal{A}(\mathbb{C}_\theta^4)(r)$; by definition the degree n component of $\mathcal{A}(\mathbb{C}_\theta^4)(r)$ is \mathcal{A}_{r+n} .

Similarly, if a given $\mathcal{A}(\mathbb{C}_\theta^4)$ -module \mathcal{E} is \mathbb{Z} -graded, we denote the degree-shifted modules by $\mathcal{E}(r)$, $r \in \mathbb{Z}$. In particular, for each finite dimensional vector space H the corresponding free right module $H \otimes \mathcal{A}(\mathbb{C}_\theta^4)$ is \mathbb{Z} -graded by the grading on $\mathcal{A}(\mathbb{C}_\theta^4)$, and the shift maps on $\mathcal{A}(\mathbb{C}_\theta^4)$ induce the shift maps on $H \otimes \mathcal{A}(\mathbb{C}_\theta^4)$.

The input data for the classical ADHM construction of $SU(2)$ instantons with topological charge k is a *monad*, by which we mean a sequence of free right modules over the algebra $\mathcal{A}(\mathbb{C}^4)$,

$$(29) \quad H \otimes \mathcal{A}(\mathbb{C}^4)(-1) \xrightarrow{\sigma_z} K \otimes \mathcal{A}(\mathbb{C}^4) \xrightarrow{\tau_z} L \otimes \mathcal{A}(\mathbb{C}^4)(1),$$

where H , K and L are complex vector spaces of dimensions k , $2k + 2$ and k respectively. The arrows σ_z and τ_z are $\mathcal{A}(\mathbb{C}^4)$ -module homomorphisms assumed to be such that σ_z is injective, τ_z is surjective and that the composition $\tau_z \sigma_z = 0$. This is the usual approach in algebraic geometry [20], although here we work with $\mathcal{A}(\mathbb{C}^4)$ -modules, *i.e.* global sections of vector bundles, rather than with locally-free sheaves.

The degree shifts signify that we think of σ_z and τ_z respectively as elements of $H^* \otimes K \otimes \mathcal{A}_1$ and $K^* \otimes L \otimes \mathcal{A}_1$, where \mathcal{A}_1 is the degree one component of $\mathcal{A}(\mathbb{C}^4)$ (the vector space spanned by the generators z_1, \dots, z_4). This means that alternatively we may think of them as linear maps

$$(30) \quad \sigma_z : H \times \mathbb{C}^4 \rightarrow K, \quad \tau_z : K \times \mathbb{C}^4 \rightarrow L,$$

thus recovering the more explicit geometric approach of [2].

Our goal in this section is to give a description of a monad of the form (29) in an algebraic framework which allows the possibility of the algebra $\mathcal{A}(\mathbb{C}_\theta^4)$ being noncommutative. In this setting, we require the maps σ_z and τ_z to be parameterised by the noncommutative space \mathbb{C}_θ^4 rather than by the classical space \mathbb{C}^4 , as was the case in equation (30). Our first task then is to find an analogue of the space of linear module maps $H \otimes \mathcal{A}(\mathbb{C}_\theta^4)(-1) \rightarrow K \otimes \mathcal{A}(\mathbb{C}_\theta^4)$.

Following a general strategy [23], we define $\mathcal{A}(\widetilde{\mathcal{M}}_\theta(H, K))$ to be the universal algebra for which there is a morphism of right $\mathcal{A}(\mathbb{C}_\theta^4)$ -modules,

$$\sigma_z : H \otimes \mathcal{A}(\mathbb{C}_\theta^4)(-1) \rightarrow \mathcal{A}(\widetilde{\mathcal{M}}_\theta(H, K)) \otimes K \otimes \mathcal{A}(\mathbb{C}_\theta^4),$$

which is linear in the generators z_1, \dots, z_4 of $\mathcal{A}(\mathbb{C}_\theta^4)$. By this, we mean that whenever \mathcal{B} is an algebra satisfying these properties, then there is a morphism of algebras from $\mathcal{A}(\widetilde{\mathcal{M}}_\theta(H, K))$ to \mathcal{B} .

Choosing a basis (u_1, \dots, u_k) for the vector space H and a basis (v_1, \dots, v_{2k+2}) for the vector space K , the algebra $\mathcal{A}(\widetilde{\mathcal{M}}_\theta(H, K))$ is generated by the matrix elements

$$\{M_{ab}^\alpha \mid a = 1, \dots, 2k+2, b = 1, \dots, k, \alpha = 1, \dots, 4\},$$

which define a map σ_z , expressed on simple tensors by

$$(31) \quad \sigma_z : u_b \otimes Z \mapsto \sum_{a,\alpha} M_{ab}^\alpha \otimes v_a \otimes z_\alpha Z, \quad Z \in \mathcal{A}(\mathbb{C}_\theta^4).$$

In more compact notation, for each α we arrange these elements into a $(2k+2) \times k$ matrix $M^\alpha = (M_{ab}^\alpha)$, so that with respect to the above bases, σ_z may be written

$$(32) \quad \sigma_z = \sum_\alpha M^\alpha \otimes z_\alpha.$$

To find the relations in the algebra $\mathcal{A}(\widetilde{\mathcal{M}}_\theta(H, K))$, let us write $(\hat{u}_1, \dots, \hat{u}_k)$ for the basis of H^* which is dual to (u_1, \dots, u_k) and write $(\hat{v}_1, \dots, \hat{v}_{2k+2})$ for the basis of K^* dual to (v_1, \dots, v_{2k+2}) . Then the map (31) has an equivalent dual description (also denoted σ_z) in terms of the dual vector spaces H^*, K^* as

$$(33) \quad \sigma_z : \hat{v}_a \otimes Z \mapsto \sum_{b,\alpha} M_{ab}^\alpha \otimes \hat{u}_b \otimes z_\alpha Z,$$

and extended as an $\mathcal{A}(\mathbb{C}_\theta^4)$ -module map. The functionals \hat{u}_b, \hat{v}_a together with their conjugates \hat{u}_b^*, \hat{v}_a^* generate the coordinate algebras of H and K respectively. It is only natural to require that (33) be an algebra map.

Proposition 4.1. *With $(\eta_{\alpha\beta})$ the matrix (5) of deformation parameters, the matrix elements M_{ab}^α enjoy the relations*

$$(34) \quad M_{ab}^\alpha M_{cd}^\beta = \eta_{\beta\alpha} M_{cd}^\beta M_{ab}^\alpha$$

for each $a, c = 1, \dots, 2k+2$, each $b, d = 1, \dots, k$ and each $\alpha, \beta = 1, \dots, 4$.

Proof. The requirement that (33) is an algebra map means that in degree one we need $\sigma_z(\hat{v}_a \hat{v}_c) = \sigma_z(\hat{v}_c \hat{v}_a)$ for all $a, c = 1, \dots, 2k+2$, which translates into the statement that

$$\sum_{b,d,\alpha,\beta} M_{ab}^\alpha M_{cd}^\beta \otimes \hat{u}_b \hat{u}_d \otimes z_\alpha z_\beta = \sum_{b,d,\alpha,\beta} M_{cd}^\beta M_{ab}^\alpha \otimes \hat{u}_d \hat{u}_b \otimes z_\beta z_\alpha$$

for all $a, c = 1, \dots, 2k+2$. Using in turn the relations (3) and the fact that the generators \hat{u}_b, \hat{u}_d commute for all values of b, d , this equation may be rearranged to give

$$\sum_{b,d,\alpha,\beta} \left(M_{ab}^\alpha M_{cd}^\beta - \eta_{\beta\alpha} M_{cd}^\beta M_{ab}^\alpha \right) \otimes \hat{u}_b \hat{u}_d \otimes z_\alpha z_\beta = 0.$$

Since for $b \leq d$ and $\alpha \leq \beta$ the quantities $\hat{u}_b \hat{u}_d \otimes z_\alpha z_\beta$ may all be taken to be independent, we must have that their coefficients are all zero, leading to the stated relations. \square

The above proposition simply says that the entries of a given matrix M^α all commute, whereas the relations between the entries of the matrices M^α and M^β are determined by the deformation parameter $\eta_{\beta\alpha}$. Hence the algebra $\mathcal{A}(\widetilde{\mathcal{M}}_\theta(H, K))$ is generated by the M_{ab}^α subject to the relations (34). The algebra $\mathcal{A}(\widetilde{\mathcal{M}}_{\theta=0}(H, K))$ is commutative and parameterises the space of all possible maps σ_z , since for each point $x \in \mathcal{M}_{\theta=0}(H, K)$ there is an evaluation map,

$$\text{ev}_x : \mathcal{A}(\mathcal{M}_{\theta=0}(H, K)) \rightarrow \mathbb{C},$$

which yields an $\mathcal{A}(\mathbb{C}^4)$ -module homomorphism

$$\begin{aligned} (\text{ev}_x \otimes \text{id})\sigma_z &: H \otimes \mathcal{A}(\mathbb{C}^4)(-1) \rightarrow K \otimes \mathcal{A}(\mathbb{C}^4), \\ (\text{ev}_x \otimes \text{id})\sigma_z &:= \sum \text{ev}_x(M_{ab}^\alpha) \otimes z_\alpha. \end{aligned}$$

When θ is different from zero, there need not be enough evaluation maps available. Nevertheless, we think of $\mathcal{A}(\widetilde{\mathcal{M}}_\theta(H, K))$ as a noncommutative family of maps parameterised by the noncommutative space $\widetilde{\mathcal{M}}_\theta(H, K)$.

