THE WAVE EQUATION ON DOMAINS WITH CRACKS GROWING ON A PRESCRIBED PATH: EXISTENCE, UNIQUENESS, AND CONTINUOUS DEPENDENCE ON THE DATA

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Abstract. Given a bounded open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary and an increasing family Γ_t, $t \in [0,T]$, of closed subsets of Ω, we analyze the scalar wave equation $\ddot{u} - \mathrm{div}(A\nabla u) = f$ in the time varying cracked domains $\Omega \setminus \Gamma_t$. Here we assume that the sets Γ_t are contained into a *prescribed* $(d-1)$-manifold of class C^2. Our approach relies on a change of variables: recasting the problem on the reference configuration $\Omega \setminus \Gamma_0$, we are led to consider a hyperbolic problem of the form $\ddot{v} - \mathrm{div}(B\nabla v) + a \cdot \nabla v - 2b \cdot \nabla \dot{v} = g$ in $\Omega \setminus \Gamma_0$. Under suitable assumptions on the regularity of the change of variables that transforms $\Omega \setminus \Gamma_t$ into $\Omega \setminus \Gamma_0$, we prove existence and uniqueness of weak solutions for both formulations. Moreover, we provide an energy equality, which gives, as a by-product, the continuous dependence of the solutions with respect to the cracks.

Keywords: wave equation, second order linear hyperbolic equations, dynamic fracture mechanics, cracking domains.

MSC 2010: 35L05, 35L20, 74R10.

Introduction

In the study of mathematical models for dynamic crack propagation, it is important to determine the behavior of the solutions of the system of equations of elastodynamics in domains with a time-dependent crack (see [8]). In this paper we give a contribution in this direction, by studying the existence, uniqueness, and continuous dependence on the cracks of the solutions of the wave equation for a scalar variable in a domain with a prescribed time-dependent crack. This corresponds to the antiplane case for elastodynamics.

We fix a bounded open set $\Omega \subset \mathbb{R}^d$, a time interval $[0,T]$, and a family Γ_t, $t \in [0,T]$, of (possibly irregular) closed subsets of $\overline{\Omega}$, increasing with respect to inclusion and contained in a given C^2 manifold Γ of dimension $d-1$. Given a tensor field A (satisfying the usual ellipticity conditions) and a forcing term f, we study the solutions of the equation

$$\ddot{u}(t,x) - \mathrm{div}_x(A(t,x)\nabla_x u(t,x)) = f(t,x)\,, \quad t \in [0,T],\ x \in \Omega_t := \Omega \setminus \Gamma_t\,, \tag{0.1}$$

supplemented by Dirichlet and Neumann boundary conditions on prescribed parts of $\partial\Omega$, and homogeneous Neumann boundary conditions on the cracks Γ_t.

The main issue here is the fact that the domain Ω_t has not a regular boundary, due to the presence of the $(d-1)$-dimensional crack Γ_t. When Ω_t is more regular, this problem has been studied in [2, 4, 11, 12].

A notion of solution of (0.1) in a domain with a growing crack, under much weaker assumptions on the cracks Γ_t, was introduced in [5] in the case of homogeneous Neumann conditions on the whole boundary of Ω_t. The existence of a solution with prescribed initial data was proved in the same paper, while the uniqueness is still an open problem under those general assumptions.

In [10], a different approach is used, based on a suitable change of variables of class C^2, which reduces the domain $\{(t,x) \in (0,T) \times \Omega :\ x \in \Omega_t\}$ to the cylinder $(0,T) \times \Omega_0$. The transformed equation reads

$$\ddot{v}(t,y) - \mathrm{div}_y(B(t,y)\nabla_y v(t,y)) + a(t,y) \cdot \nabla_y v(t,y) - 2b(t,y) \cdot \nabla_y \dot{v}(t,y) = g(t,y) \tag{0.2}$$

for $t \in [0,T]$ and $y \in \Omega_0$, with Dirichlet and Neumann conditions on some parts of $\partial\Omega$, and homogeneous Neumann conditions on the fixed crack Γ_0. The functions B, a, b, and g depend on A, f, and on the change of variables. In that paper an existence result is proved. As for the uniqueness, it appears in the statement of [10, Theorem 3.1], but the proof is missing. In our opinion, uniqueness does not follow immediately

Preprint SISSA 47/2015/MATE.

from the arguments used in the proof of that theorem. Indeed, the existence is obtained through a viscous approximation, which consists in solving the same problem with the additional term

$$\varepsilon\dot v(t,y)-\varepsilon\Delta_y\dot v(t,y)$$

in the left-hand side. The solutions v_ε of these problems, whose existence is classical (see, e.g., [6]), are proved to converge, up to a subsequence, to a solution v of (0.2), but there is no argument to show that all subsequences converge to the same solution. Moreover, the proof of [10, Theorem 3.1] does not exclude the existence of other solutions of (0.2) obtained by different methods.

In this paper we consider the same change of variables and prove existence and uniqueness of the solution of (0.2) with prescribed initial conditions (see Theorems 2.6 and 2.10). The proof of uniqueness is obtained by adapting a classical technique developed in [7] to the case of coefficients which satisfy very mild regularity assumptions with respect to time.

Moreover, we prove the energy equality

$$\begin{aligned}&\frac12\|\dot v(t)\|^2_{L^2(\Omega_0)}+\frac12\langle B(t)\nabla_y v(t),\nabla_y v(t)\rangle_{L^2(\Omega_0)}\\&=\frac12\|\dot v(0)\|^2_{L^2(\Omega_0)}+\frac12\langle B(0)\nabla_y v(0),\nabla_y v(0)\rangle_{L^2(\Omega_0)}+\mathcal R(v,t)\,,\end{aligned}\tag{0.3}$$

where $\mathcal R$ is a continuous remainder, which can be written explicitly as an integral involving $\dot v$ and $\nabla_y v$ (cf. Proposition 2.11). The proof is based on a regularization of $\dot v$ with respect to time, in the spirit of [9]. The lack of regularity of the coefficients makes the arguments rather heavy. The energy equality (0.3) is the key point for the proof of the L^2-continuity of the functions $t\mapsto\dot v(t)$ and $t\mapsto\nabla_y v(t)$.

We also prove that the solution v of (0.2) depends continuously on the coefficients and on the data (see Theorem 3.1): if $B^n\to B$, $a^n\to a$, $b^n\to b$, and $g^n\to g$ converge in a sense made precise in the paper, then the corresponding solutions v^n of (0.2) converge strongly to v, meaning that $v^n\to v$, $\dot v^n\to\dot v$, and $\nabla_y v^n\to\nabla_y v$ strongly in L^2. The proof of the continuous dependence relies on the energy equality (0.3) and on careful estimates of the solutions v_ε used in the viscous approximation of the solution v. For this reason we reproduce the existence proof of [10], which has to be adapted to our slightly less regular situation.

Finally, we prove that the solutions of (0.1) depend continuously on the cracks Γ_t (see Theorem 3.1). Given a sequence of time-dependent cracks Γ^n_t, $t\in[0,T]$, increasing with respect to t and contained in the same manifold Γ, we consider the corresponding solutions u^n of (0.1). We give some sufficient conditions on Γ^n_t which imply the convergence of the coefficients of the transformed equations (0.2). This leads to the strong convergence of the solutions v^n of (0.2), which, after a change of variables, yields the strong convergence of u^n to u.

In general, the sufficient conditions on Γ^n_t are expressed in terms of the diffeomorphisms used in the change of variables. When $d=2$, and Γ^n_t are arcs contained in a curve Γ, the hypotheses on Γ^n_t depend only on the crack tips, i.e., on the end-points of these arcs (see Example 3.3).

We expect that this continuous dependence on the cracks will be an important tool for a precise mathematical formulation of a dynamic model of crack evolution in the spirit of [8].

The paper is organized as follows. In Section 1 we fix the notation adopted throughout the paper and we list the standing assumptions on the set Ω, on the geometry of the cracks Γ_t, and on the diffeomorphisms used for the change of variables. In Definition 1.4 we specify the notion of weak solution of problem (0.1). Then, in Theorem 1.7, we prove that its existence (and uniqueness) is equivalent to that of problem (0.2), whose weak version is specified in Definition 1.5.

Section 2 is devoted to the study of problem (0.2). First, in Theorems 2.6 and 2.10, we obtain existence and uniqueness of solutions in a function space larger than that of weak solutions. Eventually, in Proposition 2.11, we provide the energy equality (0.3), which ensures that the solution found is indeed a weak solution. The energy equality gives also the continuous dependence of solutions on the data (both for problem (0.1) and problem (0.2)), which is studied in Section 3 (see Theorem 3.1).

In the Appendix, we gather some auxiliary results and all the technical lemmas.

Explicit examples of admissible growing cracks in dimension $d=2$ are presented in Examples 1.14, 2.1, and 3.3.

1. Notation and preliminary results

The space of $m\times d$ matrices with real entries is denoted by $\mathbb{R}^{m\times d}$; in case $m=d$, the subspace of symmetric matrices is denoted by $\mathbb{R}^{d\times d}_{sym}$. Given two vectors $a,b\in\mathbb{R}^d$, their scalar product is denoted by $a\cdot b$ and their tensor product is denoted by $a\otimes b$. Given two square matrices A and B in $\mathbb{R}^{d\times d}$, we write $A\cdot B$ to denote their Euclidean scalar product, namely $A\cdot B=A_{ij}B_{ij}$. Here and in the rest of the paper we adopt the convention of summation over repeated indices. We denote by A^{-1} and A^T the inverse and the transpose matrices of A, by A^{-T} the transpose of the inverse of A. We denote by $I\in\mathbb{R}^{d\times d}$ the identity matrix, and by id the identity function in $\mathbb{R}^m$, possibly restricted to a subset.

The partial derivatives with respect to the variable x_i are denoted by ∂_{x_i} or D_i. Given a function $F\colon\mathbb{R}^d\to\mathbb{R}^m$, we denote its Jacobian matrix by DF, whose components are $(DF)_{ij}=\partial_jF_i$. If $m=d$, in order to distinguish the inverse of the Jacobian from the Jacobian of the inverse, we denote the former by $(DF)^{-1}$ and latter by DF^{-1}. For a tensor field $A\in C^1(\mathbb{R}^d;\mathbb{R}^{d\times d})$, by $\operatorname{div}A$ we mean its divergence with respect to lines, namely $(\operatorname{div}A)_i:=\partial_jA_{ij}$.

We adopt standard notations for Lebesgue and Sobolev spaces on a bounded open set of $\mathbb{R}^d$. The boundary values of a Sobolev function are always intended in the sense of traces. The $(d-1)$-dimensional Hausdorff measure is denoted by $\mathcal{H}^{d-1}$. Given an open set Ω with Lipschitz boundary, we denote by ν the outer unit normal vector to $\partial\Omega$, defined a.e. on the boundary.

Given a normed vector space X and its topological dual X^*, the norm in X is denoted by $\|\cdot\|_X$ and the duality product between X^* and X is denoted by $\langle\cdot,\cdot\rangle_X$. We adopt the same notations also for vector valued functions in X. Given an interval $I\subset\mathbb{R}$ and a Banach space X, $L^p(I;X)$ is the space of L^p functions from I to X. Given $u\in L^p(I;X)$, we denote by $\dot u\in\mathcal{D}'(I;X)$ its distributional derivative. The set of continuous and absolutely continuous functions from I to X are denoted by $C^0(I;X)$ and $AC(I;X)$, respectively. When $X=\mathbb{R}^m$, we denote the uniform norm in $C^0(I;\mathbb{R}^m)$ by $\|\cdot\|_\infty$. Given two metric spaces Y and Z, $\mathrm{Lip}(Y;Z)$ is the space of Lipschitz functions from Y to Z.

Throughout the paper we shall assume the following hypotheses on the set Ω, on the geometry of the cracks Γ_t, and on the diffeomorphisms of Ω into itself mapping Γ_0 into Γ_t:

- (H1) $\Omega\subset\mathbb{R}^d$ is a bounded open set with Lipschitz boundary $\partial\Omega$;
- (H2) $\partial_D\Omega$ is a (possibly empty) Borel subset of $\partial\Omega$ and $\partial_N\Omega$ is its complement;
- (H3) Γ is a C^2 manifold of dimension $d-1$ contained in $\overline{\Omega}$ and with boundary $\partial\Gamma$;
- (H4) $\Gamma\cap\partial\Omega=\partial\Gamma$ and $\Omega\setminus\Gamma$ is union of two disjoint open sets Ω^+ and Ω^- with Lipschitz boundary;
- (H5) $T>0$;
- (H6) Γ_t, $t\in[0,T]$, is a family of closed subsets of Γ, with $\Gamma_s\subset\Gamma_t$ for every $s\le t$;
- (H7) Φ, $\Psi\colon[0,T]\times\overline{\Omega}\to\overline{\Omega}$ are continuous and the partial derivatives $\partial_t\Phi$, $\partial_t\Psi$, $\partial_i\Phi$, $\partial_i\Psi$, $\partial_i\partial_j\Phi$, $\partial_i\partial_j\Psi$, $\partial_i\partial_t\Phi=\partial_t\partial_i\Phi$, $\partial_i\partial_t\Psi=\partial_t\partial_i\Psi$ exist and are continuous for $i,j=1,\dots,d$;
- (H8) $\Phi(t,\Omega)=\Omega$, $\Phi(t,\Gamma)=\Gamma$, $\Phi(t,\Gamma_0)=\Gamma_t$, and $\Phi(t,y)=y$ for every $t\in[0,T]$ and every y in a neighborhood of $\partial\Omega$;
- (H9) $\Psi(t,\Phi(t,y))=y$ and $\Phi(t,\Psi(t,x))=x$ for every $x,y\in\overline{\Omega}$;
- (H10) $\Phi(0,y)=y$ for every $y\in\overline{\Omega}$;
- (H11) $\partial_t\Phi$, $\partial_t\Psi$, $\partial_i\Phi$, $\partial_i\Psi$, $\partial_i\partial_j\Phi$, $\partial_i\partial_j\Psi$, $\partial_i\partial_t\Phi$, $\partial_i\partial_t\Psi$ belong to $\mathrm{Lip}([0,T];C^0(\overline{\Omega};\mathbb{R}^d))$ for $i,j=1,\dots d$;
- (H12) there exists $L>0$ such that $|\partial_i\partial_t\Phi(t,x)-\partial_i\partial_t\Phi(t,y)|\le L|x-y|$ and $|\partial_i\partial_t\Psi(t,x)-\partial_i\partial_t\Psi(t,y)|\le L|x-y|$ for every $t\in[0,T]$, $x,y\in\overline{\Omega}$, and $i=1,\dots,d$.

The differential operators D, ∇ and div always refer to the space variable in Ω. We often use the notation $\dot u$ instead of ∂_tu.

Notice that from (H7) and (H9) it follows that $\det D\Phi(t,y)\neq0$ and $\det D\Psi(t,x)\neq0$ for every $t\in[0,T]$ and $x,y\in\overline{\Omega}$. Using (H10) we conclude that both the determinants are positive. Moreover, (H4), (H7), (H8), and (H10) imply that $\Phi(t,\Omega^\pm)=\Omega^\pm$.

Given a point $y\in\Gamma$, its trajectory in time is described by the function $t\mapsto\Phi(t,y)\in\Gamma$. We infer that its velocity is tangential to the manifold Γ at the point $\Phi(t,y)$, that is $\dot\Phi(t,y)\cdot\nu(\Phi(t,y))=0$, where $\nu(x)$ is the normal vector to Γ at x. By combining this equality with the relation between $\nu(\Phi(t,y))$ and $\nu(y)$

$$\nu(\Phi(t,y)) = \frac{(D\Phi(t,y))^{-T}\nu(y)}{|(D\Phi(t,y))^{-T}\nu(y)|} \quad \text{for } y \in \Gamma\,, \tag{1.1}$$

we deduce that

$$((D\Phi(t,y))^{-1}\dot\Phi(t,y))\cdot\nu(y) = 0 \text{ for } y \in \Gamma\,, \text{ or equivalently } \ \dot\Phi(t,\Psi(t,x))\cdot\nu(x) = 0 \text{ for } x \in \Gamma\,. \tag{1.2}$$

We set

$$Q_\Gamma := \{(t,x) \in (0,T)\times\Omega \ : \ x \notin \Gamma_t\}$$

and $\Omega_t := \Omega \setminus \Gamma_t$, so that $Q_\Gamma = \{(t,x) \ : \ t \in (0,T)\,,\ x \in \Omega_t\}$. We introduce the space

$$H^1_D(\Omega_t) := \{u \in H^1(\Omega_t) \ : \ u = 0 \ \mathcal{H}^{d-1}\text{- a.e. on } \partial_D\Omega\}\,,$$

where the equality on $\partial_D\Omega$ refers to the trace of u on $\partial\Omega$. The space $H^1_D(\Omega_t)$ is endowed with the norm of $H^1(\Omega_t)$ and its dual is denoted by $H^{-1}_D(\Omega_t)$. The transpose of the natural embedding $H^1_D(\Omega_t) \hookrightarrow L^2(\Omega)$ induces the embedding of $L^2(\Omega)$ into $H^{-1}_D(\Omega_t)$ defined by $\langle g,\varphi\rangle_{H^1_D(\Omega_t)} := \langle g,\varphi\rangle_{L^2(\Omega)}$ for every $g \in L^2(\Omega)$ and $\varphi \in H^1_D(\Omega_t)$.