Remark 4.2. Since we constructed $\mathcal{A}(\widetilde{\mathcal{M}}_\theta(H, K))$ through the minimal requirement that σ_z is an algebra map, it is indeed the universal algebra with the required properties. This means that our interpretation of $\mathcal{A}(\widetilde{\mathcal{M}}_\theta(H, K))$ as a noncommutative family of maps is in agreement with the approaches of [23, 25, 22] for quantum families of maps parameterised by noncommutative spaces. Moreover, it also agrees with the definition of algebras of rectangular quantum matrices discussed in [17]. It may also be viewed as a kind of ‘comeasuring’ as introduced in [18], but now for modules instead of algebras.

Thus we have a noncommutative analogue of the space of all maps σ_z . A similar construction works for the maps τ_z : there is a universal algebra $\mathcal{A}(\widetilde{\mathcal{M}}_\theta(K, L))$ generated by matrix elements N_{ba}^α for $b = 1, \dots, k$, $a = 1, \dots, 2k+2$ and $\alpha = 1, \dots, 4$, here coming from a map

$$(35) \quad \tau_z : v_a \otimes Z \mapsto \sum_{b,\alpha} N_{ba}^\alpha \otimes w_b \otimes z_\alpha Z,$$

having chosen a basis (w_1, \dots, w_k) for the vector space L . Dually, the requirement that τ_z be an algebra map from the coordinate algebra of L to the coordinate algebra of K results in relations for the generators of the algebra $\mathcal{A}(\widetilde{\mathcal{M}}_\theta(K, L))$,

$$(36) \quad N_{ba}^\alpha N_{dc}^\beta = \eta_{\beta\alpha} N_{dc}^\beta N_{ba}^\alpha,$$

which are the parallel of conditions (34) for the algebra $\mathcal{A}(\widetilde{\mathcal{M}}_\theta(H, K))$.

To complete the monad picture we finally require that the composition of the maps σ_z and τ_z be zero. In the dualised format, the composition is easily dealt with as the composition as a map from the coordinate algebra of L to that of H , with the product

appearing as part of a general procedure for ‘gluing’ quantum matrices [17]: the composition $\tau_z \sigma_z$ is an algebra-valued $k \times k$ matrix, the product of a $k \times (2k + 2)$ matrix with a $(2k + 2) \times k$ matrix. The condition that $\tau_z \sigma_z = 0$ is established by the following proposition.

Proposition 4.3. *The condition $\tau_z \sigma_z = 0$ is equivalent to the requirement that*

$$(37) \quad \sum_r (N_{br}^\alpha M_{rd}^\beta + \eta_{\beta\alpha} N_{br}^\beta M_{rd}^\alpha) = 0$$

for all $b, d = 1, \dots, k$ and all $\alpha, \beta = 1, \dots, 4$.

Proof. In terms of algebra-valued matrices the map $\tau_z \sigma_z$ is computed, as the composition of the duals of the maps (31) and (35), to be equal to

$$(\tau_z \sigma_z)_{bd} = \sum_{r, \alpha, \beta} N_{br}^\alpha M_{rd}^\beta \otimes z_\alpha z_\beta.$$

Equating to zero the coefficients of the linearly independent generators $z_\alpha z_\beta$ for $\alpha \leq \beta$, gives the relations as stated. \square

The conditions in equation (37) may be expressed more compactly in terms of products of matrices as

$$N^\alpha M^\beta + \eta_{\beta\alpha} N^\beta M^\alpha = 0,$$

for $\alpha, \beta = 1, \dots, 4$ (and as in (36) there is no sum over α and β in this expression).

Definition 4.4. Define $\mathcal{A}(\widetilde{\mathcal{M}}_{\theta; k})$ to be the algebra generated by the matrix elements M_{ab}^α and N_{ba}^β subject to the relations

$$M_{ab}^\alpha M_{cd}^\beta = \eta_{\beta\alpha} M_{cd}^\beta M_{ab}^\alpha, \quad N_{ba}^\alpha N_{dc}^\beta = \eta_{\beta\alpha} N_{dc}^\beta N_{ba}^\alpha,$$

as well as the relations

$$\sum_r (N_{dr}^\alpha M_{rb}^\beta + \eta_{\beta\alpha} N_{dr}^\beta M_{rb}^\alpha) = 0$$

for all $\alpha, \beta = 1, \dots, 4$, all $b, d = 1, \dots, k$ and all $a, c = 1, \dots, 2k + 2$.

The noncommutative algebra $\mathcal{A}(\widetilde{\mathcal{M}}_{\theta; k})$ is by construction universal amongst all algebras having the property that the resulting maps σ_z and τ_z are algebra maps which compose to zero. Our interpretation is that for fixed k the collection of monads over \mathbb{C}_θ^4 is parameterised by the noncommutative space which is ‘dual’ to this algebra.

4.2. The subspace of self-dual monads. In the classical case, the input datum of a monad is by itself insufficient to construct bundles over the four-sphere S^4 . To achieve this, one must incorporate the quaternionic structure afforded by the map J as in (15) (in the classical limit) and ensure that the monad is compatible with this extra structure. The same is true in the noncommutative case, as we shall see presently.

Given the pair of maps constructed in the previous section,

$$\begin{aligned} \sigma_z : H \otimes \mathcal{A}(\mathbb{C}_\theta^4)(-1) &\rightarrow \mathcal{A}(\widetilde{\mathcal{M}}_\theta(H, K)) \otimes K \otimes \mathcal{A}(\mathbb{C}_\theta^4), \\ \tau_z : K \otimes \mathcal{A}(\mathbb{C}_\theta^4) &\rightarrow \mathcal{A}(\widetilde{\mathcal{M}}_\theta(K, L)) \otimes L \otimes \mathcal{A}(\mathbb{C}_\theta^4)(1), \end{aligned}$$

we firstly note that the anti-algebra map J in (15) induces a new pair of maps,

$$(38) \quad \begin{aligned} \sigma_{J(z)} &: H \otimes J((\mathcal{A}(\mathbb{C}_\theta^4)(-1)) \rightarrow \mathcal{A}\left(\widetilde{\mathcal{M}}_\theta(H, K)\right) \otimes K \otimes J((\mathcal{A}(\mathbb{C}_\theta^4))), \\ \sigma_{J(z)} &:= \sum_{\alpha} M^{\alpha} \otimes J(z_{\alpha}), \end{aligned}$$

and

$$(39) \quad \begin{aligned} \tau_{J(z)} &: K \otimes J(\mathcal{A}(\mathbb{C}_\theta^4)) \rightarrow \mathcal{A}\left(\widetilde{\mathcal{M}}_\theta(K, L)\right) \otimes L \otimes J((\mathcal{A}(\mathbb{C}_\theta^4)(1))), \\ \tau_{J(z)} &:= \sum_{\alpha} N^{\alpha} \otimes J(z_{\alpha}). \end{aligned}$$

Here, $J((\mathcal{A}(\mathbb{C}_\theta^4))$ is the left $\mathcal{A}(\mathbb{C}_\theta^4)$ -module induced by the anti-algebra map J and $\sigma_{J(z)}$, $\tau_{J(z)}$ are homomorphisms of left $\mathcal{A}(\mathbb{C}_\theta^4)$ -modules. We may also take the adjoints of the above maps. To make sense of this, we need to add to our picture the matrix elements $M_{ab}^{\alpha*}$, so that the adjoint of σ_z is

$$(40) \quad \sigma_z^* : \hat{v}_a \otimes Z \mapsto \sum_{b,\alpha} M_{ab}^{\alpha*} \otimes \hat{u}_b \otimes z_{\alpha}^* Z, \quad Z \in \mathcal{A}(\mathbb{C}_\theta^4),$$

where $a = 1, \dots, 2k + 2$, $b = 1, \dots, k$ and $\alpha = 1, \dots, 4$. Let us denote by $M^{\alpha\dagger}$ the $k \times (2k + 2)$ matrix with entries $(M^{\alpha\dagger})_{ba} = M_{ab}^{\alpha*}$. Then with respect to the above choice of bases, the adjoint map σ_z^* may be written more compactly as

$$\sigma_z^* = \sum_{\alpha} M^{\alpha\dagger} \otimes z_{\alpha}^*.$$

Similarly, we add the matrix elements $N_{dc}^{\alpha*}$ and write $(N^{\alpha\dagger})_{cd} = N_{dc}^{\alpha*}$, so that the adjoint of τ_z is

$$\tau_z^* : w_b \otimes Z \mapsto \sum_{a,\alpha} N_{ba}^{\alpha*} \otimes v_a \otimes z_{\alpha}^* Z,$$

or $\tau_z^* = \sum_{\alpha} N^{\alpha\dagger} \otimes z_{\alpha}^*$ in compact notation. The elements $M_{ab}^{\alpha*}$ are the generators of the algebra $\mathcal{A}(\widetilde{\mathcal{M}}_\theta(K^*, H^*))$, whereas the elements $N_{dc}^{\alpha*}$ are the generators the algebra $\mathcal{A}(\widetilde{\mathcal{M}}_\theta(L^*, K^*))$. Applied to equations (38) and (39), all of this yields a pair of homomorphisms of right $\mathcal{A}(\mathbb{C}_\theta^4)$ -modules

$$\begin{aligned} \sigma_{J(z)}^* &: K^* \otimes J(\mathcal{A}(\mathbb{C}_\theta^4))^* \rightarrow \mathcal{A}\left(\widetilde{\mathcal{M}}_\theta(K^*, H^*)\right) \otimes H^* \otimes J(\mathcal{A}(\mathbb{C}_\theta^4))^*(1), \\ \tau_{J(z)}^* &: L^* \otimes J(\mathcal{A}(\mathbb{C}_\theta^4))^*(-1) \rightarrow \mathcal{A}\left(\widetilde{\mathcal{M}}_\theta(L^*, K^*)\right) \otimes K^* \otimes J(\mathcal{A}(\mathbb{C}_\theta^4))^*, \end{aligned}$$

defined respectively by

$$\sigma_{J(z)}^* = \sum_{\alpha} M^{\alpha\dagger} \otimes J(z_{\alpha})^*, \quad \tau_{J(z)}^* = \sum_{\alpha} N^{\alpha\dagger} \otimes J(z_{\alpha})^*.$$

Of course, we may identify the vector spaces H and L^* through the basis isomorphism $u_b \mapsto \hat{w}_b$ for each $b = 1, \dots, k$. Similarly the isomorphism $v_a \mapsto \hat{v}_a$ for $a = 1, \dots, 2k + 2$ gives an identification of the vector space K with its dual K^* . Also, the right module $J(\mathcal{A}(\mathbb{C}_\theta^4))^*$ may be identified with $\mathcal{A}(\mathbb{C}_\theta^4)$ by the composition of the map J with the involution $*$ (noting that this identification is not the identity map). Through these

identifications, we may think of $\sigma_{J(z)}^*$ and $\tau_{J(z)}^*$ as module homomorphisms

$$\begin{aligned}\tau_{J(z)}^* &: H \otimes \mathcal{A}(\mathbb{C}_\theta^4)(-1) \rightarrow \mathcal{A}(\widetilde{\mathcal{M}}_\theta(H, K)) \otimes K \otimes \mathcal{A}(\mathbb{C}_\theta^4), \\ \sigma_{J(z)}^* &: K \otimes \mathcal{A}(\mathbb{C}_\theta^4) \rightarrow \mathcal{A}(\widetilde{\mathcal{M}}_\theta(K, L)) \otimes L \otimes \mathcal{A}(\mathbb{C}_\theta^4)(1).\end{aligned}$$

It is straightforward to check that we now have $\sigma_{J(z)}^* \tau_{J(z)}^* = 0$ and so all of this means that the maps $\sigma_{J(z)}^*$ and $\tau_{J(z)}^*$ also give a parameterisation of the noncommutative space of monads, albeit a different parameterisation from the one we started with. In the classical case the above procedure applied to a given monad yields again a monad, although it is not necessarily the one we started with. In fact, in the classical case, one is interested only in the subset of monads which are invariant under the above construction, namely the monad obtained by applying J and dualising is required to be isomorphic to the one we start with (this is the sense in which we require monads to be compatible with J). We call such monads *self-dual*. In our algebraic framework, where we work not with specific monads but rather with the (possibly noncommutative) space $\widetilde{\mathcal{M}}_{\theta;k}$ of all monads, this extra requirement is encoded as follows.

Proposition 4.5. *The space of self-dual monads is parameterised by the algebra $\mathcal{A}(\widetilde{\mathcal{M}}_{\theta;k}^{SD})$, the quotient of the algebra $\mathcal{A}(\widetilde{\mathcal{M}}_{\theta;k})$ by the further relations*

$$(41) \quad N^1 = -M^{2\dagger}, \quad N^2 = M^{1\dagger}, \quad N^3 = -M^{4\dagger}, \quad N^4 = M^{3\dagger}.$$

Proof. The condition that the maps σ_z and τ_z should parameterise self-dual monads is that $\sigma_z = \tau_{J(z)}^*$, equivalently that $\tau_z = -\sigma_{J(z)}^*$. In terms of the matrices M^α , N^α , the former condition reads

$$(42) \quad \sum_\alpha M^\alpha \otimes z_\alpha = \sum_\alpha N^{\alpha\dagger} \otimes J(z_\alpha)^*.$$

Equating coefficients of generators of $\mathcal{A}(\mathbb{C}_\theta^4)$ in each of these equations yields the extra relations as stated. \square

Remark 4.6. The identification of the vector space K with its dual K^* means that the module $K \otimes \mathcal{A}(\mathbb{C}_\theta^4)$ acquires a bilinear form given by

$$(43) \quad (\xi, \eta) := \langle J\xi | \eta \rangle = \sum_a (J\xi)_a^* \eta_a$$

for $\xi = (\xi_a)$ and $\eta = (\eta_a) \in K \otimes \mathcal{A}(\mathbb{C}_\theta^4)$, with $\langle \cdot | \cdot \rangle$ the canonical Hermitian structure on $K \otimes \mathcal{A}(\mathbb{C}_\theta^4)$. The monad condition, which now reads

$$0 = \tau_z \sigma_z = -\sigma_{J(z)}^* \sigma_z,$$

translates into the more practical condition that the columns of the matrix σ_z (equivalently the rows of τ_z) are orthogonal with respect to the form (\cdot, \cdot) .

Moreover, we see that

$$\begin{aligned}0 &= \tau_{z+J(z)} \sigma_{z+J(z)} = \tau_z \sigma_z + \tau_z \sigma_{J(z)} + \tau_{J(z)} \sigma_z + \tau_{J(z)} \sigma_{J(z)} = \tau_z \sigma_{J(z)} + \tau_{J(z)} \sigma_z \\ &= -\sigma_{J(z)}^* \sigma_{J(z)} + \sigma_z^* \sigma_z\end{aligned}$$

so that in the matrix algebra $M_k(\mathbb{C}) \otimes \mathcal{A}(\widetilde{\mathcal{M}}_{\theta;k}^{SD}) \otimes \mathcal{A}(\mathbb{C}_\theta^4)$ we have also

$$\sigma_{J(z)}^* \sigma_{J(z)} = \sigma_z^* \sigma_z.$$

Remark 4.7. The above identifications of vector spaces $H \cong L^*$ and $K \cong K^*$ yield an identification of $\mathcal{A}(\widetilde{\mathcal{M}}_\theta(H, K))$ with $\mathcal{A}(\widetilde{\mathcal{M}}_\theta(L^*, K^*))$ and hence a reality structure on the generators M_{ab}^α . It follows that the space of self-dual monads is parameterised by a total of $4k(2k+2)$ generators M_{ab}^α . As already remarked, the condition $\sigma_{J(z)}^* \sigma_z = 0$ is equivalent to demanding that the columns of σ_z are pairwise orthogonal with respect to the bilinear form (\cdot, \cdot) and, since σ_z has k columns, this yields $\frac{1}{2}k(k-1)$ such orthogonality conditions. Now as in Prop. 4.3 we may equate to zero the coefficients of the products $z_\alpha z_\beta$ for $\alpha \leq \beta$, and we note that there are 10 such coefficients in each orthogonality condition. This yields a total of $5k(k-1)$ constraints on the generators M_{ab}^α .

4.3. ADHM construction of noncommutative instantons. We are ready for the construction of charge k noncommutative bundles with instanton connections. As in previous sections, we have the $(2k+2) \times k$ algebra-valued matrices

$$\begin{aligned}\sigma_z &= M^1 \otimes z_1 + M^2 \otimes z_2 + M^3 \otimes z_3 + M^4 \otimes z_4, \\ \sigma_{J(z)} &= -M^1 \otimes z_2^* + M^2 \otimes z_1^* - M^3 \otimes z_4^* + M^4 \otimes z_3^*\end{aligned}$$

which, as already observed, have the property that $\sigma_{J(z)}^* \sigma_z = 0$ and $\sigma_{J(z)}^* \sigma_{J(z)} = \sigma_z^* \sigma_z$.

Lemma 4.8. *The entries of the matrix $\rho^2 := \sigma_z^* \sigma_z = \sigma_{J(z)}^* \sigma_{J(z)}$ commute with the entries of the matrix σ_z .*

Proof. One finds that the μ, ν entry of ρ^2 is

$$(\rho^2)_{\mu\nu} = \sum_{r,\alpha,\beta} (M^{\alpha\dagger})_{\mu r} M_{r\nu}^\beta \otimes z_\alpha^* z_\beta$$

and that the a, b entry of σ_z is

$$(\sigma_z)_{ab} = \sum_\gamma M_{ab}^\gamma \otimes z_\gamma.$$

It is straightforward to check that these elements always commute using the relations (3) for $\mathcal{A}(\mathbb{C}_\theta^4)$ and the relations of Prop. 4.1 for $\mathcal{A}(\widetilde{\mathcal{M}}_\theta(H, K))$. The essential feature is that every factor of $\eta_{\beta\alpha}$ coming from the relations between the M^α 's is cancelled by a factor of $\eta_{\alpha\beta}$ coming from the relations between the z_α 's. \square

We need to enlarge slightly the matrix algebra $M_k(\mathbb{C}) \otimes \mathcal{A}(\widetilde{\mathcal{M}}_{\theta;k}^{SD}) \otimes \mathcal{A}(\mathbb{C}_\theta^4)$ by adjoining an inverse element ρ^{-2} for ρ^2 , together with a square root ρ^{-1} . That these matrices may be inverted is an assumption, even in the commutative case where doing so corresponds to the deletion of the non-generic points of the moduli space; these correspond to so-called ‘instantons of zero size’.