Given $0 \le s \le t \le T$ let $P_{st}\colon H^{-1}_D(\Omega_t) \to H^{-1}_D(\Omega_s)$ be the transpose of the natural embedding $H^1_D(\Omega_s) \hookrightarrow H^1_D(\Omega_t)$, i.e., $\langle P_{st}(g),\varphi\rangle_{H^1_D(\Omega_s)} := \langle g,\varphi\rangle_{H^1_D(\Omega_t)}$ for every $g \in H^{-1}_D(\Omega_t)$ and $\varphi \in H^1_D(\Omega_s)$. The operator P_{st} is continuous, with norm less than or equal to 1. In general it is not injective, since $H^1_D(\Omega_s)$ is not dense in $H^1_D(\Omega_t)$. Note that $P_{st}(g) = g$ for every $g \in L^2(\Omega)$.

Let $A : [0,T]\times\overline{\Omega} \to \mathbb{R}^{d\times d}_{sym}$ be a time varying tensor field in Ω such that

$$A \in \mathrm{Lip}([0,T];C^0(\overline{\Omega};\mathbb{R}^{d\times d}_{sym}))\,,\ A(t,\cdot) \in \mathrm{Lip}(\overline{\Omega})\,,\ \text{and} \quad \|\partial_i A(t,\cdot)\|_{L^\infty(\Omega)} \le C \tag{1.3}$$

for every $t \in [0,T]$, for every $i = 1,\dots,d$, and for some $C > 0$ independent of t and i. We assume that the following ellipticity condition holds for a suitable constant $c_A > 0$:

$$(A(t,x)\xi)\cdot\xi \ge c_A|\xi|^2 \quad \text{for every } t \in [0,T]\,,\ x \in \overline{\Omega}\,,\ \text{and } \xi \in \mathbb{R}^d\,. \tag{1.4}$$

Given

$$f \in L^2((0,T);L^2(\Omega))\,,\ u^0 \in H^1(\Omega_0)\,,\ u^1 \in L^2(\Omega)\,,\ \text{and } w_D \in L^2((0,T);H^{1/2}(\partial\Omega))\,, \tag{1.5}$$

we study the differential equation

$$\ddot u - \mathrm{div}(A\nabla u) = f \quad \text{in } Q_\Gamma\,, \tag{1.6}$$

with initial conditions

$$u(0) = u^0\,,\ \dot u(0) = u^1 \quad \text{in } \Omega_0\,, \tag{1.7}$$

and boundary conditions formally written as

$$u(t) = w_D(t) \quad \text{on } \partial_D\Omega \ \text{ for a.e. } t \in (0,T)\,, \tag{1.8}$$

$$(A(t)\nabla u(t))\cdot\nu = 0 \quad \text{on } \Gamma_t \cup \partial_N\Omega \ \text{ for a.e. } t \in (0,T)\,. \tag{1.9}$$

To give a precise meaning to (1.6)-(1.9), it is convenient to introduce the following notation. Given $v \in H^1(\Omega_t)$, its gradient in the sense of distributions, denoted by ∇v, belongs to $L^2(\Omega_t;\mathbb{R}^d)$. We define the function $\widehat\nabla v \in L^2(\Omega;\mathbb{R}^d)$ by setting $\widehat\nabla v = \nabla v$ on Ω_t and $\widehat\nabla v = 0$ on Γ_t. Note that $\widehat\nabla v$ is not the gradient in the sense of distributions on Ω of the function v, considered as defined almost everywhere on Ω: indeed, the equality

$$\int_\Omega \omega\cdot\widehat\nabla v\,dx = -\int_\Omega v\,\mathrm{div}\,\omega\,dx$$

holds for $\omega \in C^1_c(\Omega_t;\mathbb{R}^d)$, but in general not for $\omega \in C^1_c(\Omega;\mathbb{R}^d)$.

To prove an existence and uniqueness result for (1.6)-(1.9), we assume that there exists

$$w \in L^2((0,T);H^2(\Omega_0)) \cap H^1((0,T);H^1(\Omega_0)) \cap H^2((0,T);L^2(\Omega_0)) \tag{1.10}$$

such that

$$w(t) = w_D(t) \quad \text{on } \partial_D\Omega \ \text{ for a.e. } t \in (0,T)\,, \tag{1.11}$$

$$(A(t)\nabla w(t))\cdot\nu = 0 \quad \text{on } \Gamma_t \cup \partial_N\Omega \ \text{ for a.e. } t \in (0,T)\,, \tag{1.12}$$

$$w(0) = u^0 \text{ on } \partial_D\Omega\,, \tag{1.13}$$

where these equalities have to be considered in the appropriate sense of traces. Note that equality (1.12) for the conormal derivative must be satisfied also on $\Gamma_t \setminus \Gamma_0$.

Remark 1.1. For problems in a fixed domain, the usual assumption of w is

$$w \in L^2((0,T); H^1(\Omega_0)) \cap H^1((0,T); L^2(\Omega_0)) \cap H^2((0,T); H^{-1}(\Omega_0))\,.$$

We have to assume more regularity, both to respect to space and time, because our method of proof uses the equation satisfied by $u - w$, which has source term $g := f - \ddot{w} + \operatorname{div}(A\nabla w)$, and some estimates require that $g \in L^2((0,T); L^2(\Omega_0))$.

To give a precise meaning to (1.6)-(1.9) we consider functions u which satisfy the following regularity assumptions:

$$u \in C^1([0,T]; L^2(\Omega))\,, \tag{1.14}$$

$$u(t) - w(t) \in H^1_D(\Omega_t) \quad \text{for every } t \in [0,T]\,, \tag{1.15}$$

$$\widehat{\nabla} u \in C^0([0,T]; L^2(\Omega; \mathbb{R}^d))\,, \tag{1.16}$$

$$\dot{u} \in AC([s,T]; H^{-1}_D(\Omega_s)) \text{ for every } s \in [0,T)\,, \tag{1.17}$$

$$\tfrac{1}{h}[\dot{u}(t+h) - \dot{u}(t)] \rightharpoonup \ddot{u}(t) \;\text{ weakly in } H^{-1}_D(\Omega_t) \text{ as } h \to 0\,, \text{ for a.e. } t \in (0,T)\,, \tag{1.18}$$

$$\text{the function } t \mapsto \|\ddot{u}(t)\|_{H^{-1}_D(\Omega_t)} \text{ is integrable in } (0,T)\,. \tag{1.19}$$

The notation used for the weak limit in (1.18) is justified in the following lemma.

Lemma 1.2. *Assume that u satisfies* (1.14)-(1.19). *Then for a.e. $t \in (0,T)$ we have*

$$\frac{\dot{u}(t+h) - \dot{u}(t)}{h} \to P_{st}(\ddot{u}(t)) \;\text{ strongly in } H^{-1}_D(\Omega_s) \tag{1.20}$$

as $h \to 0$, for every $s \in [0,t)$. In particular, $t \mapsto P_{st}(\ddot{u}(t))$ is the distributional derivative of the function $t \mapsto \dot{u}(t)$ from (s,T) to $H^{-1}_D(\Omega_s)$. Moreover,

$$\dot{u}(t) - \dot{u}(s) = \int_s^t P_{s\tau}(\ddot{u}(\tau))\, d\tau \tag{1.21}$$

for every $0 \le s \le t \le T$.

Proof. Let us fix a countable dense set S of $(0,T)$. For every $s \in S$ let $\ddot{u}_s \colon (s,T) \to H^{-1}_D(\Omega_s)$ be the distributional derivative of $\dot{u}$ in the space $H^{-1}_D(\Omega_s)$. By the standard theory of absolutely continuous functions with values in a reflexive Banach space (see, e.g., [3]), there exists a negligible subset N_s of (s,T) such that

$$\frac{\dot{u}(t+h) - \dot{u}(t)}{h} \to \ddot{u}_s(t) \;\text{ strongly in } H^{-1}_D(\Omega_s) \tag{1.22}$$

for every $t \in (s,T) \setminus N_s$. Let N be a negligible subset of $(0,T)$ containing all N_s for $s \in S$ and all the t for which (1.18) is not satisfied. Comparing (1.18) and (1.22), we infer that $P_{st}(\ddot{u}(t)) = \ddot{u}_s(t)$ for every $t \in (0,T) \setminus N$ and $s \in [0,t) \cap S$; in particular, (1.20) is satisfied for such t and s. Note that, since every function $\varphi \in H^1_D(\Omega_t)$ can be approximated strongly in $H^1_D(\Omega_t)$ by functions $\varphi_s \in H^1_D(\Omega_s)$ with $s \in S$ and $s \to t^-$, we obtain that $\ddot{u}(t)$ is uniquely determined by the equality $P_{st}(\ddot{u}(t)) = \ddot{u}_s(t)$. In order to conclude the proof of (1.20), we need to check its validity for $t \in (0,T) \setminus N$ and $s_0 \in [0,t) \setminus S$: taking an intermediate value $s_1 \in (s_0,t) \cap S$ (recall that S is dense), we have the convergence (1.20) of the difference quotient to $P_{s_1 t}(\ddot{u}(t))$ strongly in $H^{-1}_D(\Omega_{s_1})$; then, by applying the continuous operator $P_{s_0 s_1}$ to both sides of such expression, we obtain the desired strong convergence in $H^{-1}_D(\Omega_{s_0})$.

Finally, by the standard theory of absolutely continuous functions, formula (1.21) is an immediate consequence of (1.17) . ☐

Remark 1.3. Properties (1.14)-(1.18) imply that the function $t \mapsto \|\ddot{u}(t)\|_{H^{-1}_D(\Omega_t)}$ is measurable. To prove this fact, we first observe that for every $0 \le s \le T$ the function $(s,T) \ni t \mapsto P_{st}(\ddot{u}(t))$ is measurable with values in $H^{-1}_D(\Omega_s)$ thanks to (1.20) and to the continuity of $\dot{u}$. Therefore the function $(s,T) \ni t \mapsto \|P_{st}(\ddot{u}(t))\|_{H^{-1}_D(\Omega_s)}$ is measurable, and so is the function G_s defined by $G_s(t) = 0$ if $t \in (0,s)$, and $G_s(t) = \|P_{st}(\ddot{u}(t))\|_{H^{-1}_D(\Omega_s)}$

if $t \in (s,T)$. Since every test function $\varphi \in H^1_D(\Omega_t)$ can be approximated strongly in $H^1_D(\Omega_t)$ by functions $\varphi_s \in H^1_D(\Omega_s)$ with $s \to t^-$, we obtain that $\|P_{st}(\ddot u(t))\|_{H^{-1}_D(\Omega_s)} \to \|\ddot u(t)\|_{H^{-1}_D(\Omega_t)}$ as $s \to t^-$. By monotonicity with respect to s, given a countable dense subset S of $(0,T)$, we obtain that $\|\ddot u(t)\|_{H^{-1}_D(\Omega_t)} = \sup_{s\in S} G_s(t)$, concluding the proof.

We are now ready to make precise the notion of solution of problem (1.6)-(1.9).

Definition 1.4. Let A, f, u^0, u^1, w be as in (1.3), (1.5), and (1.10)-(1.13). We say that u is a *weak solution* of the wave equation (1.6) with initial conditions (1.7) and boundary conditions (1.8) and (1.9) if u satisfies (1.14)-(1.19) and for a.e. $t \in (0,T)$ we have

$$\langle \ddot u(t), \varphi\rangle_{H^1_D(\Omega_t)} + \langle A(t)\nabla u(t), \nabla\varphi\rangle_{L^2(\Omega_t)} = \langle f(t), \varphi\rangle_{L^2(\Omega_t)} \quad \text{for every } \varphi \in H^1_D(\Omega_t)\,, \tag{1.23}$$

where $\ddot u(t)$ is defined in Lemma 1.2.

To prove the existence and the uniqueness of a weak solution, it is useful to perform the change of variables

$$v(t,y) = u(t,\Phi(t,y)) \quad \text{and} \quad u(t,x) = v(t,\Psi(t,x)) \tag{1.24}$$

through the diffeomorphisms introduced in (H7)-(H12). Notice that $v(t,\cdot) \in H^1(\Omega_0)$ if and only if $u(t,\cdot) \in H^1(\Omega_t)$. Therefore, with this change of variables we work in the fixed set Ω_0. This leads to consider the problem

$$\ddot v - \operatorname{div}(B\nabla v) + a\cdot\nabla v - 2b\cdot\nabla\dot v = g \quad \text{in } Q_{\Gamma_0}\,, \tag{1.25}$$

with initial conditions

$$v(0) = v^0\,,\ \dot v(0) = v^1 \quad \text{in } \Omega_0\,, \tag{1.26}$$

and boundary conditions formally written as

$$v(t) = w_D(t) \quad \text{on } \partial_D\Omega \ \text{ for a.e. } t\in(0,T)\,, \tag{1.27}$$

$$(B(t)\nabla v(t))\cdot\nu = 0 \quad \text{on } \Gamma_0\cup\partial_N\Omega \ \text{ for a.e. } t\in(0,T)\,, \tag{1.28}$$

with

$$B(t,y) := D\Psi(t,\Phi(t,y))A(t,\Phi(t,y))D\Psi(t,\Phi(t,y))^T - b(t,y)\otimes b(t,y)\,, \tag{1.29}$$

$$a(t,y) := -[B^T(t,y)\nabla(\det D\Phi(t,y)) + \partial_t(b(t,y)\det D\Phi(t,y))]\det D\Psi(t,\Phi(t,y))\,, \tag{1.30}$$

$$b(t,y) := -\dot\Psi(t,\Phi(t,y))\,, \tag{1.31}$$

$$g(t,y) := f(t,\Phi(t,y))\,, \tag{1.32}$$

$$v^0 := u^0\,,\ v^1 := u^1 + \dot\Phi(0)\cdot\nabla u^0\,. \tag{1.33}$$

To give a precise meaning to (1.25)-(1.28), we consider functions v which satisfy the following regularity assumptions:

$$v \in C^1([0,T];L^2(\Omega_0))\,, \tag{1.34}$$

$$v(t) - w(t) \in H^1_D(\Omega_0) \text{ for every } t\in[0,T]\,, \tag{1.35}$$

$$\nabla v \in C^0([0,T];L^2(\Omega_0;\mathbb{R}^d))\,, \tag{1.36}$$

$$\dot v \in AC([0,T];H^{-1}_D(\Omega_0))\,. \tag{1.37}$$

Let us specify in what sense we study problem (1.25)-(1.28).

Definition 1.5. Let A, f, u^0, u^1, and w be as in (1.3), (1.5), and (1.10)-(1.13), and let B, a, b, g, v^0, and v^1 be defined according to (1.29)-(1.33). We say that v is a *weak solution* of equation (1.25) with initial conditions (1.26) and boundary conditions (1.27) and (1.28) if v satisfies (1.34)-(1.37) and for a.e. $t \in (0,T)$ we have

$$\begin{aligned}&\langle \ddot v(t), \psi\rangle_{H^1_D(\Omega_0)} + \langle B(t)\nabla v(t), \nabla\psi\rangle_{L^2(\Omega_0)} + \langle a(t)\cdot\nabla v(t), \psi\rangle_{L^2(\Omega_0)} + 2\langle \dot v(t), \operatorname{div}(b(t)\psi)\rangle_{L^2(\Omega_0)}\\ &= \langle g(t), \psi\rangle_{L^2(\Omega_0)} \quad \text{for every } \psi \in H^1_D(\Omega_0)\,.\end{aligned} \tag{1.38}$$

Remark 1.6. Take v satisfying (1.34)-(1.37). Let us check that the scalar products in (1.38) make sense for a.e. $t \in (0,T)$. By (1.37), $\ddot v(t) \in H^{-1}_D(\Omega_0)$ for a.e. t, therefore it is in duality with $\psi \in H^1_D(\Omega_0)$. In view of (1.34) and (1.36), for every $t \in (0,T)$ we have that $\dot v(t)$ and $\nabla v(t)$ belong to $L^2(\Omega)$ and $L^2(\Omega;\mathbb{R}^d)$, respectively. Thus, to ensure that the scalar products in the left-hand side of (1.38) are well defined, we need to show that the coefficients B, a, b, and $\operatorname{div} b$ are essentially bounded in space for almost every time.