From the previous lemma the matrix ρ^2 , which is self-adjoint by construction, has entries in the centre of the algebra $\mathcal{A}(\widetilde{\mathcal{M}}_{\theta;k}^{SD}) \otimes \mathcal{A}(\mathbb{C}_\theta^4)$, so these new matrices ρ^{-1} and ρ^{-2} must also be self-adjoint with central entries. We collect the matrices $\sigma_z, \sigma_{J(z)}$ together into the $(2k+2) \times 2k$ matrix

$$(44) \quad \mathbf{U} := \begin{pmatrix} \sigma_z & \sigma_{J(z)} \end{pmatrix}$$

and we have by construction that

$$\mathbf{U}^* \mathbf{U} = \rho^2 \begin{pmatrix} \mathbb{I}_k & 0 \\ 0 & \mathbb{I}_k \end{pmatrix},$$

where \mathbb{I}_k denotes the $k \times k$ identity matrix. This of course means that the quantity

$$(45) \quad Q := U\rho^{-2}U^* = \sigma_z\rho^{-2}\sigma_z^* + \sigma_{J(z)}\rho^{-2}\sigma_{J(z)}^*$$

is automatically a projection: $Q^2 = Q = Q^*$. For convenience we denote

$$Q_z := \sigma_z\rho^{-2}\sigma_z^*, \quad Q_{J(z)} := \sigma_{J(z)}\rho^{-2}\sigma_{J(z)}^*,$$

which are themselves projections, in fact orthogonal ones, $Q_{J(z)}Q_z = 0$, due to the fact that $\sigma_{J(z)}^*\sigma_z = 0$.

Lemma 4.9. *The trace of the projection Q_z is equal to k ; likewise for $Q_{J(z)}$.*

Proof. We compute the trace as follows:

$$\begin{aligned} \text{Tr } Q_z &= \sum_{\mu} (\sigma_z\rho^{-2}\sigma_z^*)_{\mu\mu} = \sum_{\mu,r,s} (\sigma_z)_{\mu r} (\rho^{-2})_{rs} (\sigma_z^*)_{s\mu} = \sum_{\mu,r,s} (\rho^{-2})_{rs} (\sigma_z)_{\mu r} (\sigma_z^*)_{s\mu} \\ &= \sum_{\mu,r,s} (\rho^{-2})_{rs} (\sigma_z^*)_{s\mu} (\sigma_z)_{\mu r} = \sum_{r,s} (\rho^{-2})_{rs} (\sigma_z^*\sigma_z)_{sr} = \text{Tr } \mathbb{I}_k = k. \end{aligned}$$

In the third equality we have used the fact that, as said, the entries of ρ^{-2} commute with those of σ_z , whereas in the fourth equality we have used the fact that every element of $\mathcal{A}(\widetilde{\mathcal{M}}_{\theta;k}^{SD}) \otimes \mathcal{A}(\mathbb{C}P^3)$ commutes with its own adjoint. An analogous chain of equality establishes the same result for the projection $Q_{J(z)}$. \square

As a consequence the projection Q has trace $2k$.

Proposition 4.10. *The operator*

$$P := \mathbb{I}_{2k+2} - Q$$

is a projection in the algebra $M_{2k+2}(\mathcal{A}(\widetilde{\mathcal{M}}_{\theta;k}^{SD}) \otimes \mathcal{A}(S_\theta^4))$ with trace equal to 2.

Proof. The entries of the projection Q_z are in the subalgebra of $\mathcal{A}(\widetilde{\mathcal{M}}_{\theta;k}^{SD}) \otimes \mathcal{A}(\mathbb{C}P^3)$ made of $U(1)$ -invariants which, by the discussion of Sect. 3.2, is precisely $\mathcal{A}(\widetilde{\mathcal{M}}_{\theta;k}^{SD}) \otimes \mathcal{A}(\mathbb{C}P^3)$. Now recall from Sect. 2.3 that the degree one elements of $\mathcal{A}(\mathbb{C}P^3)$ of the form $Z + J(Z)$ generate the J -invariant subalgebra, which may be identified with $\mathcal{A}(S_\theta^4)$. The entries of Q_z being linear in the generators of $\mathcal{A}(\mathbb{C}P^3)$, it follows that the projection Q has entries in $\mathcal{A}(\widetilde{\mathcal{M}}_{\theta;k}^{SD}) \otimes \mathcal{A}(S_\theta^4)$. The same is true of the complementary projection P as well. Finally, since the projection Q has trace $2k$, the trace of the projector P is just 2. \square

We think of the projective right $\mathcal{A}(S_\theta^4)$ -module $\mathcal{E} := P\mathcal{A}(S_\theta^4)^{2k+2}$ as defining a family of rank two vector bundles over S_θ^4 parameterised by the noncommutative space $\widetilde{\mathcal{M}}_{\theta;k}^{SD}$. We equip this family of vector bundles with the associated family of Grassmann connections $\nabla := P \circ (\text{id} \otimes d)$, after extending the exterior derivative from $\mathcal{A}(S_\theta^4)$ to $\mathcal{A}(\widetilde{\mathcal{M}}_{\theta;k}^{SD}) \otimes \mathcal{A}(S_\theta^4)$ by $\text{id} \otimes d$. Moreover, we need also to extend the Hodge $*$ -operator as $\text{id} \otimes *_\theta$.

Proposition 4.11. *The curvature $F = P((\text{id} \otimes d)P)^2$ of the Grassmann connection $\nabla = P \circ (\text{id} \otimes d)$ is anti-self-dual.*

Proof. When $\theta = 0$ this construction is the usual ADHM construction and it is known [2] (cf. also [19]) that it produces connections whose curvature is an anti-self-dual two-form:

$$*_\theta P((\text{id} \otimes d)P)^2 = -P((\text{id} \otimes d)P)^2.$$

As observed in Sect. 2.1, the Hodge $*$ -operator is defined by the same formula as it is classically and, as vector spaces, the self-dual and anti-self-dual two-forms $\Omega_{\pm}^2(S_{\theta}^4)$ are the same as their undeformed counterparts $\Omega_{\pm}^2(S^4)$. This identification survives the tensoring by $\mathcal{A}(\widetilde{\mathcal{M}}_{\theta,k}^{SD})$ which yields $\mathcal{A}(\widetilde{\mathcal{M}}_{\theta,k}^{SD}) \otimes \Omega_{\pm}^2(S_{\theta}^4)$ to be isomorphic, as vector spaces, to $\mathcal{A}(\widetilde{\mathcal{M}}_k^{SD}) \otimes \Omega_{\pm}^2(S^4)$. Thus the anti-self-duality holds also when $\theta \neq 0$. \square

Remark 4.12. One may alternatively verify the anti-self-duality via a complex structure as follows. Indeed, there is an (almost) complex structure $\gamma : \Omega^1(\mathbb{CP}_{\theta}^3) \rightarrow \Omega^1(\mathbb{CP}_{\theta}^3)$ given by $\gamma(dz_l) := (d \circ J)(z_l)$, $l = 1, \dots, 4$, the operator J being the one defined in (15), for which we declare the forms dz_l to be holomorphic and the forms dz_l^* to be anti-holomorphic. For instance, on generators of $\mathcal{A}(\mathbb{CP}_{\theta}^3)$ we have

$$d(z_j z_l^*) = \eta_{lj} z_l^* dz_j + z_j dz_l^*$$

from the Leibniz rule and the relations (3), and we write $d = \partial + \bar{\partial}$ with respect to this decomposition into holomorphic and anti-holomorphic forms. Since as vector spaces the various graded components of the differential algebra $\Omega(\mathbb{CP}_{\theta}^3)$ are undeformed, these operators $\partial, \bar{\partial}$ extend to a full Dolbeault complex with $\partial^2 = \bar{\partial}^2 = \bar{\partial}\partial + \partial\bar{\partial} = 0$. The algebra inclusion $\mathcal{A}(S_{\theta}^4) \hookrightarrow \mathcal{A}(\mathbb{CP}_{\theta}^3)$ extends to an inclusion of differential graded algebras $\Omega(S_{\theta}^4) \hookrightarrow \Omega(\mathbb{CP}_{\theta}^3)$ and the Hodge operator $*_{\theta}$ is, as in the classical case, defined in such a way that a two-form $\omega \in \Omega^2(S_{\theta}^4)$ is anti-self-dual if and only if its image in $\Omega^2(\mathbb{CP}_{\theta}^3)$ is of type (1, 1). Thus, to check that the curvature $P((\text{id} \otimes d)P)^2$ is anti-self-dual, we use this inclusion of forms (*i.e.* we express everything in terms of dz_j, dz_j^*) and check that each component $F_{ad} = P_{ab}((\text{id} \otimes d)P_{bc}) \wedge (\text{id} \otimes d)P_{cd}$ of the curvature is a sum of terms of type (1, 1). This approach to noncommutative twistor theory, including a more explicit description of the noncommutative Penrose-Ward Transform, will be discussed in more detail elsewhere [6].

We next turn to the computation of the topological charge of the family of bundles $\mathcal{E} := P\mathcal{A}(S_{\theta}^4)^{2k+2}$ given above. To this end we observe that the matrix σ_z has k linearly independent columns (since if not, it would not be injective) and that the columns of $\sigma_{J(z)}$ are obtained from those of σ_z by applying the map J . Clearly we are free to rearrange the columns of the matrix U (since this will not alter the class of the projection P), whence we may as well arrange them as

$$U = \left(\sigma_z^{(1)} \quad J(\sigma_z^{(1)}) \quad \sigma_z^{(2)} \quad J(\sigma_z^{(2)}) \quad \dots \quad \sigma_z^{(k)} \quad J(\sigma_z^{(k)}) \right),$$

where $\sigma_z^{(l)}$ denotes the l -th column of σ_z and $J(\sigma_z^{(l)})$ denotes the l -th column of $\sigma_{J(z)}$. For fixed l , we denote the entries of the column $\sigma_z^{(l)}$ (together with their conjugates) by

$$w_{\mu}^{(l)} := \sum_{\alpha} M_{\mu l}^{\alpha} \otimes z_{\alpha}, \quad (w_{\mu}^{(l)})^* := \sum_{\alpha} M_{\mu l}^{\alpha*} \otimes z_{\alpha}^*, \quad \mu = 1, \dots, 2k + 2.$$

The entries of the column $J(\sigma_z^{(l)})$ are obtained from those of $\sigma_z^{(l)}$ by applying the map J , and one clearly has $J((w_{\mu}^{(l)})^*) = (J(w_{\mu}^{(l)}))^*$. In the classical limit $\theta = 0$, we could evaluate the parameters M_{ab}^{α} as fixed numerical values: this would identify the columns $\sigma_z^{(l)}$ and $J(\sigma_z^{(l)})$ as spanning a quaternionic line in \mathbb{H}^{k+1} , where the latter is defined by the $2k + 2$ complex coordinates $w_{\mu}^{(l)}$ and their conjugates, equipped with an anti-involution J . In the noncommutative case although we lack the evaluation of the parameters M_{ab}^{α}

we continue to interpret the columns $\sigma_z^{(l)}$ and $J(\sigma_z^{(l)})$ as spanning a ‘one-dimensional’ quaternionic line.

As already observed in Rem. 4.6, the columns of σ_z are orthogonal, as are the columns of $\sigma_{J(z)}$; whence the rank $2k$ projection Q in (45) decomposes as a sum of projections

$$Q = Q_1 + \dots + Q_k,$$

where $Q_l := \tilde{\Psi}_l \tilde{\Psi}_l^*$ is the rank two projection defined by the $(2k+2) \times 2$ matrix $\tilde{\Psi}_l$ comprised of the columns $\sigma_z^{(l)}$ and $J(\sigma_z^{(l)})$, appropriately normalised by ρ^{-1} . Explicitly, this matrix is

$$\tilde{\Psi}_l := \left(\sum_{r,\alpha} (M_{\mu r}^\alpha \otimes z_\alpha) (\rho^{-1})_{rl} \quad \sum_{s,\beta} (M_{\mu s}^\beta \otimes J(z_\beta)) (\rho^{-1})_{sl} \right)_{\mu=1,\dots,2k+2},$$

and a direct check yields $\tilde{\Psi}_l^* \tilde{\Psi}_l = \mathbb{I}_2$ so that Q_l is indeed a projection for each $l = 1, \dots, k$. Hence the matrix U in (44) has $2k$ columns which we interpret as spanning k quaternionic lines, with the same being true of the normalised matrix $U\rho^{-1}$. The computation of the topological charge of the projection Q therefore boils down to the computation of the charge of each of the projections Q_l , for $l = 1, \dots, k$.

Lemma 4.13. *For each $l = 1, \dots, k$ the projection Q_l is Murray-von Neumann equivalent to the projection $1 \otimes p$ in the algebra $M_{2k+2}(\mathcal{A}(\widetilde{\mathcal{M}}_{\theta;k}^{SD}) \otimes \mathcal{A}(S_\theta^4))$, where p is the basic projection defined in equation (27).*

Proof. From equations (26) and (27) we know that $p = \Psi\Psi^*$. Then, for each $l = 1, \dots, k$ define a partial isometry V_l in $M_{2k+2,4}(\mathcal{A}(\widetilde{\mathcal{M}}_{\theta;k}^{SD}) \otimes \mathcal{A}(S_\theta^4))$ by

$$V_l := \tilde{\Psi}_l (1 \otimes \Psi^*), \quad V_l^* := (1 \otimes \Psi) \tilde{\Psi}_l^*.$$

Straightforward computations show that $V_l V_l^* = Q_l$ and $V_l^* V_l = 1 \otimes p$. \square

We invoke the strategy of [16] to compute the topological charge of the family of bundles defined by each Q_l . Indeed, the charge of the projection p was shown in [14] be equal to 1, given as a pairing between the second Chern class $\text{ch}_2(p)$, which lives in the cyclic homology group $\text{HC}_4(\mathcal{A}(S_\theta^4))$, with the fundamental class of S_θ^4 , which lives in the cyclic cohomology $\text{HC}^4(\mathcal{A}(S_\theta^4))$. Although the class $\text{ch}_2(Q_l)$, being an element in $\text{HC}_4(\mathcal{A}(\widetilde{\mathcal{M}}_{\theta;k}^{SD}) \otimes \mathcal{A}(S_\theta^4))$, may not *a priori* be paired with the fundamental class of S_θ^4 , Kasparov’s KK-theory is used to show that in fact there is a well-defined pairing between the K-theory $K_0(\mathcal{A}(\widetilde{\mathcal{M}}_{\theta;k}^{SD}) \otimes \mathcal{A}(S_\theta^4))$ and the K-homology $K^0(\mathcal{A}(S_\theta^4))$. Since by the previous lemma the projections $1 \otimes p$ and Q_l define the same class in the K-theory of $\mathcal{A}(\widetilde{\mathcal{M}}_{\theta;k}^{SD}) \otimes \mathcal{A}(S_\theta^4)$, it follows as in [16] that the topological charge of each projection Q_l is equal to 1.

Proposition 4.14. *The topological charge of the family of bundles $\mathcal{E} = P\mathcal{A}(S_\theta^4)^{2k+2}$ is equal to $-k$.*

Proof. By the above argument the projections Q_l have topological charge equal to 1 for each $l = 1, \dots, k$. The projection $Q = Q_1 + \dots + Q_k$ therefore has charge k , whence P must have charge equal to $-k$. \square

We finish this section by remarking that the construction given above in the section has an interpretation in terms of ‘universal connections’, as described in [3]. As already said, the classical quaternion vector space \mathbb{H}^{k+1} may be identified with the complex vector space \mathbb{C}^{2k+2} equipped with the quaternionic structure J . Points of the Grassmannian manifold $\text{Gr}_k(\mathbb{H}^{k+1})$ of quaternionic k -dimensional subspaces of \mathbb{H}^{k+1} may thus be identified with $2k$ -dimensional subspaces of \mathbb{C}^{2k+2} which are invariant under the involution J . Following the general strategy of [5] for the coordinatisation of Grassmannians, the algebra of functions on $\text{Gr}_k(\mathbb{H}^{k+1})$ is given by functions taking values in the set of rank $2k$ projectors $P = (P^\mu_\nu)$ on \mathbb{C}^{2k+2} which are J -invariant, *viz.*

$$\begin{aligned} \mathcal{A}(\text{Gr}_k(\mathbb{H}^{k+1})) &:= \\ &= \mathbb{C} \left[P^\mu_\nu \mid \sum_\lambda P^\mu_\lambda P^\lambda_\nu = P^\mu_\nu, (P^\mu_\nu)^* = P^\nu_\mu, \sum_\mu P^\mu_\mu = 2k, J(P^\mu_\nu) = P^\mu_\nu \right], \end{aligned}$$

where $\mu, \nu = 1, \dots, 2k + 2$. In the classical case, when $\theta = 0$, the projection Q in (45) realises $\mathcal{A}(S_{\theta=0}^4)$ as a subalgebra of $\mathcal{A}(\text{Gr}_k(\mathbb{H}^{k+1}))$, whence this construction should be viewed as the dual of an embedding $S^4 \hookrightarrow \text{Gr}_k(\mathbb{H}^{k+1})$, as given in [3]. We expect that, in the deformed case, the projection Q views $\mathcal{A}(S_\theta^4)$ as a subalgebra of a suitably-deformed version of $\mathcal{A}(\text{Gr}_k(\mathbb{H}^{k+1}))$. For fixed k , the set of monads is bound to parameterise the set of such ‘algebra embeddings’.

4.4. ADHM construction of charge one instantons. As a way of illustration we briefly verify that the above ADHM construction of noncommutative families of instantons gives back the family constructed in [16] when performed for the charge one case.

The starting point is the basic instanton on S_θ^4 described in Sect. 2.2 and which arises via a monad construction as follows. The monad we consider is the sequence

$$(46) \quad \mathcal{A}(\mathbb{C}_\theta^4)(-1) \xrightarrow{\sigma_z} \mathbb{C}^4 \otimes \mathcal{A}(\mathbb{C}_\theta^4) \xrightarrow{\tau_z} \mathcal{A}(\mathbb{C}_\theta^4)(1),$$

where the arrows are the maps

$$\sigma_z = (z_1 \ z_2 \ z_3 \ z_4)^t, \quad \tau_z = \sigma_{J(z)}^* = (-z_2^* \ z_1^* \ -z_4^* \ z_3^*).$$

Since $\tau_z \sigma_z = \sigma_{J(z)}^* \sigma_z = 0$, it is clear that this is a monad with $k = 1$; by construction it is self-dual. In the present case $\rho^2 = \sigma_z^* \sigma_z = \sum_j z_j^* z_j = r^2$, which we already assumed was invertible (corresponding to the deletion of the origin in \mathbb{C}_θ^4). One computes that

$$\text{UU}^* = \frac{1}{2} r^{-2} \begin{pmatrix} r^2 + x & 0 & \alpha & \beta \\ 0 & r^2 + x & -\mu\beta^* & \bar{\mu}\alpha^* \\ \alpha^* & -\bar{\mu}\beta & r^2 - x & 0 \\ \beta^* & \mu\alpha & 0 & r^2 - x \end{pmatrix}$$

which is just the projector p of equation (27). This is the ‘tautological’ monad construction given in [5]. The anti-self-dual version is then the projector $P = 1 - \text{UU}^*$, in agreement with the ADHM construction above.