In view of (H7), (H11), (H12), and (1.3), it is easy to check that the tensor fields $D\Psi(t,\Phi(t,\cdot))$ and $A(t,\Phi(t,\cdot))$, the vector field $\dot\Psi(t,\Phi(t,\cdot))$, and the function $\operatorname{div}(\dot\Psi(t,\Phi(t,\cdot)))$ are Lipschitz continuous from $[0,T]$ to $L^\infty(\Omega;\mathbb{R}^{d\times d}_{sym})$, $L^\infty(\Omega;\mathbb{R}^d)$, and $L^\infty(\Omega)$, respectively. In particular, we deduce that the coefficients B and b introduced in (1.29) and (1.31) satisfy

$$B \in \mathrm{Lip}([0,T];L^\infty(\Omega;\mathbb{R}^{d\times d}_{sym}))\,,\quad b \in \mathrm{Lip}([0,T];L^\infty(\Omega;\mathbb{R}^d))\,,\quad \operatorname{div} b \in \mathrm{Lip}([0,T];L^\infty(\Omega))\,. \tag{1.39}$$

As for the coefficient a defined in (1.30), we split it into the sum $a = a_1 + a_2$, with

$$a_1(t,y) := -[B^T(t,y)\nabla(\det D\Phi(t,y)) + b(t,y)\partial_t(\det D\Phi(t,y))]\det D\Psi(t,\Phi(t,y))\,, \tag{1.40}$$

$$a_2(t,y) := -\dot b(t,y)\,. \tag{1.41}$$

In view of the discussion above, we infer that $a_1 \in \mathrm{Lip}([0,T];L^\infty(\Omega;\mathbb{R}^d))$, while a_2, being the distributional derivative of a function in $\mathrm{Lip}([0,T];L^\infty(\Omega;\mathbb{R}^d))$, is an element of $L^\infty((0,T);L^2(\Omega;\mathbb{R}^d))$, moreover there exists $C > 0$ such that $\|a_2(t,\cdot)\|_{L^\infty(\Omega)} \le C$ for a.e. $t \in (0,T)$.

Eventually, since by assumption $f \in L^2((0,T);L^2(\Omega))$, also g defined in (1.32) is an element of $L^2((0,T);L^2(\Omega))$ and the right-hand side of (1.38) makes sense for a.e. $t \in (0,T)$.

The relation between problems (1.6)-(1.9) and (1.25)-(1.28) is given by the following theorem.

Theorem 1.7. *Under the assumptions of Definition 1.5, a function u is a weak solution of problem* (1.6)-(1.9) *if and only if the corresponding function v introduced in* (1.24) *is a weak solution of problem* (1.25)-(1.28).

Before proving the theorem, in the following lemmas we investigate the regularity properties of the functions u and v.

Lemma 1.8. *Suppose that u and v are related by* (1.24) *and that u satisfies* (1.14)-(1.19). *Then v satisfies* (1.34)-(1.37).

Proof. For brevity, throughout the proof, C will denote a positive constant independent of time, whose value may vary from line to line. The proof is divided into several steps.

Step 1. v satisfies (1.35). By (1.15) we have that for every $t \in [0,T]$ the function $u(t,\cdot)$ belongs to $H^1(\Omega_t)$. Moreover, by (H7)-(H9), $\Phi(t,\cdot)$ is a bi-Lipschitz diffeomorphism from Ω_0 into Ω_t. Therefore the composition $v(t,\cdot) = u(t,\Phi(t,\cdot))$ belongs to $H^1(\Omega_0)$. Eventually, since $\Phi(t,\cdot)$ restricted to the boundary $\partial\Omega$ is the identity, we get $v(t,\cdot) - w(t,\cdot) = u(t,\cdot) - w(t,\cdot) = 0$ $\mathcal{H}^{n-1}$-a.e. on $\partial_D\Omega$, concluding the proof of (1.35).

Step 2. v satisfies (1.36). In view of the previous step, $v(t,\cdot) \in H^1(\Omega_0)$ for every $t \in [0,T]$ and, by the chain rule in Sobolev spaces, we have $\nabla v(t,\cdot) = D\Phi(t,\cdot)^T\nabla u(t,\Phi(t,\cdot)) \in L^2(\Omega_0;\mathbb{R}^d)$. By applying Lemma 4.4 with $f = \widehat\nabla u(t,\cdot)$ and $\Lambda = \Phi$, we infer that $t \mapsto \nabla v(t,\cdot)$ is continuous from $[0,T]$ to $L^2(\Omega_0;\mathbb{R}^d)$.

Step 3. v is Lipschitz from $[0,T]$ to $L^2(\Omega_0)$. Let $0 \le s \le t \le T$ be fixed. By the triangle inequality, we may write

$$\|v(t,\cdot) - v(s,\cdot)\|_{L^2(\Omega_0)} \le \|u(t,\Phi(t,\cdot)) - u(s,\Phi(t,\cdot))\|_{L^2(\Omega_0)} + \|u(s,\Phi(t,\cdot)) - u(s,\Phi(s,\cdot))\|_{L^2(\Omega_0)}\,. \tag{1.42}$$

Exploiting the change of variables $x = \Phi(t,y)$, the first difference in the right-hand side can be estimated as follows:

$$\|u(t,\Phi(t,\cdot)) - u(s,\Phi(t,\cdot))\|_{L^2(\Omega_0)} \le \sup_{\tau\in[0,T]} \|\det D\Psi(\tau,\cdot)\|^{1/2}_{L^\infty(\Omega)}\|u(t,\cdot) - u(s,\cdot)\|_{L^2(\Omega)} \le C|t-s|\,, \tag{1.43}$$

where the last inequality follows from the C^1 regularity of Ψ and from assumption (1.14) on u. By applying Lemma 4.5 with $f = u(s,\cdot)$ and $\Lambda = \Phi$, and exploiting the continuity (1.16) of $\widehat\nabla u$, we obtain

$$\|u(s,\Phi(t,\cdot)) - u(s,\Phi(s,\cdot))\|_{L^2(\Omega_0)} \le C\max_{\tau\in[0,T]}\|\widehat\nabla u(\tau,\cdot)\|_{L^2(\Omega)}|t-s| \le C|t-s|\,. \tag{1.44}$$

Finally, by combining (1.42)-(1.44), we conclude that $v \in \mathrm{Lip}([0,T];L^2(\Omega_0))$.

Step 4. v satisfies (1.34). Since v is Lipschitz continuous, it can be reconstructed by integrating its time derivative, which exists almost everywhere. Therefore, to prove C^1 regularity, it is enough to show that the right derivative exists everywhere and is continuous. We claim that for every $t \in [0,T]$

$$\frac{1}{h}[v(t+h,\cdot)-v(t,\cdot)] \to \dot u(t,\Phi(t,\cdot)) + \widehat\nabla u(t,\Phi(t,\cdot))\cdot\dot\Phi(t,\cdot) \quad \text{strongly in } L^2(\Omega_0) \text{ as } h\to 0^+\,, \tag{1.45}$$

and that the function

$$t \mapsto \dot u(t,\Phi(t,\cdot)) + \widehat\nabla u(t,\Phi(t,\cdot))\cdot\dot\Phi(t,\cdot) \tag{1.46}$$

is continuous from $[0,T]$ to $L^2(\Omega_0)$.

We start by showing (1.45). Fix $t\in[0,T]$. By (1.24), we can write

$$\frac{1}{h}[v(t+h,\cdot)-v(t,\cdot)] = \frac{1}{h}[u(t+h,\Phi(t+h,\cdot))-u(t,\Phi(t+h,\cdot))] + \frac{1}{h}[u(t,\Phi(t+h,\cdot))-u(t,\Phi(t,\cdot))]\,. \tag{1.47}$$

By (1.14) we have that $(u(t+h,\cdot)-u(t,\cdot))/h \to \dot u(t,\cdot)$ strongly in $L^2(\Omega)$. By a change of variables, it is easy to see that

$$\frac{1}{h}[u(t+h,\Phi(t+h,\cdot))-u(t,\Phi(t+h,\cdot))] - \dot u(t,\Phi(t+h,\cdot)) \to 0 \quad \text{strongly in } L^2(\Omega_0) \text{ as } h\to 0^+\,,$$

moreover, by applying Lemma 4.3 with $f=\dot u(t,\cdot)$ and $\Lambda=\Phi$, we obtain $\dot u(t,\Phi(t+h,\cdot))-\dot u(t,\Phi(t,\cdot))\to 0$ strongly in $L^2(\Omega_0)$, hence

$$\frac{1}{h}[u(t+h,\Phi(t+h,\cdot))-u(t,\Phi(t+h,\cdot))] \to \dot u(t,\Phi(t,\cdot)) \quad \text{strongly in } L^2(\Omega_0) \text{ as } h\to 0^+\,. \tag{1.48}$$

On the other hand, by Lemma 4.6 with $f=u(t,\cdot)$ and $\Lambda=\Phi$, we infer that

$$\frac{1}{h}[u(t,\Phi(t+h,\cdot))-u(t,\Phi(t,\cdot))] \to \widehat\nabla u(t,\Phi(t,\cdot))\cdot\dot\Phi(t,\cdot) \quad \text{strongly in } L^2(\Omega_0) \text{ as } h\to 0^+\,. \tag{1.49}$$

By combining (1.47) with (1.48) and (1.49) we get (1.45). Finally, Lemma 4.4 gives the desired continuity of the function (1.46). Thus we conclude that v satisfies (1.34) and that its time derivative is given by

$$\dot v(t,\cdot) = \dot u(t,\Phi(t,\cdot)) + \widehat\nabla u(t,\Phi(t,\cdot))\cdot\dot\Phi(t,\cdot)\,. \tag{1.50}$$

Step 5. $\dot v \in AC([0,T];H^{-1}_D(\Omega_0))$. By (1.19) it is enough to prove that there exists a constant $C>0$ such that

$$\|\dot v(s,\cdot)-\dot v(t,\cdot)\|_{H^{-1}_D(\Omega_0)} \le C\int_s^t (\|\ddot u(\tau,\cdot)\|_{H^{-1}_D(\Omega_\tau)}+1)\,d\tau \tag{1.51}$$

for every $0\le s\le t\le T$. Let $\psi\in H^1_D(\Omega_0)$ with $\|\psi\|_{H^1_D(\Omega_0)}\le 1$. Then we have

$$\begin{aligned}\langle \dot v(s,\cdot)-\dot v(t,\cdot),\psi\rangle_{L^2(\Omega_0)} &= \langle \dot u(s,\Phi(s,\cdot))-\dot u(t,\Phi(s,\cdot)),\psi\rangle_{L^2(\Omega_0)} + \langle \dot u(t,\Phi(s,\cdot))-\dot u(t,\Phi(t,\cdot)),\psi\rangle_{L^2(\Omega_0)}\\ &\quad + \langle \widehat\nabla u(s,\Phi(s,\cdot))\cdot\dot\Phi(s,\cdot)-\widehat\nabla u(s,\Phi(t,\cdot))\cdot\dot\Phi(t,\cdot),\psi\rangle_{L^2(\Omega_0)}\\ &\quad + \langle \widehat\nabla u(s,\Phi(t,\cdot))\cdot\dot\Phi(t,\cdot)-\widehat\nabla u(t,\Phi(t,\cdot))\cdot\dot\Phi(t,\cdot),\psi\rangle_{L^2(\Omega_0)}\,.\end{aligned} \tag{1.52}$$

Using (1.21) after a change of variables we obtain

$$\begin{aligned}\langle \dot u(s,\Phi(s,\cdot))-\dot u(t,\Phi(s,\cdot)),\psi\rangle_{L^2(\Omega_0)} &\le C\|\dot u(s,\cdot)-\dot u(t,\cdot)\|_{H^{-1}_D(\Omega_s)}\|\psi(\Psi(s,\cdot))\|_{H^1_D(\Omega_s)}\\ &\le C\int_s^t \|P_{s\tau}(\ddot u(\tau,\cdot))\|_{H^{-1}_D(\Omega_s)}\,d\tau \le C\int_s^t \|\ddot u(\tau,\cdot))\|_{H^{-1}_D(\Omega_\tau)}\,d\tau\,.\end{aligned} \tag{1.53}$$

Exploiting assumption (1.14), the Lipschitz regularity of $D\Psi$ considered in (H11), and Lemma 4.5 with $f=\psi$ and $\Lambda=\Psi$, we get

$$\begin{aligned}\langle \dot u(t,\Phi(s,\cdot))-\dot u(t,\Phi(t,\cdot)),\psi\rangle_{L^2(\Omega_0)} &\le C\|\dot u(t,\cdot)\|_{L^2(\Omega)}\|\det D\Psi(s,\cdot)-\det D\Psi(t,\cdot)\|_{L^\infty(\Omega)}\\ &\quad + C\|\dot u(t,\cdot)\|_{L^2(\Omega)}\|\psi(\Psi(s,\cdot))-\psi(\Psi(t,\cdot))\|_{L^2(\Omega)} \le C|t-s|\,.\end{aligned} \tag{1.54}$$

Similarly, by the Lipschitz regularity of $\dot\Phi(t,\Psi(t,\cdot))$ ensured by (H11) and by (1.16), we have

$$\langle \widehat\nabla u(s,\Phi(s,\cdot))\cdot\dot\Phi(s,\cdot)-\widehat\nabla u(s,\Phi(t,\cdot))\cdot\dot\Phi(t,\cdot),\psi\rangle_{L^2(\Omega_0)} \le C|t-s|\,. \tag{1.55}$$

Since $s \le t$, we have $H^1(\Omega_s) \subset H^1(\Omega_t)$, in particular $u(t,\cdot), u(s,\cdot)$, and $\psi(\Psi(t,\cdot))$ all belong to $H^1(\Omega_t)$, therefore we have

$$\begin{aligned}\langle \widehat{\nabla} u(s,\Phi(t,\cdot)) - \widehat{\nabla} u(t,\Phi(t,\cdot)), \dot\Phi(t,\cdot)\psi\rangle_{L^2(\Omega_0)} &= \langle \nabla(u(s,\cdot)-u(t,\cdot)), \dot\Phi(t,\Psi(t,\cdot))\psi(\Psi(t,\cdot))\rangle_{L^2(\Omega_t)}\\ &= -\langle u(s,\cdot)-u(t,\cdot), \operatorname{div}[\dot\Phi(t,\Psi(t,\cdot))\psi(\Psi(t,\cdot))]\rangle_{L^2(\Omega_t)}\\ &\le C\|u(s,\cdot)-u(t,\cdot)\|_{L^2(\Omega)} \le C|t-s|\,, \qquad (1.56)\end{aligned}$$

where we have used the equality $\dot\Phi(t,\Psi(t,\cdot))\cdot\nu = 0$ on $\partial\Omega_t$ (see (H8) and (1.2)) in the integration by parts, and assumptions (1.14), (H11), and (H12) in the last inequality. By combining (1.52) with (1.53)-(1.56), by the arbitrariness of the test function ψ, we obtain (1.51).

□

Lemma 1.9. *Under the assumptions of Lemma 1.8, for a.e. $t\in(0,T)$ we have*

$$\begin{aligned}\langle \ddot v(t,\cdot),\psi\rangle_{H^1_D(\Omega_0)} &= \langle \ddot u(t,\cdot), \psi(\Psi(t,\cdot))\det D\Psi(t,\cdot)\rangle_{H^1_D(\Omega_t)}\\ &\quad + \langle \dot u(t,\cdot), \partial_t[\psi(\Psi(t,\cdot))\det D\Psi(t,\cdot)] - \operatorname{div}[\psi(\Psi(t,\cdot))\dot\Phi(t,\Psi(t,\cdot))\det D\Psi(t,\cdot)]\rangle_{L^2(\Omega)}\\ &\quad + \langle \widehat{\nabla} u(t,\cdot), \partial_t[\psi(\Psi(t,\cdot))\dot\Phi(t,\Psi(t,\cdot))\det D\Psi(t,\cdot)]\rangle_{L^2(\Omega)} \qquad (1.57)\end{aligned}$$

for every $\psi\in H^1_D(\Omega_0)$.

Remark 1.10. Notice that by Lemma 4.5 with $f{=}\psi$ and $\Lambda{=}\Psi$, we deduce that $\psi(\Psi(t,\cdot)){\in}\mathrm{Lip}([0,T];L^2(\Omega))$; moreover, by (H11), we have $\dot\Phi(t,\Psi(t,\cdot)){\in}\mathrm{Lip}([0,T];L^\infty(\Omega;\mathbb{R}^d))$ and $\det D\Psi{\in}\mathrm{Lip}([0,T];L^\infty(\Omega))$. Therefore the products $\psi(\Psi(t,\cdot))\det D\Psi(t,\cdot)$ and $\psi(\Psi(t,\cdot))\dot\Phi(t,\Psi(t,\cdot))\det D\Psi(t,\cdot)$ are elements of $\mathrm{Lip}([0,T];L^2(\Omega))$ and $\mathrm{Lip}([0,T];L^2(\Omega;\mathbb{R}^d))$, respectively, and their distributional time derivatives belong to $L^\infty((0,T);L^2(\Omega))$ and $L^\infty((0,T);L^2(\Omega;\mathbb{R}^d))$, respectively.