The monad (46) may be rewritten in the form

$$(47) \quad \sigma_z = (1, 0, 0, 0)^t \otimes z_1 + (0, 1, 0, 0)^t \otimes z_2 + (0, 0, 1, 0)^t \otimes z_3 + (0, 0, 0, 1)^t \otimes z_4,$$

with τ_z defined as its dual. With the strategy of [16] one generates new instantons by coacting on the generators z_1, \dots, z_4 with the quantum conformal group $\mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))$.

Using the formula (18) for the coaction, the monad map (47) transforms into

$$(48) \quad \sigma_{\Delta_L(z)} = \begin{pmatrix} a_1 \\ a_2 \\ c_1 \\ c_2 \end{pmatrix} \otimes z_1 + \begin{pmatrix} -a_2^* \\ a_1^* \\ -c_2^* \\ c_1^* \end{pmatrix} \otimes z_2 + \begin{pmatrix} b_1 \\ b_2 \\ d_1 \\ d_2 \end{pmatrix} \otimes z_3 + \begin{pmatrix} -b_2^* \\ b_1^* \\ -d_2^* \\ d_1^* \end{pmatrix} \otimes z_4,$$

and these four column vectors are the columns of the matrix (17) which defines the algebra $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H}))$. If we write

$$\begin{aligned} \widehat{M}^1 &= (a_1 \ a_2 \ c_1 \ c_2)^t, & \widehat{M}^2 &= (-a_2^* \ a_1^* \ -c_2^* \ c_1^*)^t, \\ \widehat{M}^3 &= (b_1 \ b_2 \ d_1 \ d_2)^t, & \widehat{M}^4 &= (-b_2^* \ b_1^* \ -d_2^* \ d_1^*)^t, \end{aligned}$$

then we have the algebra relations $\widehat{M}_j^\alpha \widehat{M}_l^\beta = \eta_{jl} \eta_{\beta\alpha} \widehat{M}_l^\beta \widehat{M}_j^\alpha$ coming from the relations (20) for $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H}))$. We thus think of the algebra generated by the \widehat{M}_j^α as parameterising the set of charge one instantons, since the map $\sigma_{\Delta_L(z)}$ may be used to construct the family (28) of projections with topological charge equal to 1 and hence a family of Grassmann connections with anti-self-dual curvature, just as in [16].

In contrast, the ADHM construction of Sect. 4.3 for the case $k = 1$ says that the charge one monads are parameterised by the algebra $\mathcal{A}(\widetilde{\mathcal{M}}_{\theta;k})$ generated by the matrix elements M_j^α , with $j, \alpha = 1, \dots, 4$, subject in particular to the relations $M_j^\alpha M_l^\beta = \eta_{\beta\alpha} M_l^\beta M_j^\alpha$.

We see that these two approaches seem to give different parameterisations of the set of monads for the case $k = 1$, and hence of the set of charge one instantons. The discrepancy has its root in that the ADHM construction requires generators lying in the same row of the matrix (A_{ij}) to commute, whereas the ‘coaction approach’ given above says that such generators do not commute.

However, the discrepancy fades away when we pass to the ‘true’ parameter space for the families. On the one hand, as observed in [16], the coaction (24) of the quantum subgroup $\mathcal{A}(\mathrm{Sp}_\theta(2))$ of $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H}))$ leaves the basic one-form (11) invariant. We think of the latter coaction as generating gauge-equivalent instantons, so that the ‘true’ parameter space for this family is rather the subalgebra of $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H}))$ of coinvariants under the coaction of $\mathcal{A}(\mathrm{Sp}_\theta(2))$. The generators of this algebra are computed to be

$$\widehat{m}_{\alpha\beta} := \sum_l \widehat{M}_l^{\alpha*} \widehat{M}_l^\beta, \quad \alpha, \beta = 1 \dots, 4,$$

whose relations are easily found to be

$$(49) \quad \widehat{m}_{\alpha\beta} \widehat{m}_{\mu\nu} = \eta_{\beta\mu} \eta_{\nu\beta} \eta_{\mu\alpha} \eta_{\alpha\nu} \widehat{m}_{\mu\nu} \widehat{m}_{\alpha\beta},$$

and which certainly do not depend on the rows of the matrix (A_{ij}) . On the other hand, gauge equivalence for the ADHM family parameterised by the M_i^α is generated by the action of the classical group $\mathrm{Sp}(2)$ (we borrow this result from Prop. 5.2 in the next section), and here the invariant subalgebra is generated by elements of the form

$$m_{\alpha\beta} := \sum_l M_l^{\alpha*} M_l^\beta, \quad \alpha, \beta = 1 \dots, 4.$$

The relations in this algebra are just as in equation (49), so that these two families of charge one instantons are just the same.

5. GAUGE EQUIVALENCE OF NONCOMMUTATIVE INSTANTONS

Classically, a way to think of a gauge transformation of a vector bundle E over S^4 is as a unitary change of basis in each fibre E_x in a way which depends smoothly on $x \in S^4$. Two connections on E are said to be gauge equivalent if they are related by a gauge transformation in this way. Now, rather than being interested in the set of all instantons on S^4 , one is interested in the collection of gauge equivalence classes, that is to say classes of instantons modulo gauge transformations.

It is therefore necessary to have an analogue of the notion of gauge equivalence also for the noncommutative families of instantons constructed previously. In fact, noncommutative geometry is a very natural setting for the study of gauge transformations, as we shall see in this section; we refer in particular to [7, 8] (*cf.* also [13]).

5.1. Gauge equivalence for families of instantons. Recall that a first order differential calculus on a unital $*$ -algebra A is a pair $(\Omega^1 A, d_A)$, where $\Omega^1 A$ is an A - A -bimodule giving the space of one-forms and $d_A : A \rightarrow \Omega^1 A$ is a linear map satisfying the Leibniz rule,

$$d_A(xy) = x(d_A y) + (d_A x)y \quad \text{for all } x, y \in A.$$

One also assumes that the map $x \otimes y \rightarrow x(d_A y)$ is surjective. We say that $\Omega^1 A$ is a $*$ -calculus if for elements $x_j, y_j \in A$ one has that $\sum_j x_j dy_j = 0$ implies $\sum_j d(y_j^*)x_j^* = 0$: it follows from this condition that there is [26] a unique $*$ -structure on $\Omega^1 A$ such that $(d_A a)^* = d_A(a^*)$ for all $a \in A$. The differential calculi on $\mathcal{A}(S_\theta^4)$ and $\mathcal{A}(S_\theta^7)$ in Sect. 2.1 are examples of first order differential $*$ -calculi on noncommutative spaces.

Let us fix a choice of $*$ -calculus on A . And let \mathcal{E} be a finitely generated projective right A -module endowed with an A -valued Hermitian structure denoted $\langle \cdot | \cdot \rangle$. A connection on \mathcal{E} is a linear map $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_A \Omega^1 A$ satisfying the Leibniz rule

$$\nabla(\xi x) = (\nabla \xi)x + \xi \otimes d_A x \quad \text{for all } \xi \in \mathcal{E}, x \in A.$$

The connection ∇ is said to be compatible with the Hermitian structure on \mathcal{E} if it obeys

$$\langle \nabla \xi | \eta \rangle + \langle \xi | \nabla \eta \rangle = d_A \langle \xi | \eta \rangle \quad \text{for all } \xi \in \mathcal{E}, \eta \in \mathcal{E}.$$

On \mathcal{E} there is at least one compatible connection, the so-called Grassmann connection ∇_0 . If $P \in \text{End}_A(\mathcal{E})$, $P^2 = P = P^*$, is the projection which defines \mathcal{E} as a direct summand of a free module, that is $\mathcal{E} = P(\mathbb{C}^N \otimes A)$, then $\nabla_0 = P \circ d$. Any other connection on \mathcal{E} is of the form $\nabla = \nabla_0 + \omega$, where ω is an element of $\text{Hom}_A(\mathcal{E}, \mathcal{E} \otimes_A \Omega^1 A)$.

The gauge group of \mathcal{E} is defined to be

$$\mathcal{U}(\mathcal{E}) := \{U \in \text{End}_A(\mathcal{E}) \mid \langle U\xi | U\eta \rangle = \langle \xi | \eta \rangle \text{ for all } \xi, \eta \in \mathcal{E}\}.$$