Proof of Lemma 1.9. In view of Lemma 1.8, the function v satisfies (1.34)-(1.37). By the absolute continuity (1.37) of $\dot v$, we infer that its distributional derivative $\ddot v$ from $(0,T)$ to $H^{-1}_D(\Omega_0)$ is an element of $L^1(0,T;H^{-1}_D(\Omega_0))$; moreover, for a.e. $t\in(0,T)$, the action of $\ddot v(t)$ against any test function $\psi\in H^1_D(\Omega_0)$ can be deduced by the identity

$$\langle \ddot v(t,\cdot),\psi\rangle_{H^1_D(\Omega_0)} = \lim_{h\to 0^-}\frac{1}{h}\langle \dot v(t+h,\cdot)-\dot v(t,\cdot),\psi\rangle_{L^2(\Omega)}\,. \qquad (1.58)$$

Let $h<0$. Exploiting (1.50) we can write

$$\begin{aligned}\dot v(t+h,\cdot)-\dot v(t,\cdot) &= [\dot u(t+h,\Phi(t,\cdot)) - \dot u(t,\Phi(t,\cdot))] + [\dot u(t+h,\Phi(t+h,\cdot)) - \dot u(t+h,\Phi(t,\cdot))]\\ &\quad + [\widehat{\nabla} u(t+h,\Phi(t+h,\cdot))\cdot\dot\Phi(t+h,\cdot) - \widehat{\nabla} u(t+h,\Phi(t,\cdot))\cdot\dot\Phi(t,\cdot)]\\ &\quad + [\widehat{\nabla} u(t+h,\Phi(t,\cdot))\cdot\dot\Phi(t,\cdot) - \widehat{\nabla} u(t,\Phi(t,\cdot))\cdot\dot\Phi(t,\cdot)]\,. \qquad (1.59)\end{aligned}$$

In view of Lemma 1.2, for a.e. $t\in(0,T)$ we get

$$\begin{aligned}\frac{1}{h}\langle \dot u(t+h,\Phi(t,\cdot)) - \dot u(t,\Phi(t,\cdot)),\psi\rangle_{L^2(\Omega)} &= \frac{1}{h}\langle \dot u(t+h,\cdot)-\dot u(t,\cdot), \psi(\Psi(t,\cdot))\det D\Psi(t,\cdot)\rangle_{L^2(\Omega)}\\ &\to \langle \ddot u(t,\cdot),\psi(\Psi(t,\cdot))\det D\Psi(t,\cdot)\rangle_{H^1_D(\Omega_t)} \quad \text{as } h\to 0^-\,, \qquad (1.60)\end{aligned}$$

since $\psi(\Psi(t,\cdot))\det D\Psi(t,\cdot)$ is an element of $H^1_D(\Omega_t)$. By assumption (1.14) and Remark 1.10, we infer that

$$\begin{aligned}&\frac{1}{h}\langle \dot u(t+h,\Phi(t+h,\cdot)) - \dot u(t+h,\Phi(t,\cdot)),\psi\rangle_{L^2(\Omega)}\\ &= \frac{1}{h}\langle \dot u(t+h,\cdot), \psi(\Psi(t+h,\cdot))\det D\Psi(t+h,\cdot) - \psi(\Psi(t,\cdot))\det D\Psi(t,\cdot)\rangle_{L^2(\Omega)}\\ &\to \langle \dot u(t,\cdot), \partial_t[\psi(\Psi(t,\cdot))\det D\Psi(t,\cdot)]\rangle_{L^2(\Omega)} \quad \text{as } h\to 0^-\,. \qquad (1.61)\end{aligned}$$

Similarly, again by Remark 1.10 and by (1.16), we obtain

$$\begin{aligned}&\frac{1}{h}\langle \widehat{\nabla} u(t+h,\Phi(t+h,\cdot))\cdot\dot\Phi(t+h,\cdot) - \widehat{\nabla} u(t+h,\Phi(t,\cdot))\cdot\dot\Phi(t,\cdot),\psi\rangle_{L^2(\Omega)}\\ &= \frac{1}{h}\langle \widehat{\nabla} u(t+h,\cdot), \dot\Phi(t+h,\Psi(t+h,\cdot))\psi(\Psi(t+h,\cdot))\det D\Psi(t+h,\cdot) - \dot\Phi(t,\Psi(t,\cdot))\psi(\Psi(t,\cdot))\det D\Psi(t,\cdot)\rangle_{L^2(\Omega)}\end{aligned}$$

$$\to \langle \widehat{\nabla} u(t,\cdot), \partial_t[\psi(\Psi(t,\cdot))\dot{\Phi}(t,\Psi(t,\cdot)) \det D\Psi(t,\cdot)]\rangle_{L^2(\Omega)} \quad \text{as } h \to 0^- \,. \tag{1.62}$$

To treat the last term in (1.59) we need to perform an integration by parts: if $h < 0$ the functions $u(t+h,\cdot)$, $u(t,\cdot)$, and $\psi(\Psi(t,\cdot))$ all belong to $H^1(\Omega_t)$, therefore we have

$$\begin{aligned}
&\frac{1}{h}\langle \widehat{\nabla} u(t+h,\Phi(t,\cdot)) \cdot \dot{\Phi}(t,\cdot) - \widehat{\nabla} u(t,\Phi(t,\cdot)) \cdot \dot{\Phi}(t,\cdot), \psi\rangle_{L^2(\Omega)} \\
&= \frac{1}{h}\langle \widehat{\nabla} u(t+h,\cdot) - \widehat{\nabla} u(t,), \psi(\Psi(t,\cdot))\dot{\Phi}(t,\Psi(t,\cdot)) \det D\Psi(t,\cdot)\rangle_{L^2(\Omega_t)} \\
&\to \langle \dot{u}(t,\cdot), -\operatorname{div}[\psi(\Psi(t,\cdot))\dot{\Phi}(t,\Psi(t,\cdot)) \det D\Psi(t,\cdot)]\rangle_{L^2(\Omega)} \quad \text{as } h \to 0^- \,,
\end{aligned} \tag{1.63}$$

where, in the integration by parts, we have used the equality $\dot{\Phi}(t,\Psi(t,\cdot)) \cdot \nu = 0$ on $\partial\Omega_t$ (see (H8) and (1.2)). By comparing the limits found in (1.60)-(1.63) with (1.59) and (1.58), we conclude that the distributional derivative $\ddot{v}$ is characterized by (1.57). □

Conversely, the regularity of u can be deduced by the regularity of v, as we state in the two following lemmas. Both results can be readily obtained by following the same procedure adopted in the proof of Lemmas 1.8 and 1.9, exchanging the role of u and v. Therefore we omit the proofs.

Lemma 1.11. *Suppose that u and v are related by* (1.24) *and that v satisfies* (1.34)-(1.37). *Then u satisfies* (1.14)-(1.19).

Lemma 1.12. *Under the assumptions of Lemma 1.11, for a.e. $t \in (0,T)$ we have*

$$\begin{aligned}
\langle \ddot{u}(t,\cdot), \varphi\rangle_{H^1_D(\Omega_t)} &= \langle \ddot{v}(t,\cdot), \varphi(\Phi(t,\cdot)) \det D\Phi(t,\cdot)\rangle_{H^1_D(\Omega_0)} \\
&+ \langle \dot{v}(t,\cdot), \partial_t[\varphi(\Phi(t,\cdot)) \det D\Phi(t,\cdot)] - \operatorname{div}[\varphi(\Phi(t,\cdot))\dot{\Psi}(t,\Phi(t,\cdot)) \det D\Phi(t,\cdot)]\rangle_{L^2(\Omega)} \\
&+ \langle \widehat{\nabla} v(t,\cdot), \partial_t[\varphi(\Phi(t,\cdot))\dot{\Psi}(t,\Phi(t,\cdot)) \det D\Phi(t,\cdot)]\rangle_{L^2(\Omega)}
\end{aligned}$$

for every $\varphi \in H^1_D(\Omega_t)$.

We are now in a position to prove Theorem 1.7.

Proof of Theorem 1.7. First, we assume that u is a weak solution of problem (1.6)-(1.9). In view of Lemma 1.8 the function v satisfies (1.34)-(1.37). Let $\psi \in H^1_D(\Omega_0)$ be an arbitrary test function. For every $t \in [0,T]$ the function $\psi(\Psi(t,\cdot)) \det D\Psi(t,\cdot)$ belongs to $H^1_D(\Omega_t)$, thus, by (1.23), we have

$$\begin{aligned}
\langle \ddot{u}(t,\cdot), \psi(\Psi(t,\cdot)) \det D\Psi(t,\cdot)\rangle_{H^1_D(\Omega_t)} = &- \langle A(t,\cdot)\widehat{\nabla} u(t,\cdot), \nabla[\psi(\Psi(t,\cdot)) \det D\Psi(t,\cdot)]\rangle_{L^2(\Omega)} \\
&+ \langle f(t,\cdot), \psi(\Psi(t,\cdot)) \det D\Psi(t,\cdot)\rangle_{L^2(\Omega)} \,.
\end{aligned}$$

Inserting this expression into (1.57), we get

$$\begin{aligned}
\langle \ddot{v}(t,\cdot), \psi\rangle_{H^1_D(\Omega_0)} &= -\langle A(t,\cdot)\widehat{\nabla} u(t,\cdot), \nabla[\psi(\Psi(t,\cdot)) \det D\Psi(t,\cdot)]\rangle_{L^2(\Omega)} \\
&+ \langle \widehat{\nabla} u(t,\cdot), \partial_t[\psi(\Psi(t,\cdot))\dot{\Phi}(t,\Psi(t,\cdot)) \det D\Psi(t,\cdot)]\rangle_{L^2(\Omega)} \\
&+ \langle \dot{u}(t,\cdot), \partial_t[\psi(\Psi(t,\cdot)) \det D\Psi(t,\cdot)] - \operatorname{div}[\psi(\Psi(t,\cdot))\dot{\Phi}(t,\Psi(t,\cdot)) \det D\Psi(t,\cdot)]\rangle_{L^2(\Omega)} \\
&+ \langle f(t,\cdot), \psi(\Psi(t,\cdot)) \det D\Psi(t,\cdot)\rangle_{L^2(\Omega)} \,.
\end{aligned} \tag{1.64}$$

In view of the relation (1.24) and formula (1.50), we can write ∇u and $\dot{u}$ in terms of v as follows:

$$\widehat{\nabla} u(t,\cdot) = D\Psi^T(t,\cdot)\widehat{\nabla} v(t,\Psi(t,\cdot)) \,, \quad \dot{u}(t,\cdot) = \dot{v}(t,\Psi(t,\cdot)) + \widehat{\nabla} v(t,\Psi(t,\cdot)) \cdot \dot{\Psi}(t,\cdot) \,. \tag{1.65}$$

Inserting the expressions (1.65) into (1.64), we obtain that v satisfies (1.38). The initial conditions (1.26) follow from the regularity property (1.34) of v and the initial conditions (1.7) satisfied by u. Finally, the Neumann boundary condition (1.28) for v is readily verified: for every $t \in (0,T)$ and $\mathcal{H}^{d-1}$-a.e. $y \in \Gamma_0 \cup \partial_N\Omega$ we have

$$\begin{aligned}
B(t,y)\nabla v(t,y) \cdot \nu(y) = &[A(t,\Phi(t,y))\nabla u(t,\Phi(t,y))] \cdot [D\Psi^T(t,\Phi(t,y))\nu(y)] \\
&- [\dot{\Psi}(t,\Phi(t,y)) \cdot \nabla v(t,y)]\,[\dot{\Psi}(t,\Phi(t,y)) \cdot \nu(y)] \,.
\end{aligned}$$

Both terms in the last expression are zero for a.e. t and y: the first one vanishes thanks to the Neumann boundary condition (1.9) satisfied by u, combined with the relation (1.1) between $\nu(y)$ and $\nu(t,\Phi(t,y))$;

while the second term vanishes on Γ_0 in view of (1.2), and on $\partial_N\Omega$ in view of (H8). In an analogous way, by applying Lemmas 1.11 and 1.12, it is easy to show that if v is a weak solution of problem (1.25)-(1.28) then u is a weak solution of problem (1.6)-(1.9). □

Remark 1.13. Notice that for weak solutions u of (1.6)-(1.9) the integrability condition (1.19) can be improved. Indeed, by (1.23), by the continuity (1.16) of $\widehat{\nabla} u$, and by the Lipschitz regularity of A, we infer that for a.e. $t \in (0,T)$

$$\|\ddot{u}(t)\|_{H_D^{-1}(\Omega_t)} \leq C + \|f(t)\|_{L^2(\Omega)} \tag{1.66}$$

for some constant $C > 0$ independent of t. Therefore, since $f \in L^2((0,T);L^2(\Omega))$, the function $t \mapsto \|\ddot{u}(t)\|_{H_D^{-1}(\Omega_t)}$ belongs to $L^2(0,T)$. If in addition $f \in L^p((0,T);L^2(\Omega))$ with $p \in (2,+\infty]$, then the function $t \mapsto \|\ddot{u}(t)\|_{H_D^{-1}(\Omega_t)}$ belongs to $L^p(0,T)$. The same holds true for a weak solution v of (1.25)-(1.28): indeed, taking in (1.38) an arbitrary test function $\psi \in H_D^1(\Omega_0)$ with norm 1, we obtain that if the source term f belongs to $L^p((0,T);L^2(\Omega))$ then $\ddot{v} \in L^p((0,T);H_D^{-1}(\Omega_0))$. Here, in order to derive an estimate like (1.66) for $\|\ddot{v}(t)\|_{H_D^{-1}(\Omega_0)}$, we exploit the continuity properties (1.34) and (1.36) of $\dot{v}$ and ∇v, respectively, and the regularity of the coefficients (1.29)-(1.32) discussed in Remark 1.6.

We conclude the section by presenting a possible construction of the diffeomorphisms Φ and Ψ in dimension $d = 2$.

Example 1.14. Let $d = 2$ and assume that Γ is a $C^{2,1}$ simple curve in the planar domain Ω. More precisely, we assume that Γ is injectively parametrized by arc-length through a function $\gamma\colon [0,\ell] \to \mathbb{R}^2$, such that $\gamma(0), \gamma(\ell) \in \partial\Omega$, and $\gamma(s) \in \Omega$ for every $s \in (0,\ell)$. We assume that $\Gamma_0 = \gamma([0,s_0])$ with $s_0 \in (0,\ell)$, and that $\Gamma_t = \gamma([0,s(t)])$, where $s(\cdot)$ is a nondecreasing function of class $C^{1,1}([0,T])$, with $s(0) = s_0$ and $s(T) =: s_T < \ell$.

We claim that there exist $\widehat{\Phi}$, $\widehat{\Psi}\colon [s_0,s_T] \times \overline{\Omega} \to \overline{\Omega}$ with the following properties, for $i,j = 1,2$:

(i) $\widehat{\Phi}$ and $\widehat{\Psi}$ are of class C^1 on $[s_0,s_T] \times \overline{\Omega}$ and the partial derivatives $\partial_i\partial_j\widehat{\Phi}$, $\partial_i\partial_j\widehat{\Psi}$, $\partial_i\partial_s\widehat{\Phi} = \partial_s\partial_i\widehat{\Phi}$, $\partial_i\partial_s\widehat{\Psi} = \partial_s\partial_i\widehat{\Psi}$ exist and are continuous;
(ii) for every $s \in [s_0,s_T]$ there hold $\widehat{\Phi}(s,\Omega) = \Omega$, $\widehat{\Phi}(s,\Gamma) = \Gamma$, $\widehat{\Phi}(s,\Gamma_0) = \gamma([0,s])$, and $\widehat{\Phi}(s,\cdot) = id$ in a neighborhood of $\partial\Omega$;
(iii) $\widehat{\Phi}(s_0,\cdot) = id$ in $\overline{\Omega}$;
(iv) for every $s \in [s_0,s_T]$, $\widehat{\Psi}(s,\cdot)$ is the inverse of $\widehat{\Phi}(s,\cdot)$ on $\overline{\Omega}$;
(v) $\partial_s\widehat{\Phi}$, $\partial_s\widehat{\Psi}$, $\partial_i\widehat{\Phi}$, $\partial_i\widehat{\Psi}$, $\partial_i\partial_j\widehat{\Phi}$, $\partial_i\partial_j\widehat{\Psi}$, $\partial_i\partial_s\widehat{\Phi}$, $\partial_i\partial_s\widehat{\Psi}$ belong to $\mathrm{Lip}([s_0,s_T];C^0(\overline{\Omega};\mathbb{R}^2))$;
(vi) there exists $L > 0$ such that $|\partial_i\partial_s\widehat{\Phi}(s,x) - \partial_i\partial_s\widehat{\Phi}(s,y)| \leq L|x-y|$ and $|\partial_i\partial_s\widehat{\Psi}(s,x) - \partial_i\partial_s\widehat{\Psi}(s,y)| \leq L|x-y|$ for every $s \in [s_0,s_T]$ and every $x,y \in \overline{\Omega}$.

Once proved the claim it is easy to see that the composite functions

$$\Phi(t,y) := \widehat{\Phi}(s(t),y) \quad \text{and} \quad \Psi(t,y) := \widehat{\Psi}(s(t),y) \tag{1.67}$$

satisfy (H7)-(H12).

We conclude by constructing $\widehat{\Phi}$. First we consider the case of a crack growing on a straight segment: more precisely, we assume that $\Gamma \cap \Omega = (0,\ell) \times \{0\}$ and that Ω contains a rectangular neighborhood R of the segment $(s_0,s_T) \times \{0\}$ of the form $R := (s_1,s_2) \times (-\rho_0,\rho_0)$, for some $0 \leq s_1 < s_0 < s_T < s_2 \leq \ell$ and $0 < \rho_0 < 1$. For $\varepsilon > 0$ small enough, we can construct a C^∞ function $\lambda : [s_0,s_T] \times [s_1,s_2] \to [s_1,s_2]$ satisfying the following properties:

(vii) $\lambda(s_0,z) = z$ for every $z \in [s_1,s_2]$;
(viii) $\lambda(s,z) = z$ for every $s \in [s_0,s_T]$ and $z \in [s_1,s_1+\varepsilon] \cup [s_2-\varepsilon,s_2]$;
(ix) $\lambda(s,s_0) = s$ for every $s \in [s_0,s_T]$;
(x) $|\partial_s\lambda| \leq 1+\varepsilon$ and $\partial_z\lambda \geq \varepsilon$.

Let $\theta\colon [-\rho_0,\rho_0] \to [0,1]$ be a cut-off function which vanishes in a neighborhood of $\{\pm\rho_0\}$ and is identically 1 in a neighborhood of $\{0\}$. Finally, let $F\colon [s_0,s_T] \times R \to R$ be defined as $F(s,y) := ((1-\theta(y_2))y_1 + \theta(y_2)\lambda(s,y_1), y_2)$, where y_1 and y_2 are the coordinates of y, and let $\widehat{\Phi}(s,y) = F(s,y)$ if $y \in R$ while $\widehat{\Phi}(s,y) = y$ if $y \in \overline{\Omega} \setminus R$. It is easy to verify that $\widehat{\Phi}$, and its inverse $\widehat{\Psi}$, satisfy (i)-(vi).

Let us now consider the general case. We can construct a C^∞ function $\eta\colon [s_1,s_2]\to\mathbb{R}^2$ such that $\eta(s)\notin T_{\gamma(s)}\Gamma$ for every $s\in[s_1,s_2]$, and we define $\Lambda(s,\rho):=\gamma(s)+\rho\eta(s)$, for $s\in[s_1,s_2]$ and $\rho\in\mathbb{R}$. It is easy to see that the restriction of Λ to the rectangle $R=(s_1,s_2)\times(-\rho_0,\rho_0)$ defines a $C^{2,1}$ diffeomorphism of R into an open neighborhood U of $\gamma((s_0,s_T))$, provided that $\rho_0\in(0,1)$ is chosen small enough.

Eventually, if Λ^{-1} is the inverse of the restriction of Λ to R, for every $s\in[s_0,s_T]$ we take $\widehat\Phi(s,\cdot):=\Lambda\circ F(s,\cdot)\circ\Lambda^{-1}$ in U, and we set $\widehat\Phi(s,y)=y$ for $y\in\overline\Omega\setminus U$. It is easy to see that this function, and its inverse $\widehat\Psi$, satisfy (i)-(vi).

2. Existence and uniqueness results

In this section we prove existence and uniqueness of weak solutions, both for problem (1.6)-(1.9) and for problem (1.25)-(1.28), under an additional assumption on the velocity $\dot\Phi$ of the diffeomorphism. More precisely, we require that there exists a constant $\delta>0$ such that

$$(\dot\Phi(t,y)\cdot\xi)^2+\delta|\xi|^2\le(A(t,\Phi(t,y))\xi)\cdot\xi\quad\text{for every } t\in[0,T]\,,\ y\in\overline\Omega\,,\ \text{and } \xi\in\mathbb{R}^d\,. \tag{2.1}$$

Condition (2.1) ensures that problem (1.25) is still hyperbolic: recalling the definition (1.29) of B and exploiting (1.4), we infer that there exists a constant $c_B>0$ such that

$$(B(t,y)\xi)\cdot\xi\ge c_B|\xi|^2\quad\text{for every } t\in[0,T]\,,\ y\in\overline\Omega\,,\ \text{and } \xi\in\mathbb{R}^d\,. \tag{2.2}$$

Note that (2.1) is satisfied if

$$|\dot\Phi(t,y)|^2\le c_A-\delta\quad\text{for every } t\in[0,T]\ \text{ and every } y\in\overline\Omega\,, \tag{2.3}$$

where c_A is the constant which appears in (1.4).

In dimension 2, an example of diffeomorphisms Φ and Ψ satisfying (H7)-(H12) and (2.3) can be easily obtained as follows.

Example 2.1. Let Γ, Ω, γ, s, ρ_0, λ, η, Λ, U, and $\widehat\Phi$ be defined as in Example 1.14. We claim that the composite function $\Phi(s,y):=\widehat\Phi(s(t),y)$ introduced in (1.67) satisfies condition (2.3) provided that

$$|\dot s(t)|^2\le c_A-2\delta\,, \tag{2.4}$$

that the constant $\varepsilon>0$ appearing in (vii)-(x) is sufficiently small, and that $|\eta(s)|+|\eta'(s)|\le\varepsilon$ for every s. Let us prove the claim. Exploiting condition (x) and the bound for η and η', it is easy to show that, if ε is small enough,

$$|\partial_s\widehat\Phi(s,y)|^2\le 1+\delta/(c_A-2\delta) \tag{2.5}$$

for every $s\in[s_0,s_T]$ and every $y\in\overline\Omega$. Condition (2.3) follows by combining (1.67), (2.4), and (2.5).

In the previous section we have already shown that problems (1.6)-(1.9) and (1.25)-(1.28) are equivalent. Here we give the complete proof of the following result.

Theorem 2.2. *Let be given A,f,u^0,u^1 as in* (1.3) *and* (1.5)*, and assume the existence of w satisfying* (1.10)-(1.13)*. Let B,a,b,g,v^0,v^1 be defined according to* (1.29)-(1.33)*. Then problem* (1.25)-(1.28) *admits a unique weak solution v.*

In view of Theorem 1.7, we readily obtain from Theorem 2.2 the following result.

Corollary 2.3. *Let be given A,f,u^0,u^1 as in* (1.3) *and* (1.5)*, and assume the existence of w satisfying* (1.10)-(1.13)*. Then problem* (1.6)-(1.9) *admits a unique weak solution u.*

First, in Theorems 2.6 and 2.10, we obtain existence and uniqueness of solutions to (1.38) in the larger class of functions $v\in L^2((0,T);H^1(\Omega_0))$ such that $\dot v\in L^2((0,T);L^2(\Omega_0))$ and $\ddot v\in L^2((0,T);H^{-1}_D(\Omega_0))$, which is the standard space for the study of hyperbolic equations on $(0,T)\times\Omega_0$. Eventually, in Proposition 2.11, we provide an energy equality, which ensures that the solution we have found is a weak solution of problem (1.25)-(1.28), namely it satisfies the regularity conditions (1.34)-(1.37). The energy equality will give also the continuous dependence on the data, which will be explored in the next section.

In order to prove the existence result, it is convenient to consider the function

$$z(t,y):=w(t,\Phi(t,y))\,, \tag{2.6}$$

where w is a function satisfying (1.10)-(1.13). The properties of z are summarized in the following lemma, whose proof is postponed to the Appendix.

Lemma 2.4. *Assume that w satisfies* (1.10)-(1.13)*. Then the function z defined by* (2.6) *satisfies*

$$z \in L^2((0,T); H^2(\Omega_0)) \cap H^1((0,T); H^1(\Omega_0)) \cap H^2((0,T); L^2(\Omega_0)), \tag{2.7}$$

$$z(t) = w_D(t) \quad \text{on } \partial_D\Omega \text{ for a.e. } t \in (0,T), \tag{2.8}$$

$$(B(t)\nabla z(t)) \cdot \nu = 0 \quad \text{on } \Gamma_0 \cup \partial_N\Omega \text{ for a.e. } t \in (0,T), \tag{2.9}$$

$$z(0) = v^0 \quad \text{on } \partial_D\Omega, \tag{2.10}$$

where the last three equalities are satisfied in the sense of traces.

In the following definition we introduce a notion of solution of (1.25), which is weaker that the one considered in Definition 1.5, and is useful to obtain the existence and uniqueness results.

Definition 2.5. Let A, f, u^0, u^1, and w be as in (1.3), (1.5), and (1.10)-(1.13), and let B, a, b, g, v^0, v^1, and z be defined according to (1.29)-(1.33) and (2.6). We say that v is a *generalized solution* of (1.25) with initial data (1.26) and boundary conditions (1.27) and (1.28) if $v \in L^\infty((0,T); H^1(\Omega_0))$, $v-z \in L^\infty((0,T); H^1_D(\Omega_0))$, $\dot v \in L^\infty((0,T); L^2(\Omega_0))$, $\ddot v \in L^2((0,T); H^{-1}_D(\Omega_0))$, and

$$\begin{aligned}&\langle \ddot v(t), \psi\rangle_{H^1_D(\Omega_0)} + \langle B(t)\nabla v(t), \nabla\psi\rangle_{L^2(\Omega_0)} + \langle a(t)\cdot\nabla v(t), \psi\rangle_{L^2(\Omega_0)} + 2\langle \dot v(t), \operatorname{div}(b(t)\psi)\rangle_{L^2(\Omega_0)}\\ &= \langle g(t), \psi\rangle_{L^2(\Omega_0)},\end{aligned} \tag{2.11}$$

for a.e. $t \in (0,T)$ and every $\psi \in H^1_D(\Omega_0)$.

We are now in a position to state the first existence result.

Theorem 2.6 (Existence)**.** *Under the assumptions of Definition 2.5, there exists a generalized solution of* (1.25)*, satisfying the initial conditions* (1.26) *and the boundary conditions* (1.27) *and* (1.28)*.*

Remark 2.7. Let us clarify the meaning of the initial conditions (1.26) for generalized solutions. To this aim, given a reflexive Banach space X we introduce the space of weakly continuous functions

$$C_w([0,T]; X) := \{\eta\colon [0,T] \to X : \forall x^* \in X^* \text{ the function } t \mapsto \langle x^*, \eta(t)\rangle_X \text{ is continuous}\}.$$

If Y is another Banach space such that $X \hookrightarrow Y$ with continuous injection, then (see, e.g., [6, Chapitre XVIII, §5, Lemme 6])

$$C_w([0,T]; Y) \cap L^\infty((0,T); X) \subset C_w([0,T]; X).$$

In particular, when $z = 0$, by taking $X = H^1_D(\Omega_0)$ and $Y = L^2(\Omega_0)$, we may apply this property to a generalized solution v: since $v \in C^0([0,T]; L^2(\Omega_0)) \cap L^\infty((0,T); H^1_D(\Omega_0))$, then it also belongs to $C_w([0,T]; H^1_D(\Omega_0))$. Therefore $v(0)$ is an element of $H^1_D(\Omega_0)$. Similarly, taking now $X = L^2(\Omega_0)$ and $Y = H^{-1}_D(\Omega_0)$, we have that the derivative $\dot v \in C^0([0,T]; H^{-1}_D(\Omega_0)) \cap L^\infty((0,T); L^2(\Omega_0))$ also belongs to $C_w([0,T]; L^2(\Omega_0))$. Therefore $\dot v(0)$ is an element of $L^2(\Omega_0)$. We deduce that the initial conditions (1.26) make sense if $v^0 \in H^1_D(\Omega_0)$ and $v^1 \in L^2(\Omega_0)$. In general, when $z \neq 0$, the previous argument applies to the difference $v - z$. Thus, since by (2.7) $z \in C^0([0,T]; H^1(\Omega_0)) \cap C^1([0,T]; L^2(\Omega_0))$, we conclude that also the initial position and velocity of $v = (v - z) + z$ are well defined in $H^1_D(\Omega_0) + z(0)$ and $L^2(\Omega_0)$, respectively.

Remark 2.8. Note that equality (2.11) can be recast in the framework of the duality between $L^2((0,T); H^{-1}_D(\Omega_0))$ and $L^2((0,T); H^1_D(\Omega_0))$: indeed, by the density in $L^2((0,T); H^1_D(\Omega_0))$ of the vector space generated by $\mathcal{D}(0,T) \otimes H^1_D(\Omega_0)$, it is easy to see that (2.11) is equivalent to

$$\begin{aligned}&\int_0^T \Big[\langle \ddot v(t), \xi(t)\rangle_{H^1_D(\Omega_0)} + \langle B(t)\nabla v(t), \nabla\xi(t)\rangle_{L^2(\Omega_0)} + \langle a(t)\cdot\nabla v(t), \xi(t)\rangle_{L^2(\Omega_0)}\Big]\, dt\\ &+ \int_0^T 2\langle \dot v(t), \operatorname{div}(b(t)\xi(t))\rangle_{L^2(\Omega_0)}\, dt = \int_0^T \langle g(t), \xi(t)\rangle_{L^2(\Omega_0)}\, dt\end{aligned}$$

for every $\xi \in L^2((0,T); H^1_D(\Omega_0))$.

Proof of Theorem 2.6. Without loss of generality, we may restrict ourselves to the case of homogeneous Dirichlet-Neumann boundary conditions: indeed, if v satisfies the statement, in view of the properties (2.7)-(2.10) of z, we infer that the difference $v-z$ is a generalized solution of (1.25) with initial conditions $\hat{v}^0 := v^0 - z(0) \in H^1_D(\Omega_0)$ and $\hat{v}^1 := v^1 - \dot{z}(0) \in L^2(\Omega_0)$, homogeneous Dirichlet-Neumann boundary conditions on $\partial_D\Omega$ and $\Gamma_0 \cup \partial_N\Omega$, respectively, and source term $\hat{g} := g - h$, where

$$h := \ddot{z} - \operatorname{div}(B\nabla z) + a\cdot\nabla z - 2b\cdot\nabla\dot{z} \in L^2((0,T);L^2(\Omega_0)). \tag{2.12}$$

Therefore, from now on we assume that $z=0$.

The proof is based on a perturbation argument. Following the procedure adopted in [6, Chapitre XVIII, §5], in Step 1 we study equation (2.11) with the additional terms

$$\varepsilon\langle \dot{v}(t),\psi\rangle_{L^2(\Omega_0)} + \varepsilon\langle \nabla\dot{v}(t),\nabla\psi\rangle_{L^2(\Omega_0)}\,, \quad \varepsilon>0\,.$$

Then, in Step 2, we let the viscosity parameter ε tend to zero.

Step 1. The perturbed problem. Let $\varepsilon>0$ be fixed. We want to show that there exists a solution $v_\varepsilon \in H^1((0,T);H^1_D(\Omega_0))$, with $\ddot{v}_\varepsilon \in L^2((0,T);H^{-1}_D(\Omega_0))$, of the equation

$$\begin{aligned}&\langle \ddot{v}_\varepsilon(t),\psi\rangle_{H^1_D(\Omega_0)} + \langle B(t)\nabla v_\varepsilon(t),\nabla\psi\rangle_{L^2(\Omega_0)} + \langle a(t)\cdot\nabla v_\varepsilon(t),\psi\rangle_{L^2(\Omega_0)} - 2\langle b(t)\cdot\nabla\dot{v}_\varepsilon(t),\psi\rangle_{L^2(\Omega_0)}\\ &+\varepsilon\langle \dot{v}_\varepsilon(t),\psi\rangle_{L^2(\Omega_0)} + \varepsilon\langle\nabla\dot{v}_\varepsilon(t),\nabla\psi\rangle_{L^2(\Omega_0)} = \langle g(t),\psi\rangle_{L^2(\Omega_0)}\,,\end{aligned} \tag{2.13}$$

for a.e. $t\in(0,T)$ and every $\psi\in H^1_D(\Omega_0)$. In order to study (2.13) we shall use a theorem, proved in [6], which is stated under the assumption that the coefficients are more regular with respect to time. To this aim, we regularize our coefficients by using a sequence of mollifiers $\rho_n \in C^\infty_c(\mathbb{R})$ satisfying $\rho_n \geq 0$, $\operatorname{spt}\rho_n \subset [-1/n,1/n]$, $\int\rho_n = 1$, and we introduce three families of bilinear forms over $H^1_D(\Omega_0)\times H^1_D(\Omega_0)$ as follows: for every $\eta,\xi\in H^1_D(\Omega_0)$ and every $t\in[0,T]$, we set

$$\begin{aligned}\mathcal{B}^n(t;\eta,\xi) &:= \langle (B*\rho_n)(t)\nabla\eta,\nabla\xi\rangle_{L^2(\Omega_0)}\,,\\ \mathcal{A}^n_1(t;\eta,\xi) &:= \langle (a*\rho_n)(t)\cdot\nabla\eta,\xi\rangle_{L^2(\Omega_0)}\,,\\ \mathcal{A}_2(t;\eta,\xi) &:= \langle b(t)\cdot\nabla\eta,\xi\rangle_{L^2(\Omega_0)}\,.\end{aligned}$$

In order to define the convolutions, we have to extend B and a to a neighborhood of $[0,T]$. The function B is extended by setting $B(t)=B(0)$ for $t<0$ and $B(t)=B(T)$ for $t>T$. As for a, we consider the decomposition $a=a_1+a_2$, where a_1,a_2 are defined in (1.40) and (1.41); a_1 is extended by setting $a_1(t)=a_1(0)$ for $t<0$ and $a_1(t)=a_1(T)$ for $t>T$, while a_2 is set to be 0 outside $[0,T]$. In view of (H7), (H11), (H12), and (2.2), it is easy to show that $\mathcal{B}^n,\mathcal{A}^n_1$ and $\mathcal{A}_2$ satisfy the conditions (i)-(viii) in the Appendix.