If ∇ is a compatible connection on \mathcal{E} , each element U of the gauge group $\mathcal{U}(\mathcal{E})$ induces a ‘new’ connection by the action

$$\nabla^U := U\nabla U^*.$$

Of course, ∇^U is not really a different connection, it simply expresses ∇ in terms of the transformed bundle $U\mathcal{E}$, hence one says that a pair of connections ∇_1, ∇_2 on \mathcal{E} are *gauge equivalent* if they are related by such a gauge transformation U . In terms of the decomposition $\nabla = \nabla_0 + \omega$, one finds that $\nabla^U = \nabla_0 + \omega^U$, where

$$\omega^U := U(\nabla_0 U^*) + U\omega U^*.$$

A choice of gauge would be a choice of partial isometry $\Psi : \mathcal{E} \rightarrow \mathcal{A}^N$ such that $\Psi^*\Psi = \text{Id}_{\mathcal{E}}$ and $\Psi\Psi^* = P$. Any other gauge is then given by an element U of the gauge group of \mathcal{E} : the partial isometry Ψ gets replaced by $U\Psi$, for which we indeed have

$$(U\Psi)^*(U\Psi) = \Psi^*\Psi = \text{Id}_{\mathcal{E}}.$$

and the projection P gets transformed to

$$(U\Psi)(U\Psi)^* = U(\Psi\Psi^*)U^* = UPU^*,$$

an operation that does not change the equivalence class of P . In the fixed gauge the Grassmann connection $\nabla_0 = P \circ d$ naturally acts on ‘equivariant maps’ $\varphi = \Psi F$ where $F \in \mathcal{A}^N$. The result is an ‘equivariant one-form’

$$\nabla_0(\Psi F) = (\Psi\Psi^*)d(\Psi F) = \Psi\left(dF + \Psi^*d(\Psi)F\right),$$

and identifies the *gauge potential* to be given by

$$A = \Psi^*(d\Psi).$$

Under the transformation $\Psi \mapsto U\Psi$, the gauge potential transforms as

$$\Psi^*d\Psi \mapsto \Psi^*(d\Psi) + \Psi^*U^*(dU)\Psi.$$

We now turn back to the construction of instantons. Gauge equivalence being defined as above by unitary module endomorphisms means that we are free to act on the right $\mathcal{A}(\mathbb{C}_\theta^4)$ -module $\mathcal{K} = K \otimes \mathcal{A}(\mathbb{C}_\theta^4)$ by a unitary element of the matrix algebra $M_{2k+2}(\mathbb{C}) \otimes \mathcal{A}(\mathbb{C}_\theta^4)$. In order to preserve the instanton construction, we must do so in a way preserving the bilinear form (\cdot, \cdot) of equation (43) which comes from the identification of K with its dual K^* . Hence the map σ_z in (31) (or in (33)) is defined up to a transformation $A \in \text{End}_{\mathcal{A}(\mathbb{C}_\theta^4)}(\mathcal{K})$, which is unitary and is required to commute with the quaternion structure J . Similarly, we are free to change basis in the modules $\mathcal{H} = H \otimes \mathcal{A}(\mathbb{C}_\theta^4)$ and $\mathcal{L} = L \otimes \mathcal{A}(\mathbb{C}_\theta^4)$, provided we preserve the fact that we identify $J(\mathcal{H})^*$ and \mathcal{L} . This means that the map τ_z of (35) is defined up to an invertible transformation $B \in \text{End}_{\mathcal{A}(\mathbb{C}_\theta^4)}(\mathcal{H})$.

All this is saying is that the monad maps σ_z and τ_z were expressed as matrices with respect to a choice of basis for each of the vector spaces H , K and L ; it is natural to question the extent to which the resulting Grassmann connection ∇ depends on the choice of these bases. We denote by $\text{GL}(\mathcal{H})$ the set of automorphisms of \mathcal{H} and by $\text{Sp}(\mathcal{K})$ the set of all unitary endomorphisms of \mathcal{K} respecting the quaternion structure:

$$\text{Sp}(\mathcal{K}) := \{A \in \text{End}_{\mathcal{A}(\mathbb{C}_\theta^4)}(\mathcal{K}) \mid \langle A\xi | A\xi \rangle = \langle \xi | \xi \rangle, J(A\xi) = AJ(\xi) \text{ for all } \xi \in \mathcal{K}\}.$$

Given $A \in \text{Sp}(\mathcal{K})$ and $B \in \text{GL}(\mathcal{H})$, the gauge freedom is to map $\sigma_z \mapsto A\sigma_z B$.

Proposition 5.1. *For all $B \in \text{GL}(\mathcal{H})$, under the transformation $\sigma_z \mapsto \sigma_z B$ the projection P of Prop. 4.10 is left invariant.*

Proof. One first checks that $\rho^2 \mapsto (\sigma_z B)^*(\sigma_z B) = B^*\rho^2 B$ under this transformation, so that

$$Q_z \mapsto \sigma_z B (B^*\rho^2 B)^{-1} B^* \sigma_z^* = \sigma_z B (B^{-1} \rho^{-2} (B^*)^{-1}) B^* \sigma_z^* = Q_z,$$

whence the projection P is unchanged. \square

Proposition 5.2. *For all $A \in \text{Sp}(\mathcal{K})$, under the transformation $\sigma_z \mapsto A\sigma_z$ the projection P of Prop. 4.10 transforms as $P \mapsto APA^*$.*

Proof. Replacing σ_z by $A\sigma_z$ leaves ρ^2 invariant (since A is unitary) and so has the effect that

$$Q_z \mapsto A\sigma_z\rho^{-2}\sigma_z^*A^* = AQ_zA^*,$$

whence it follows that P is mapped to APA^* . \square

These results give the general gauge freedom on monads, although from the point of view of computing the number of constraints on the algebra generators M_{ab}^α we need only consider the effect of these transformations on the vector spaces H , K and L , *i.e.* we it is enough to consider the groups of ‘constant’ automorphisms. This means the group $\mathrm{Sp}(K) = \mathrm{Sp}(k+1) \subset \mathrm{Sp}(\mathcal{K})$ and the group $\mathrm{GL}(k, \mathbb{R}) \subset \mathrm{GL}(\mathcal{H})$ (the latter because we must preserve the identification of $J(\mathcal{H})^*$ with \mathcal{L} , and complex linear transformations of H would interfere with the tensor product in $J(\mathcal{H})^*$). In fact, it is known in the classical case that these constant transformations are sufficient to generate all gauge symmetries of the instanton bundles produced by the ADHM construction.

We conclude that also in the noncommutative case the gauge equivalence imposes a further $(k+1)(2(k+1)+1)$ constraints due to $\mathrm{Sp}(k+1)$ and a further k^2 constraints due to $\mathrm{GL}(k, \mathbb{R})$. From Rem. 4.7, the total number of generators minus the total number of constraints is thus computed to be

$$(8k^2 + 8k) - 5k(k-1) - (3k^2 + 5k + 3) = 8k - 3,$$

just as for the classical case, a result which is some how reassuring.

5.2. Morita equivalent geometries and gauge theory. It is a known idea that Morita equivalent algebras describe the same topological space. The simplest case is that of a one-point space $X = \{*\}$: the matrix algebras $M_n(\mathbb{C})$ for any positive integer n all have the same one-point spectrum. More generally, if X is a compact Hausdorff space, the algebras $C(X) \otimes M_n(\mathbb{C})$ are all Morita equivalent and all have the same spectrum X .

With this in mind, gauge theory arises naturally out of the consideration of how to transfer differential structures between Morita equivalent algebras. If one takes such structures to be defined by a Dirac operator and associated spectral triple, then the method for doing this is discussed in [7, 8]. Here we discuss a more general framework, where algebras may be equipped with differential calculi not necessarily coming from a spectral triple.

Let A be a unital $*$ -algebra and suppose that the $*$ -algebra B is Morita equivalent to A via the B - A -bimodule \mathcal{E} , that is to say $B \simeq \mathrm{End}_A(\mathcal{E})$. In addition, on \mathcal{E} there are compatible A -valued and B -valued Hermitian structures¹. Then a choice of a connection ∇ on \mathcal{E} , viewed as a right A -module, yields a differential calculus on B . First of all, the operator on B given by

$$d_B^\nabla(x) := [\nabla, x], \quad x \in B,$$

is easily seen to be a derivation: $d_B^\nabla(xy) = x(d_B^\nabla y) + (d_B^\nabla x)y$, for $x, y \in B$. The B - B -bimodule $\Omega^1 B$ of one-forms is then defined by

$$\Omega^1 B := B(d_B^\nabla(B))B.$$

¹We shall also require the Hermitian structures to be self-dual, *i.e.* every right A -module homomorphism $\varphi : \mathcal{E} \rightarrow A$ is represented by an element of $\eta \in \mathcal{E}$ by the assignment $\varphi(\cdot) = \langle \eta | \cdot \rangle$. A similar property holds for the second Hermitian structure as well.

For this to define a $*$ -calculus we need that the connection ∇ be compatible with the A -valued Hermitian structure on \mathcal{E} in the sense that

$$\langle \nabla \xi | \eta \rangle + \langle \xi | \nabla \eta \rangle = d_A \langle \xi | \eta \rangle$$

for all $a \in A$ and $\xi, \eta \in \mathcal{E}$. If this condition is satisfied, the assumption $\sum_j x_j d_B^\nabla y_j = 0$ translates into $\sum_j (x_j \nabla y_j) \xi = \sum_j x_j y_j (\nabla \xi)$ for all $x_j, y_j \in B$ and all $\xi \in \mathcal{E}$. This implies, that for all $\xi, \eta \in \mathcal{E}$ and all $x_j, y_j \in B$:

$$\begin{aligned} \sum_j \langle d_B^\nabla (y_j^*) x_j^* \xi | \eta \rangle &= \sum_j \langle \nabla (x_j^* y_j^* \xi) - x_j^* \nabla (y_j^* \xi) | \eta \rangle \\ &= \sum_j -\langle x_j^* y_j^* \xi | \nabla \eta \rangle + d_A \langle x_j^* y_j^* \xi | \eta \rangle + \langle y_j^* \xi | \nabla (x_j \eta) \rangle - d_A \langle y_j^* \xi | x_j \eta \rangle \\ &= \sum_j -\langle x_j^* y_j^* \xi | \nabla \eta \rangle + \langle \xi | y_j x_j \nabla \eta \rangle, \end{aligned}$$

whence it follows that $\sum_j d_B^\nabla (y_j^*) x_j^* = 0$ as it should be for a $*$ -calculus. We interpret the passage $d_A \rightarrow d_B^\nabla$ as an *inner fluctuation* of the geometry which results in a ‘Morita equivalent’ first order calculus $(\Omega^1 B, d_B^\nabla)$, now for the algebra B .