Therefore we are in a position to apply Theorem 4.1, with forcing term g, initial conditions v^0 and v^1, and $k:=\varepsilon$. For every $n\in\mathbb{N}$ let $v^n_\varepsilon \in H^1((0,T);H^1_D(\Omega_0))$ be a solution to (4.1). Taking $\dot{v}^n_\varepsilon$ as test function in (4.1) and integrating over $(0,t)$ we obtain

$$\begin{aligned}&\int_0^t \Big[\langle \ddot{v}^n_\varepsilon(s),\dot{v}^n_\varepsilon(s)\rangle_{H^1_D(\Omega_0)} + \langle (B*\rho_n)(s)\nabla v^n_\varepsilon(s),\nabla\dot{v}^n_\varepsilon(s)\rangle_{L^2(\Omega_0)} + \varepsilon\|\dot{v}^n_\varepsilon(s)\|^2_{H^1(\Omega_0)}\Big]\,ds\\ &+\int_0^t \big[\langle (a*\rho_n)(s)\cdot\nabla v^n_\varepsilon(s),\dot{v}^n_\varepsilon(s)\rangle_{L^2(\Omega_0)} - 2\langle b(s)\cdot\nabla\dot{v}^n_\varepsilon(s),\dot{v}^n_\varepsilon(s)\rangle_{L^2(\Omega_0)}\big]\,ds = \int_0^t\langle g(s),\dot{v}_\varepsilon(s)\rangle_{L^2(\Omega_0)}\,ds\,.\end{aligned} \tag{2.14}$$

Integrating by parts with respect to time, we may simplify the first two terms as follows:

$$\int_0^t \langle \ddot{v}^n_\varepsilon(s),\dot{v}^n_\varepsilon(s)\rangle_{H^1_D(\Omega_0)}\,ds = \frac{1}{2}\|\dot{v}^n_\varepsilon(t)\|^2_{L^2(\Omega_0)} - \frac{1}{2}\|v^1\|^2_{L^2(\Omega_0)}\,, \tag{2.15}$$

$$\begin{aligned}\int_0^t \langle (B*\rho_n)(s)\nabla v^n_\varepsilon(s),\nabla\dot{v}^n_\varepsilon(s)\rangle_{L^2(\Omega_0)}\,ds &= \frac{1}{2}\langle (B*\rho_n)(t)\nabla v^n_\varepsilon(t),\nabla v^n_\varepsilon(t)\rangle_{L^2(\Omega_0)}\\ &-\frac{1}{2}\langle (B*\rho_n)(0)\nabla v^0,\nabla v^0\rangle_{L^2(\Omega_0)} - \frac{1}{2}\int_0^t\langle \partial_s(B*\rho_n)(s)\nabla v_\varepsilon(s),\nabla v_\varepsilon(s)\rangle_{L^2(\Omega_0)}\,ds\,.\end{aligned} \tag{2.16}$$

Moreover, in view of (2.2), we infer that

$$\frac{1}{2}\langle (B*\rho_n)(t)\nabla v^n_\varepsilon(t),\nabla v^n_\varepsilon(t)\rangle_{L^2(\Omega_0)}\,ds \geq \frac{c_B}{2}\|\nabla v^n_\varepsilon(t)\|^2_{L^2(\Omega_0)}\,. \tag{2.17}$$

Let us now bound from above the remaining integral terms in (2.14) and (2.16): exploiting the properties of the convolution and the regularity of the coefficients discussed in Remarks 1.6, we can easily prove that there exists a constant $C > 0$ independent of n, ε, and t such that

$$\int_0^t \langle (a*\rho_n)(s)\cdot\nabla v_\varepsilon^n(s), \dot v_\varepsilon^n(s)\rangle_{L^2(\Omega_0)}\,ds \le C\int_0^t [\|\nabla v_\varepsilon^n(s)\|^2_{L^2(\Omega_0)} + \|\dot v_\varepsilon^n(s)\|^2_{L^2(\Omega_0)}]\,ds\,,$$
$$\int_0^t \langle g(s), \dot v_\varepsilon^n(s)\rangle_{L^2(\Omega_0)}\,ds \le C\|f\|^2_{L^2(0,T;L^2(\Omega_0))} + \int_0^t \|\dot v_\varepsilon^n(s)\|^2_{L^2(\Omega_0)}\,ds\,,$$
$$\int_0^t \langle \partial_s(B*\rho_n)(s)\nabla v_\varepsilon^n(s), \nabla v_\varepsilon^n(s)\rangle_{L^2(\Omega_0)}\,ds \le C\int_0^t \|\nabla v_\varepsilon^n(s)\|^2_{L^2(\Omega_0)}\,ds\,.$$

Note that by (1.2) we have for every $\eta \in H^1_D(\Omega_0)$

$$2\langle b(s)\cdot\nabla\eta, \eta\rangle_{L^2(\Omega_0)} = \langle b(s), \nabla|\eta|^2\rangle_{L^1(\Omega_0)} = -\langle \operatorname{div} b(s), |\eta|^2\rangle_{L^1(\Omega_0)}\,, \tag{2.18}$$

therefore there exists a constant $C > 0$ independent of n, ε, and t such that

$$-2\int_0^t \langle b(s)\cdot\nabla\dot v_\varepsilon^n(s), \dot v_\varepsilon^n(s)\rangle_{L^2(\Omega_0)}\,ds \le C\int_0^t \|\dot v_\varepsilon^n(s)\|^2_{L^2(\Omega_0)}\,ds\,. \tag{2.19}$$

By combining (2.14)-(2.19), we conclude that

$$\begin{aligned}&\|\dot v_\varepsilon^n(t)\|^2_{L^2(\Omega_0)} + c_B\|\nabla v_\varepsilon^n(t)\|^2_{L^2(\Omega_0)} + \varepsilon\int_0^t \|\dot v_\varepsilon^n(s)\|^2_{H^1_D(\Omega_0)}\,ds\\ &\le C_1 + C_2\int_0^t [\|\dot v_\varepsilon^n(s)\|^2_{L^2(\Omega_0)} + \|\nabla v_\varepsilon^n(s)\|^2_{L^2(\Omega_0)}]\,ds\,,\end{aligned}$$

for some constants $C_i > 0$ independent of n, ε, and t.

Eventually, in view of the Gronwall's Lemma, we infer that

$$\nabla v_\varepsilon^n \quad \text{is bounded in } L^\infty((0,T);L^2(\Omega_0;\mathbb{R}^d))\,, \tag{2.20}$$
$$\dot v_\varepsilon^n \quad \text{is bounded in } L^\infty((0,T);L^2(\Omega_0))\,, \tag{2.21}$$
$$\sqrt{\varepsilon}\dot v_\varepsilon^n \quad \text{is bounded in } L^2((0,T);H^1_D(\Omega_0))\,, \tag{2.22}$$

uniformly with respect to n and ε. From these properties, using equation (4.1), we obtain also that

$$\ddot v_\varepsilon^n \quad \text{is uniformly bounded in } L^2((0,T);H^{-1}_D(\Omega_0))\,. \tag{2.23}$$

Moreover, the boundedness of $\dot v_\varepsilon^n(t)$ in $L^2(\Omega_0)$ implies the boundedness of $v_\varepsilon^n(t)$ in $L^2(\Omega_0)$, indeed, it is enough to remark that

$$v_\varepsilon^n(t) = v^0 + \int_0^t \dot v_\varepsilon^n(s)\,ds \quad \Rightarrow \quad \|v_\varepsilon^n(t)\|^2_{L^2(\Omega_0)} \le 2\|v^0\|^2_{L^2(\Omega_0)} + 2t\int_0^t \|\dot v_\varepsilon^n(s)\|^2_{L^2(\Omega_0)}\,ds\,. \tag{2.24}$$

Therefore (2.20) and (2.21) imply that

$$v_\varepsilon^n \text{ is bounded in } L^\infty((0,T);H^1_D(\Omega_0)) \tag{2.25}$$

uniformly with respect to n and ε. By (2.22) and (2.25), for fixed $\varepsilon > 0$ a subsequence of v_ε^n, not relabeled, converges weakly in $H^1((0,T);H^1_D(\Omega_0))$ to some v_ε as $n \to +\infty$. Moreover, by (2.23), we infer that $\ddot v_\varepsilon^n$ converges weakly in $L^2((0,T);H^{-1}_D(\Omega_0))$ to $\ddot v_\varepsilon$.

Let $\psi \in H^1_D(\Omega_0)$ be a test function of the equation (2.13). We observe that as $n \to +\infty$

$$(B*\rho_n)(t)\nabla\psi \to B(t)\nabla\psi \quad \text{strongly in } L^2(\Omega;\mathbb{R}^d) \text{ for every } t \in (0,T)\,, \tag{2.26}$$
$$(a*\rho_n)(t)\psi \to a(t)\psi \quad \text{strongly in } L^2(\Omega;\mathbb{R}^d) \text{ for a.e. } t \in (0,T)\,. \tag{2.27}$$

Passing to the limit as $n \to +\infty$ in the PDE solved by v_ε^n, exploiting the strong convergences (2.26) and (2.27), and the weak convergence of v_ε^n, $\dot v_\varepsilon^n$, and $\ddot v_\varepsilon^n$, we infer that the weak limit v_ε solves equation (2.13), with initial conditions $v_\varepsilon(0) = v^0$ and $\dot v_\varepsilon(0) = v^1$.

Step 2. Vanishing viscosity. As already done in Step 1 for the sequence v_ε^n, taking as test function in (2.13) the velocity of v_ε itself, we derive the energy equality

$$\frac{1}{2}\|\dot v_\varepsilon(t)\|^2_{L^2(\Omega_0)}+\frac{1}{2}\langle B(t)\nabla v_\varepsilon(t),\nabla v_\varepsilon(t)\rangle_{L^2(\Omega_0)}+\varepsilon\int_0^t\|\dot v_\varepsilon(s)\|^2_{H^1_D(\Omega_0)}\,ds$$
$$=\frac{1}{2}\|v^1\|^2_{L^2(\Omega_0)}+\frac{1}{2}\langle B(0)\nabla v^0,\nabla v^0\rangle_{L^2(\Omega_0)}+\int_0^t\Big[\frac{1}{2}\langle \dot B(s)\nabla v_\varepsilon(s),\nabla v_\varepsilon(s)\rangle_{L^2(\Omega_0)}-\langle a(s)\nabla v_\varepsilon(s),\dot v_\varepsilon(s)\rangle_{L^2(\Omega_0)}\Big]\,ds$$
$$+\int_0^t\big[2\langle b(s)\cdot\nabla\dot v_\varepsilon(s),\dot v_\varepsilon(s)\rangle_{L^2(\Omega_0)}+\langle g(s),\dot v_\varepsilon(s)\rangle_{L^2(\Omega_0)}\big]\,ds\,, \tag{2.28}$$

and, by the uniform ellipticity (2.2) of B, the estimate

$$\|\dot v_\varepsilon(t)\|^2_{L^2(\Omega_0)}+\|\nabla v_\varepsilon(t)\|^2_{L^2(\Omega_0)}+\varepsilon\int_0^t\|\dot v_\varepsilon(s)\|^2_{H^1_D(\Omega_0)}\,ds\le C_1+C_2\int_0^t[\|\dot v_\varepsilon(s)\|^2_{L^2(\Omega_0)}+\|\nabla v_\varepsilon(s)\|^2_{L^2(\Omega_0)}]\,ds\,, \tag{2.29}$$

for some constants $C_i>0$ independent of ε and t. Thus, by applying Gronwall's Lemma, we conclude that for every $t\in[0,T]$

$$\|\dot v_\varepsilon(t)\|^2_{L^2(\Omega_0)}+\|\nabla v_\varepsilon(t)\|^2_{L^2(\Omega_0)}\le C_1e^{C_2T}\,. \tag{2.30}$$

As already done in (2.24) for v^n_ε, we deduce from (2.30) a uniform L^2 boundedness for $v_\varepsilon(t)$. Therefore, there exists a subsequence v_ε (not relabeled) which converges as $\varepsilon\to0$ to some $v\in L^2((0,T);H^1_D(\Omega_0))\cap H^1((0,T);L^2(\Omega_0))$, in the following weak sense:

$$v_\varepsilon\to v\ \text{ strongly in } L^2((0,T);L^2(\Omega_0))\,, \tag{2.31}$$
$$\nabla v_\varepsilon\rightharpoonup\nabla v\ \text{ weakly in } L^2((0,T);L^2(\Omega_0;\mathbb{R}^d))\,, \tag{2.32}$$
$$\dot v_\varepsilon\rightharpoonup\dot v\ \text{ weakly in } L^2((0,T);L^2(\Omega_0))\,, \tag{2.33}$$

moreover, $v\in L^\infty((0,T);H^1_D(\Omega_0))$ and $\dot v\in L^\infty((0,T);L^2(\Omega_0))$. Notice that a priori the weak limit v is not unique, but might depend on the particular subsequence chosen.

Let us show that v solves equation (2.11). For every $\varepsilon>0$ we take as test function in (2.13) the product $\alpha\psi$, where α and ψ are arbitrary elements of $C^1_c((0,T))$ and $H^1_D(\Omega_0)$, respectively (see Remark 2.8), and we obtain

$$\int_0^T[\langle\ddot v_\varepsilon(t),\psi\rangle_{H^1_D(\Omega_0)}+\langle B(t)\nabla v_\varepsilon(t),\nabla\psi\rangle_{L^2(\Omega_0)}+\langle a(t)\cdot\nabla v_\varepsilon(t),\psi\rangle_{L^2(\Omega_0)}+2\langle\dot v_\varepsilon(t),\operatorname{div}(b(t)\psi)\rangle_{L^2(\Omega_0)}]\alpha(t)\,dt$$
$$+\varepsilon\int_0^T\big[\langle\nabla\dot v_\varepsilon(t),\nabla\psi\rangle_{L^2(\Omega_0)}+\langle\dot v_\varepsilon(t),\psi\rangle_{L^2(\Omega_0)}\big]\,\alpha(t)\,dt=\int_0^T\langle g(t),\psi\rangle_{L^2(\Omega_0)}\alpha(t)\,dt\,. \tag{2.34}$$

Let us study separately the asymptotic behavior of the terms appearing in the equality above: exploiting (2.31)-(2.33), as $\varepsilon\to0$ we have

$$\int_0^T\langle\ddot v_\varepsilon(t),\psi\rangle_{H^1_D(\Omega_0)}\alpha(t)\,dt=-\int_0^T\langle\dot v_\varepsilon(t),\psi\rangle_{L^2(\Omega_0)}\dot\alpha(t)\,dt\to-\int_0^T\langle\dot v(t),\psi\rangle_{L^2(\Omega_0)}\dot\alpha(t)\,dt\,,$$
$$\int_0^T\langle B(t)\nabla v_\varepsilon(t),\nabla\psi\rangle_{L^2(\Omega_0)}\alpha(t)\,dt\to\int_0^T\langle B(t)\nabla v(t),\nabla\psi\rangle_{L^2(\Omega_0)}\alpha(t)\,dt\,,$$
$$\int_0^T\langle a(t)\cdot\nabla v_\varepsilon(t),\psi\rangle_{L^2(\Omega_0)}\alpha(t)\,dt\to\int_0^T\langle a(t)\cdot\nabla v(t),\psi\rangle_{L^2(\Omega_0)}\alpha(t)\,dt\,,$$
$$\int_0^T\langle b(t)\cdot\nabla\dot v_\varepsilon(t),\psi\rangle_{L^2(\Omega_0)}\alpha(t)\,dt=-\int_0^T\langle\dot v_\varepsilon(t),\operatorname{div}(b(t)\psi)\rangle_{L^2(\Omega_0)}\alpha(t)\,dt\to-\int_0^T\langle\dot v(t),\operatorname{div}(b(t)\psi)\rangle_{L^2(\Omega_0)}\alpha(t)\,dt\,.$$