A natural application of this is to think of the algebra A as being Morita equivalent to itself, so that $\mathcal{E} = A$ as a right A -module and $B = A$. In this case, any Hermitian connection on \mathcal{E} is necessarily of the form

$$(50) \quad \nabla \xi = d_A \xi + \omega \xi, \quad \xi \in \mathcal{E},$$

with $\omega = -\omega^* \in \Omega^1 A$ a skew-adjoint one-form. The corresponding differential on $B = A$ is computed to be

$$(d_A^\nabla b) \xi = [\nabla, b] \xi = \nabla(b \xi) - b \nabla \xi = d_A(b \xi) + \omega b \xi - b d_A \xi - b \omega \xi = (d_A b) \xi + [\omega, b] \xi,$$

using the Leibniz rule for d_A . The passage

$$d_A \rightarrow d_A^\nabla = d_A + [\omega, \cdot]$$

is once again interpreted as an inner fluctuation of the geometry, although when A is commutative there are no non-trivial inner fluctuations and thus no new degrees of freedom generated by the above self-Morita mechanism. However, in the noncommutative situation there is an interesting special case where ω is taken to be of the form $\omega = u^* d_A u$, for u a unitary element of the algebra A . Such a fluctuation is unitarily equivalent to acting on A by the inner automorphism

$$\alpha_u : A \rightarrow A, \quad \alpha_u(a) = u a u^*,$$

since for all $a \in A$ we have that $d_A^\nabla(a) = u^* d_A(\alpha_u(a)) u$. It therefore follows that inner fluctuations defined by inner automorphisms generate gauge theory on A .

5.3. Gauge theory from quantum symmetries. We now consider a slightly different type of gauge equivalence for our instanton construction which is not present in the classical case and is a purely quantum (*i.e.* noncommutative) phenomenon.

We consider the case where A is a comodule $*$ -algebra under a left coaction of a Hopf algebra H , so that A is isomorphic to its image $B = \Delta_L(A)$. To transfer a calculus on A to one on B , a possible strategy is as follows. We take the B - A -bimodule given by $\mathcal{E} := B = \Delta_L(A)$ with left B -action and right A -action defined by

$$b \triangleright \xi := b \xi, \quad \xi \triangleleft a = \xi \Delta_L(a)$$

for $\xi \in \mathcal{E}$, $a \in A$, $b \in B$. We also assume that the calculus $\Omega^1 A$ is left H -covariant, so that Δ_L extends to a coaction on $\Omega^1 A$ as a bimodule map such that d_A is an intertwiner, whence the above bimodule structure on \mathcal{E} extends to one-forms in the natural way. This also canonically equips B with a $*$ -calculus $\Omega^1 B$, where the differential is $d_B = \text{id} \otimes d_A$.

We choose an arbitrary Hermitian connection on the right A -module \mathcal{E} for the calculus $(\Omega^1 A, d_A)$, which is necessarily of the form

$$\nabla \xi = (\text{id} \otimes d_A)\xi + \tilde{\omega}\xi, \quad \xi \in \mathcal{E}$$

with $\tilde{\omega} = \Delta_L(\omega)$ for some $\omega = -\omega^* \in \Omega^1 A$ a skew-adjoint one-form. The corresponding differential on B is again defined by

$$(d_B^{\nabla} b)\xi = [\nabla, b \triangleright] \xi = \nabla(b \triangleright \xi) - b \triangleright \nabla \xi = d_A(b \triangleright \xi) + \omega(b \triangleright \xi) - b \triangleright (d_A \xi + \omega \xi),$$

and works out to be

$$d_B b = (\text{id} \otimes d_A)b + [\tilde{\omega}, b].$$

Note also that for all $b \in B$ we have $b = \Delta_L(a)$ for some $a \in A$ and so it follows that

$$d_B b = \Delta_L(d_A a) + \Delta_L([\omega, a]),$$

so that the coaction commutes with inner fluctuations. Moreover, in the case where A is noncommutative, there are non-trivial inner automorphisms of A and hence a non-trivial gauge theory which carries over from A to $\Delta_L(A)$.

In particular, we apply this to the case $A = \mathcal{A}(S_\theta^4)$, with $H = \mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))$ the quantum conformal group of S_θ^4 . The above discussion means that the coaction of $\mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))$ on $\mathcal{A}(S_\theta^4)$ by conformal transformations in itself generates gauge freedom. The natural way to extend the exterior derivative d_A on $\mathcal{A}(S_\theta^4)$ to $\Delta_L(\mathcal{A}(S_\theta^4))$ is as $\text{id} \otimes d_A$: this corresponds to taking $\tilde{\omega} = 0$ and is the choice made in [16]. However, in general we have the freedom to make the transition

$$d_A \rightarrow (\text{id} \otimes d_A) + [\tilde{\omega}, \cdot]$$

for some $\tilde{\omega} = \Delta_L(u^* d_A u)$, where u is some unitary element of $\mathcal{A}(S_\theta^4)$. Since the group of inner automorphisms of A is trivial when A is commutative, this is a feature of gauge theory which is certainly not present in the classical case and is unique to the noncommutative paradigm. More on this will be reported elsewhere.

REFERENCES

- [1] Atiyah M.F., Ward R.S.: Instantons and Algebraic Geometry. *Comm. Math. Phys.* **55**, 117-124 (1977)
- [2] Atiyah M.F., Drinfel'd V.G., Hitchin N.J., Manin Yu.I.: Construction of Instantons. *Phys. Lett.* **65A** No. 3, 185-187 (1978)
- [3] Atiyah M.F.: *Geometry of Yang-Mills Fields*. Fermi Lectures, Scuola Normale Pisa, 1979
- [4] Barth W.: Moduli of Vector Bundles on the Projective Plane. *Invent. Math.* **42**, 63-91 (1977)
- [5] Brain S.J., Majid S.: Quantisation of Twistor Theory by Cocycle Twist. *Comm. Math. Phys.* **284** No. 3, 713-774 (2008)
- [6] Brain S.J., Majid S.: Forthcoming Paper.
- [7] Connes A.: Gravity Coupled with Matter and the Foundation of Noncommutative Geometry. *Comm. Math. Phys.* **182** No. 1, 155-176 (1996)
- [8] Connes A.: On the Spectral Characterization of Manifolds. [[arXiv:0810.2088](https://arxiv.org/abs/0810.2088)]

- [9] Connes A., Landi G.: Noncommutative Manifolds, the Instanton Algebra and Isospectral Deformations. *Comm. Math. Phys.* **221** No. 1, 141-159 (2001)
- [10] Cuntz J., Quillen D.: Algebra Extensions and Nonsingularity. *J. Amer. Math. Soc.* **8** No. 2, 251-289 (1995)
- [11] Dubois-Violette M., Georgelin Y.: Gauge Theory in Terms of Projector-Valued Fields. *Phys. Lett.* **82B** (1979)
- [12] Horrocks G.: Vector Bundles on the Punctured Spectrum of a Ring. *Proc. London Math. Soc.* **14** No. 3, 689-713 (1964)
- [13] Landi G.: *An Introduction to Noncommutative Spaces and their Geometry*. Lecture Notes in Physics **51**. Springer-Verlag, Berlin, 1997
- [14] Landi G., van Suijlekom W.D.: Principal Fibrations from Noncommutative Spheres. *Comm. Math. Phys.* **260** No. 1, 203-225 (2005)
- [15] Landi G., van Suijlekom W.D.: Noncommutative Instantons from Twisted Conformal Symmetries. *Comm. Math. Phys.* **271** No. 3, 591-634 (2007)
- [16] Landi G., Pagani C., Reina C., van Suijlekom W.D.: Noncommutative Families of Instantons. *Int. Math. Res. IMRN* **12**, Art. ID rnn038 (2008)
- [17] Majid S., Markl M.: Glueing Operations for R-Matrices, Quantum Groups and Link-Invariants of Hecke Type. *Math. Proc. Camb. Phil. Soc.* **119**, 139-166 (1996)
- [18] Majid S.: Quantum and Braided Diffeomorphism Groups. *J. Geom. Phys.* **28**, 94-128 (1998)
- [19] Mason L.J., Woodhouse N.M.J.: *Integrability, Self-Duality and Twistor Theory*. Oxford University Press, 1996
- [20] Okonek C., Schneider M., Spindler H.: *Vector Bundles on Complex Projective Spaces*. Birkhauser, Boston, 1980
- [21] Penrose R., Rindler W.: *Spinors in Space-Time* vol. 2. Cambridge University Press, 1986
- [22] Soltan P.M.: Quantum Families of Maps and Quantum Semigroups on Quantum Finite Spaces. [[arXiv:0610922](https://arxiv.org/abs/0610922)]
- [23] Wang S.: Quantum Symmetry Groups of Finite Spaces. *Commun. Math. Phys.* **195**, 195-211 (1998)
- [24] Ward R.S.: On Self-Dual Gauge Fields. *Phys. Lett.* **61A** No. 2, 81-82 (1977)
- [25] Woronowicz S.L.: Pseudogroups, Pseudospaces and Pontryagin Duality. In *Proceedings of the International Conference on Mathematical Physics, Lausanne 1979*, Ed. K. Osterwalder. Lecture Notes in Physics **116**, Springer, 407-412, 1980
- [26] Woronowicz S.L.: Differential Calculus on Compact Matrix Pseudogroups (Quantum Groups). *Comm. Math. Phys.* **122**, 125-170 (1989)

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