Moreover, by combining (2.29) and (2.30), we infer that also $\varepsilon\int_0^T\|\dot v_\varepsilon\|^2_{H^1_D(\Omega_0)}\,dt$ is bounded by a constant $C>0$ independent of ε and t, therefore we get

$$\left|\varepsilon\int_0^T\big[\langle\nabla\dot v_\varepsilon(t),\nabla\psi\rangle_{L^2(\Omega_0)}+\langle\dot v_\varepsilon(t),\psi\rangle_{L^2(\Omega_0)}\big]\,\alpha(t)\,dt\right|\le\sqrt{\varepsilon}\int_0^T\sqrt{\varepsilon}\|\dot v_\varepsilon(t)\|_{H^1_D(\Omega_0)}\|\psi\|_{H^1_D(\Omega_0)}|\alpha(t)|\,dt$$

$$\leq \sqrt{\varepsilon}\|\psi\|_{H^1_D(\Omega_0)}\|\alpha\|_{L^2(0,T)}\left(\int_0^T \varepsilon\|\dot v_\varepsilon(t)\|^2_{H^1_D(\Omega_0)}\,dt\right)^{1/2} \leq \sqrt{\varepsilon}C \to 0\,.$$

The last properties, together with equality (2.34), give that the distributional derivative $\ddot v$ is an element of $L^2(0,T;H^{-1}_D(\Omega_0))$, whose action against the test function $\alpha\psi$ is given by

$$\int_0^T \langle \ddot v(t),\psi\rangle_{H^1_D(\Omega_0)}\alpha(t)\,dt = -\int_0^T [\langle B(t)\nabla v(t),\nabla\psi\rangle_{L^2(\Omega_0)} + 2\langle \dot v(t),\operatorname{div}(b(t)\psi)\rangle_{L^2(\Omega_0)}]\alpha(t)\,dt$$
$$+\int_0^T \langle[-a(t)\cdot\nabla v(t),\psi\rangle_{L^2(\Omega_0)} + \langle g(t),\psi\rangle_{L^2(\Omega_0)}]\alpha(t)\,dt\,,$$

that is, v solves (2.11) (see also Remark 2.8). The validity of the initial conditions of v is readily verified by taking $\alpha\in C^1([0,T])$ vanishing at T, $\psi\in H^1_D(\Omega_0)$, and passing to the limit as $\varepsilon\to 0$ in the products

$$\int_0^T \langle \dot v_\varepsilon(t),\alpha(t)\psi\rangle_{L^2(\Omega_0)}\,dt\,,\qquad \int_0^T \langle \ddot v_\varepsilon(t),\alpha(t)\psi\rangle_{H^1_D(\Omega_0)}\,dt\,.$$

This concludes the proof. □

Remark 2.9. Let v be a generalized solution of (1.25) and let v_ε be its viscous approximation solving (2.13). By the weak lower semicontinuity of the norm, the estimate (2.30) passes to the limit, namely

$$\|\dot v(t)\|^2_{L^2(\Omega_0)} + \|\nabla v(t)\|^2_{L^2(\Omega_0)} \leq \liminf_\varepsilon[\|\dot v_\varepsilon(t)\|^2_{L^2(\Omega_0)} + \|\nabla v_\varepsilon(t)\|^2_{H^1_D(\Omega_0)}] \leq C\,, \tag{2.35}$$

for some constant $C>0$ independent of t. Let now $u\colon Q_\Gamma\to\mathbb{R}$ be defined according to (1.24), i.e., $u(t,x):=v(t,\Psi(t,x))$. In view of (2.35) and formulas (1.65) it is immediate to check that for every $t\in[0,T]$ we have

$$\|\dot u(t)\|^2_{L^2(\Omega)} + \|\widehat\nabla u(t)\|^2_{L^2(\Omega)} \leq C\,, \tag{2.36}$$

for some constant $C>0$ independent of t.

The uniqueness of solutions relies on a standard technique due to Ladyzenskaya [7], which consists in taking as test function in (2.11) the primitive of a solution.

Theorem 2.10 (Uniqueness)**.** *Under the assumptions of Definition 2.5, there is at most one generalized solution of* (1.25)*, satisfying the initial conditions* (1.26) *and the boundary conditions* (1.27) *and* (1.28)*.*

Proof. As already pointed out at the beginning of the proof of Theorem 2.6, we may restrict ourselves to the case in which $z=0$. Moreover, by linearity, it is enough to show that the sole generalized solution v to the problem (1.25) with

$$z=g=v^0=v^1=0$$

is $v\equiv 0$. The proof is recursive: first, we show uniqueness in a small time interval $[0,t_0]$; the same argument applies to $[t_0,2t_0]$ and, in a finite number of steps, to all $[0,T]$.

Step 1. Let $s\in(0,T)$ be fixed and let $\xi\in L^2(0,T;H^1_D(\Omega_0))$ be defined as follows:

$$\xi(t):=\begin{cases} -\int_t^s v(\tau)\,d\tau & \text{if } t\in[0,s]\,,\\ 0 & \text{if } t\in[s,T]\,.\end{cases}$$

Note that

$$\xi(T)=\xi(s)=0\,,$$

moreover, $\dot\xi\in L^2(0,T;L^2(\Omega_0))$, indeed,

$$\dot\xi(t)=\begin{cases} v(t) & \text{if } t\in[0,s)\,,\\ 0 & \text{if } t\in(s,T]\,.\end{cases}$$

By taking ξ as test function in (2.11), we get

$$\int_0^s \Big[\langle \ddot v(t), \xi(t)\rangle_{H^1_D(\Omega_0)} + \langle B(t)\nabla v(t), \nabla \xi(t)\rangle_{L^2(\Omega_0)}\Big]\,dt + \int_0^s \big[\langle a(t)\cdot \nabla v(t), \xi(t)\rangle_{L^2(\Omega_0)} + 2\langle \dot v(t), \operatorname{div}(b(t)\xi(t))\rangle_{L^2(\Omega_0)}\big]\,dt = 0\,. \tag{2.37}$$

Integrating by parts with respect to time we may write

$$\int_0^s \langle \ddot v, \xi\rangle_{H^1_D(\Omega_0)}\,dt = -\int_0^s \langle \dot v, v\rangle_{L^2(\Omega_0)}\,dt = -\frac{1}{2}\|v(s)\|^2_{L^2(\Omega_0)}\,, \tag{2.38}$$

where we have used the assumption $v^1 = 0$.

Let us rewrite the term involving B. As already noticed in (1.39), $B \in \mathrm{Lip}([0,T]; L^\infty(\Omega;\mathbb{R}^{d\times d}_{sym}))$; furthermore, by the definition of generalized solution (recall Definition 2.5), it is easy to see that $\xi \in \mathrm{Lip}([0,T]; H^1_D(\Omega_0))$. Therefore the product $B\nabla\xi$ belongs to $\mathrm{Lip}([0,T]; L^2(\Omega_0))$. Integrating by parts with respect to time, we obtain

$$\int_0^s \langle B(t)\nabla v(t), \nabla \xi(t)\rangle_{L^2(\Omega_0)}\,dt = \int_0^s \langle \nabla \dot\xi(t), B(t)\nabla\xi(t)\rangle_{L^2(\Omega_0)}\,dt = -\frac{1}{2}\,\langle B(0)\nabla\xi(0), \nabla\xi(0)\rangle_{L^2(\Omega_0)} - \frac{1}{2}\int_0^s \langle \dot B(t)\nabla\xi(t), \nabla\xi(t)\rangle_{L^2(\Omega_0)}\,dt\,, \tag{2.39}$$

since by construction $v^0 = \xi(s) = 0$ in $H^1_D(\Omega_0)$. Inserting (2.38) and (2.39) into (2.37), we get

$$\frac{1}{2}\,\|v(s)\|^2_{L^2(\Omega_0)} + \frac{1}{2}\,\langle B(0)\nabla\xi(0), \nabla\xi(0)\rangle_{L^2(\Omega_0)} = \int_0^s \Big[-\frac{1}{2}\langle \dot B(t)\nabla\xi(t), \nabla\xi(t)\rangle_{L^2(\Omega_0)} + \langle a(t)\cdot\nabla v(t), \xi(t)\rangle_{L^2(\Omega_0)} + 2\langle \dot v(t), \operatorname{div}(b(t)\xi(t))\rangle_{L^2(\Omega_0)}\Big]\,dt\,. \tag{2.40}$$

Let us now bound form above the scalar products in the right-hand side of (2.40). By the Lipschitz regularity of B, there exists $C > 0$ such that $\|\dot B(s,\cdot)\|_{L^\infty(\Omega_0)} \le C$ for a.e. $t \in (0,T)$, in particular

$$\int_0^s \langle \dot B(t)\nabla\xi(t), \nabla\xi(t)\rangle_{L^2(\Omega_0)}\,dt \le C\int_0^s \|\xi(t)\|^2_{H^1_D(\Omega_0)}\,dt\,. \tag{2.41}$$

We split $\operatorname{div}(b\xi)$ into the sum $\xi \operatorname{div} b + \nabla\xi\cdot b$. As already pointed out in (1.39) $\operatorname{div} b \in \mathrm{Lip}([0,T]; L^\infty(\Omega))$, therefore we may argue for $\operatorname{div} b$ as for B: integrating by parts with respect to time and exploiting the equalities $v^0 = \xi(s) = 0$, we obtain

$$\int_0^s \langle \dot v(t), \xi(t)\operatorname{div} b(t)\rangle_{L^2(\Omega_0)}\,dt = -\int_0^s \langle v(t), v(t)\operatorname{div} b(t)\rangle_{L^2(\Omega_0)}\,dt - \int_0^s \langle v(t), \xi(t)\partial_t(\operatorname{div} b)(t)\rangle_{L^2(\Omega_0)}\,dt \le C\int_0^s [\|\xi(t)\|^2_{L^2(\Omega_0)} + \|v(t)\|^2_{L^2(\Omega_0)}]\,dt\,, \tag{2.42}$$

for some constant $C > 0$ independent of s. Performing first an integration by parts with respect to time and then with respect to space, exploiting in the former the assumption $v^1 = \xi(s) = 0$ and in the latter the equality $b\cdot n = 0$ on $\Gamma_0 \cup \partial\Omega$ (see (H8) and (1.2)), we infer that

$$\int_0^s \langle \dot v(t), \nabla\xi(t)\cdot b(t)\rangle_{L^2(\Omega_0)}\,dt = \frac{1}{2}\int_0^s \Big[\langle \operatorname{div} b(t), |\dot v(t)|^2\rangle_{L^1(\Omega_0)} - 2\langle v(t), \nabla\xi(t)\cdot \dot b(t)\rangle_{L^2(\Omega_0)}\Big]\,dt \le C\int_0^s [\|v(t)\|^2_{L^2(\Omega_0)} + \|\xi(t)\|^2_{H^1_D(\Omega_0)}]\,dt\,, \tag{2.43}$$

for some constant $C > 0$ independent of s.

As already done in Remark 1.6, we split a into $a = a_1 + a_2$, with a_1 and a_2 defined according to (1.40) and (1.41), respectively. We recall that a_1 belongs to $\mathrm{Lip}([0,T]; L^\infty(\Omega;\mathbb{R}^d))$, therefore $\dot a_1 \in L^\infty((0,T); L^2(\Omega))$ and there exists $C > 0$ such that $\|\dot a_1(t,\cdot)\|_{L^\infty(\Omega)} \le C$ for a.e. $t \in (0,T)$. Integrating by parts with respect to time and exploiting the equalities $a_1(0) = \xi(s) = 0$, we get

$$\int_0^s \langle a_1(t)\cdot\nabla v(t), \xi(t)\rangle_{L^2(\Omega_0)}\,dt = \int_0^s \langle \nabla\dot\xi(t), a_1(t)\xi(t)\rangle_{L^2(\Omega_0)}\,dt$$

$$
\begin{aligned}
&= -\int_0^s \langle \nabla\xi(t), \dot{a_1}(t)\xi(t)\rangle_{L^2(\Omega_0)}\,dt - \int_0^s \langle \nabla\xi(t), a_1(t)\dot{\xi}(t)\rangle_{L^2}\,dt \\
&= -\int_0^s \langle \nabla\xi(t), \dot{a_1}(t)\xi(t)\rangle_{L^2(\Omega_0)}\,dt - \int_0^s \langle \nabla\xi(t), a_1(t)v(t)\rangle_{L^2(\Omega_0)}\,dt \\
&\le C\int_0^s [\|\xi(t)\|^2_{H^1_D(\Omega_0)} + \|v(t)\|^2_{L^2(\Omega_0)}]\,dt\,, \qquad (2.44)
\end{aligned}
$$

for some constant $C > 0$ independent of s. On the other hand, performing an integration by parts with respect to the space variable we obtain

$$
\begin{aligned}
\int_0^s \langle \nabla v(t), a_2(t)\xi(t)\rangle_{L^2(\Omega_0)}\,dt &= -\int_0^s \big[\langle \operatorname{div} a_2(t), v(t)\xi(t)\rangle_{L^2(\Omega_0)} + \langle v(t)a_2(t), \nabla\xi(t)\rangle_{L^2(\Omega_0)}\big]\,dt \\
&\le C\int_0^s [\|v(t)\|^2_{L^2(\Omega_0)} + \|\xi(t)\|^2_{H^1_D(\Omega_0)}]\,dt\,, \qquad (2.45)
\end{aligned}
$$

for some constant $C > 0$ independent of s. To derive (2.45) we have used the property $a_2(t)\cdot\nu = 0$ on $\Gamma_0 \cup \partial\Omega$, which follows from the definition $a_2 := -\dot{b}$ and from the equality $b(t)\cdot\nu \equiv 0$ on $\Gamma_0\cup\partial\Omega$.

By combining (2.40) with the coercivity property (2.2) of B and the upper bounds (2.41)-(2.45), we conclude that

$$
\|v(s)\|^2_{L^2(\Omega_0)} + c_B\|\nabla\xi(0)\|^2_{L^2(\Omega_0)} \le \overline{C}\int_0^s [\|v(t)\|^2_{L^2(\Omega_0)} + \|\xi(t)\|^2_{H^1_D(\Omega_0)}]\,dt\,, \qquad (2.46)
$$

where the constant $\overline{C} > 0$ does not depend on the parameter s chosen. Now, introducing

$$
z(s) := \int_0^s v(\tau)\,d\tau\,,
$$

we can rewrite $\xi(t) = z(t) - z(s)$ for every $t \in [0,s]$, in particular

$$
\|\nabla\xi(0)\|^2_{L^2(\Omega_0)} = \|\nabla z(s)\|^2_{L^2(\Omega_0)}\,, \quad \int_0^s \|\xi(t)\|^2_{H^1_D(\Omega_0)}\,dt \le 2s\|z(s)\|^2_{H^1_D(\Omega_0)} + 2\int_0^s \|z(t)\|^2_{H^1_D(\Omega_0)}\,dt\,. \qquad (2.47)
$$

Moreover, as already done in (2.24), by the definition of z it is easy to show that

$$
\|z(s)\|^2_{L^2(\Omega_0)} \le 2T\int_0^s \|\dot{z}(t)\|^2_{L^2(\Omega_0)}\,dt = 2T\int_0^s \|v(t)\|^2_{L^2(\Omega_0)}\,dt\,. \qquad (2.48)
$$

Therefore, by combining (2.46) with (2.47) and (2.48), we obtain

$$
\|v(s)\|^2_{L^2(\Omega_0)} + (c_B - 2\overline{C}s)\|z(s)\|^2_{H^1_D(\Omega_0)} \le (2TC_B + 2\overline{C})\int_0^s [\|v(t)\|^2_{L^2(\Omega_0)} + \|z(t)\|^2_{H^1_D(\Omega_0)}]\,dt\,.
$$

If s is small enough, e.g. $s = t_0 := c_B/(4\overline{C})$, we can apply Gronwall's lemma and obtain that

$$
v \equiv 0 \ \text{ in } [0,t_0]\,.
$$

Step 2. The strategy adopted in Step 1 can be repeated in some time interval $[t_0,t_1]$, and in a finite number of steps, in the whole $[0,T]$. Notice that in the previous proof we have used as a key tool the fact that at time 0 the diffeomorphism was the identity; therefore in the following step we possibly have to restate the problem starting at the last endpoint t_1, considering as initial set Ω_{t_1}. □

In order to state the next result, we need to introduce the following energy: given $\eta \in L^\infty((0,T);H^1(\Omega_0))$ with distributional time derivative $\dot\eta \in L^\infty((0,T);L^2(\Omega_0))$, we set for a.e. $t\in(0,T)$

$$
\mathcal{E}_B(\eta,t) := \frac{1}{2}\|\dot\eta(t)\|^2_{L^2(\Omega_0)} + \frac{1}{2}\langle B(t)\nabla\eta(t), \nabla\eta(t)\rangle_{L^2(\Omega_0)}\,, \qquad (2.49)
$$

where B is the tensor field defined in (1.29).

Proposition 2.11 (Energy equality)**.** *Under the assumptions of Definition 2.5, let v be the (unique) generalized solution of* (1.25)*, satisfying the initial conditions* (1.26) *and the boundary conditions* (1.27) *and* (1.28)*. Then the energy $\mathcal{E}_B(v,\cdot)$ is a continuous function from $[0,T]$ to $\mathbb{R}$. Moreover, in case $z=0$, it reads*

$$\mathcal{E}_B(v,t) = \mathcal{E}_B(v,0) + \mathcal{R}(v,t)\,, \tag{2.50}$$

where $\mathcal{R}$ is the integral remainder

$$\begin{aligned}\mathcal{R}(v,t) := &\int_0^t [\frac{1}{2}\langle \dot{B}(s)\nabla v(s), \nabla v(s)\rangle_{L^2(\Omega_0)} - \langle a(s)\cdot\nabla v(s), \dot v(s)\rangle_{L^2(\Omega_0)}]ds\\ &+\int_0^t \left[-\langle \operatorname{div} b(s), |\dot v(s)|^2\rangle_{L^1(\Omega_0)} + \langle g(s), \dot v(s)\rangle_{L^2(\Omega_0)}\right] ds\,.\end{aligned}$$

Note that if the solution v were smooth enough, then we could take $\dot v$ as test function in (2.11), and (2.50) would be straightforward. In our case, the proof is rather technical: roughly speaking, we approach $\dot v$ with $H^1_D(\Omega_0)$-valued functions by means of a double regularization process, in the same spirit of [9, Chapter 8, Lemma 8.3].

Proof of Proposition 2.11. The energy $\mathcal{E}_B(v,t)$ can be written as

$$\mathcal{E}_B(v,t) = \mathcal{E}_B(v-z,t) + \mathcal{E}_B(z,t) - \langle \dot v(t), \dot z(t)\rangle_{L^2(\Omega_0)} - \langle B(t)\nabla v(t), \nabla z(t)\rangle_{L^2(\Omega_0)}\,,$$

where z is defined according to (2.6). By the strong continuity of z (see (2.7)) and the weak continuity of v (see Remark 2.7), we infer that $\mathcal{E}_B(z,t)$, $\langle \dot v(t), \dot z(t)\rangle_{L^2(\Omega_0)}$, and $\langle B(t)\nabla v(t), \nabla z(t)\rangle_{L^2(\Omega_0)}$ are continuous functions from $[0,T]$ to $\mathbb{R}$. Thus, $\mathcal{E}_B(v,\cdot)$ is continuous if and only if so is $\mathcal{E}_B(v-z,\cdot)$. Therefore, it is enough to prove the statement in the case of homogeneous Dirichlet boundary condition. From now on we will take $z=0$.

Let $t=t_0$ be fixed. Let θ_0 denote the characteristic function of the time interval $(0,t_0)$. We want to approximate $v\theta_0\colon \mathbb{R}\to H^1_D(\Omega_0)$ by means of a suitable sequence of functions belonging to $C^\infty_c(\mathbb{R}; H^1_D(\Omega_0))$. To this aim, we first need to define v for every $t\in\mathbb{R}$: by time reflection, we construct an extension (not relabeled) $v\in L^2(\mathbb{R}; H^1_D(\Omega_0))$, still satisfying $\dot v\in L^2(\mathbb{R}; L^2(\Omega_0))$ and $\ddot v\in L^2(\mathbb{R}; H^{-1}_D(\Omega_0))$. Similarly, we extend also the coefficients of the PDE (2.11).

For every $\delta>0$, we call $\theta_\delta\colon\mathbb{R}\to\mathbb{R}$ the function which equals 1 in $[\delta, t_0-\delta]$, 0 outside $[0,t_0]$ and which is linear in $[0,\delta]$ and $[t_0-\delta, t_0]$. As $\delta\to 0$, $\theta_\delta\to\theta_0$ in $L^1(\mathbb{R})$. Let $\rho_m\in C^\infty_c(\mathbb{R})$ be a sequence of mollifiers. For brevity, in the following, we will omit the indices δ and m.

In view of the definitions above, for every m and δ fixed, it holds

$$\rho*(\theta v)\in C^\infty_c(\mathbb{R}; H^1_D(\Omega_0))\,,$$

where the compact support is due to the presence of θ, and the regularity follows from the equality

$$\frac{d^k}{ds^k}[\rho*(\theta v)] = \Big(\frac{d^k\rho}{ds^k}\Big)*(\theta v)$$

and the fact that v is $H^1_D(\Omega_0)$-valued. Similarly, it holds

$$\rho*(\theta\dot v)\in C^\infty_c(\mathbb{R}; L^2(\Omega_0))\,,$$

in particular

$$\int_{\mathbb{R}}\frac{d}{ds}\|\rho*(\theta\dot v)\|^2_{L^2(\Omega_0)}(s)\,ds = 0\,. \tag{2.51}$$

By differentiating the integrand in (2.51) and exploiting the properties of the convolution, we get

$$0 = \int_{\mathbb{R}}\Big[\langle \rho*(\dot\theta\dot v), \rho*(\theta\dot v)\rangle_{L^2(\Omega_0)} + \langle \rho*(\theta\ddot v), \rho*(\theta\dot v)\rangle_{L^2(\Omega_0)}\Big]\,ds\,, \tag{2.52}$$

where $\rho*(\theta\ddot v)$ stands for the difference

$$\rho*(\theta\ddot v) := \dot\rho*(\theta\dot v) - \rho*(\dot\theta\dot v)\in L^2(\mathbb{R}; L^2(\Omega_0))\,.$$

Let us now study separately the behavior of each term in (2.52) as $\delta \to 0$, keeping m fixed. The first term has the following asymptotics:

$$\begin{aligned}\lim_{\delta\to 0}\int_{\mathbb{R}}\langle \rho*(\dot\theta\dot v),\rho*(\theta\dot v)\rangle_{L^2(\Omega_0)}\,ds &= \lim_{\delta\to 0}\int_{\mathbb{R}}\langle \rho*(\dot\theta\dot v),\rho*(\theta_0\dot v)\rangle_{L^2(\Omega_0)}\,ds = \lim_{\delta\to 0}\int_{\mathbb{R}}\dot\theta\langle \dot v,\rho*\rho*(\theta_0\dot v)\rangle_{L^2(\Omega_0)}\,ds\\ &= -\langle \dot v,\rho*\rho*(\theta_0\dot v)\rangle_{L^2(\Omega_0)}(t_0)+\langle \dot v,\rho*\rho*(\theta_0\dot v)\rangle_{L^2(\Omega_0)}(0)\,. \end{aligned} \tag{2.53}$$

Here we have split θ as $(\theta-\theta_0)+\theta_0$ and used the properties

$$\begin{aligned}&\rho*(\dot\theta\dot v)\text{ uniformly bounded in }L^1(\mathbb{R};L^2(\Omega_0))\,,\\ &\rho*((\theta-\theta_0)\dot v)\to 0\text{ strongly in }L^\infty(\mathbb{R};L^2(\Omega_0))\,,\\ &s\mapsto\langle \dot v,\rho*\rho*(\theta_0\dot v)\rangle_{L^2(\Omega_0)}(s)\text{ is continuous in }\mathbb{R}\,,\end{aligned}$$

where the last property holds true since $\rho*\rho*(\theta_0\dot v)\in C^0(\mathbb{R};L^2(\Omega_0))$ and $\dot v\in C_w(\mathbb{R};L^2(\Omega_0))\cap L^\infty(\mathbb{R};L^2(\Omega_0))$ (for the weak continuity of $\dot v$ see Remark 2.7). The second term of (2.52) satisfies

$$\lim_{\delta\to 0}\int_{\mathbb{R}}\langle \rho*(\theta\ddot v),\rho*(\theta\dot v)\rangle_{L^2(\Omega_0)}\,ds=\int_{\mathbb{R}}\langle \rho*(\theta_0\ddot v),\rho*(\theta_0\dot v)\rangle_{L^2(\Omega_0)}\,ds\,. \tag{2.54}$$

This follows by direct computation: it turns out that $\rho*(\theta\ddot v)$ is uniformly bounded in $L^2(\mathbb{R};L^2(\Omega_0))$, moreover it converges strongly in $L^2(\mathbb{R};H^{-1}_D(\Omega_0))$ to $\rho*(\theta_0\ddot v)$ (which, again by difference, is an element of $L^2(\mathbb{R};L^2(\Omega_0))$). By combining (2.52), (2.53) and (2.54), we infer that

$$0=-\langle \dot v,\rho*\rho*(\theta_0\dot v)\rangle_{L^2(\Omega_0)}(t_0)+\langle \dot v,\rho*\rho*(\theta_0\dot v)\rangle_{L^2(\Omega_0)}(0)+\int_{\mathbb{R}}\langle \rho*(\theta_0\ddot v),\rho*(\theta_0\dot v)\rangle_{L^2(\Omega_0)}\,ds\,. \tag{2.55}$$

We now apply the same argument to the function

$$\langle B\rho*(\theta\nabla v),\rho*(\theta\nabla v)\rangle_{L^2(\Omega_0)}\in W^{1,\infty}(\mathbb{R})\,.$$

Starting from the identity

$$\int_{\mathbb{R}}\frac{d}{ds}\langle B\rho*(\theta\nabla v),\rho*(\theta\nabla v)\rangle_{L^2(\Omega_0)}(s)\,ds=0$$

we infer that

$$\begin{aligned}0=\ &\int_{\mathbb{R}}\Big[\langle \dot B\rho*(\theta\nabla v),\rho*(\theta\nabla v)\rangle_{L^2(\Omega_0)}+2\langle \rho*(B\theta\nabla v),\rho*(\theta\nabla\dot v)\rangle_{L^2(\Omega_0)}\Big]\,ds\\ &+\int_{\mathbb{R}}\Big[2\langle \rho*(B\theta\nabla v),\rho*(\dot\theta\nabla v)\rangle_{L^2(\Omega_0)}+2\langle B\rho*(\theta\nabla v)-\rho*(B\theta\nabla v),\dot\rho*(\theta\nabla v)\rangle_{L^2(\Omega_0)}\Big]\,ds\,,\end{aligned} \tag{2.56}$$

where $\rho*(\theta\nabla\dot v)$ is well defined in $L^2(\mathbb{R};L^2(\Omega_0;\mathbb{R}^d))$ as the difference between $\dot\rho*(\theta\nabla v)$ and $\rho*(\dot\theta\nabla v)$. Notice that since $B\in\mathrm{Lip}(\mathbb{R};L^\infty(\Omega;\mathbb{R}^{d\times d}_{sym}))\subset\mathrm{Lip}(\mathbb{R};L^2(\Omega_0;\mathbb{R}^{d\times d}_{sym}))$, we have that $\dot B\in L^\infty(\mathbb{R};L^2(\Omega_0;\mathbb{R}^{d\times d}_{sym}))$, moreover there exists $C>0$ such that $\|\dot B(s,\cdot)\|_{L^\infty(\Omega_0)}\le C$ for a.e. $s\in\mathbb{R}$. We now pass to the limit as $\delta\to 0$: exploiting in (2.56) the properties

$$\begin{aligned}&\dot B\rho*((\theta-\theta_0)\nabla v)\rightharpoonup 0\text{ weakly in }L^2(\mathbb{R};L^2(\Omega_0;\mathbb{R}^d))\,,\\ &\rho*((\theta-\theta_0)\nabla v)\to 0\text{ strongly in }L^2(\mathbb{R};L^2(\Omega_0;\mathbb{R}^d))\,,\\ &\rho*((\theta-\theta_0)\nabla\dot v)\rightharpoonup 0\text{ weakly in }L^2(\mathbb{R};L^2(\Omega_0;\mathbb{R}^d))\,,\\ &\rho*(B(\theta-\theta_0)\nabla v)\to 0\text{ strongly in }L^\infty(\mathbb{R};L^2(\Omega_0;\mathbb{R}^d))\,,\\ &\rho*(\dot\theta\nabla v)\text{ uniformly bounded in }L^1(\mathbb{R};L^2(\Omega_0;\mathbb{R}^d))\,,\\ &s\mapsto\langle \rho*(\rho*(B\theta_0\nabla v)),\nabla v\rangle_{L^2(\Omega_0)}(s)\text{ is continuous in }\mathbb{R}\,,\\ &\dot\rho*(\theta-\theta_0)\nabla v\to 0\text{ strongly in }L^2(\mathbb{R};L^2(\Omega_0;\mathbb{R}^d))\,,\end{aligned}$$

we obtain

$$
\begin{aligned}
0 &= \int_{\mathbb{R}} \Big[\langle \dot{B}\rho * (\theta_0 \nabla v), \rho * (\theta_0 \nabla v)\rangle_{L^2(\Omega_0)} + 2\langle \rho * (B\theta_0 \nabla v), \rho * (\theta_0 \nabla \dot{v})\rangle_{L^2(\Omega_0)} \Big]\, ds \\
&\quad -2\langle \rho * (\rho * (B\theta_0 \nabla v)), \nabla v\rangle_{L^2(\Omega_0)}(t_0) + 2\langle \rho * (\rho * (B\theta_0 \nabla v)), \nabla v\rangle_{L^2(\Omega_0)}(0) \\
&\quad + \int_{\mathbb{R}} \big[2\langle B\rho * (\theta_0 \nabla v) - \rho * (B\theta_0 \nabla v), \dot{\rho} * (\theta_0 \nabla v)\rangle_{L^2(\Omega_0)} \big]\, ds\,.
\end{aligned} \tag{2.57}
$$

Since the restriction $\theta_0 v$ solves the PDE (2.11), we infer that for every $\xi \in L^2(\mathbb{R}; H^1_D(\Omega_0))$ it holds

$$
\begin{aligned}
&\int_{\mathbb{R}} \Big[\langle \theta_0 \ddot{v}, \xi\rangle_{H^1_D(\Omega_0)} + \langle B\nabla(\theta_0 v), \nabla \xi\rangle_{L^2(\Omega_0)} + \langle a \cdot \nabla(\theta_0 v), \xi\rangle_{L^2(\Omega_0)} + 2\langle \theta_0 \dot{v}, \operatorname{div}(b\xi)\rangle_{L^2(\Omega_0)} \Big]\, ds \\
&= \int_{\mathbb{R}} \langle \theta_0 g, \xi\rangle_{L^2(\Omega_0)}\, ds
\end{aligned}
$$

(see Remark 2.8), in particular, by taking $\xi = \rho * (\rho * (\theta_0 \dot{v}))$ and exploiting the properties of the convolution, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}} \big[\langle \rho * (\theta_0 \ddot{v}), \rho * (\theta_0 \dot{v})\rangle_{L^2(\Omega_0)} + \langle \rho * (B\theta_0 \nabla v), \rho * (\theta_0 \nabla \dot{v})\rangle_{L^2(\Omega_0)} \big]\, ds \\
&= \int_{\mathbb{R}} \big[-\langle \rho * (\theta_0 a \cdot \nabla v), \rho * (\theta_0 \dot{v})\rangle_{L^2(\Omega_0)} - 2\langle \rho * (b\theta_0 \dot{v}), \rho * (\theta_0 \nabla \dot{v})\rangle_{L^2(\Omega_0)} \big]\, ds \\
&+ \int_{\mathbb{R}} \big[-2\langle \rho * (\theta_0 \dot{v} \operatorname{div} b), \rho * (\theta_0 \dot{v})\rangle_{L^2(\Omega_0)} + \langle \rho * (\theta_0 g), \rho * (\theta_0 \dot{v})\rangle_{L^2(\Omega_0)} \big]\, ds\,.
\end{aligned} \tag{2.58}
$$

By combining (2.55), (2.57) and (2.58) we conclude that

$$
\begin{aligned}
&\langle \dot{v}, \rho * \rho * (\theta_0 \dot{v})\rangle_{L^2(\Omega_0)}(t_0) - \langle \dot{v}, \rho * \rho * (\theta_0 \dot{v})\rangle_{L^2(\Omega_0)}(0) \\
&+\langle \nabla v, \rho * (\rho * (B\theta_0 \nabla v))\rangle_{L^2(\Omega_0)}(t_0) - \langle \nabla v, \rho * (\rho * (B\theta_0 \nabla v))\rangle_{L^2(\Omega_0)}(0) \\
&= \int_{\mathbb{R}} \Big[\frac{1}{2}\langle \dot{B}\rho * (\theta_0 \nabla v), \rho * (\theta_0 \nabla v)\rangle_{L^2(\Omega_0)} + \langle B\rho * (\theta_0 \nabla v) - \rho * (B\theta_0 \nabla v), \dot{\rho} * (\theta_0 \nabla v)\rangle_{L^2(\Omega_0)} \Big]\, ds \\
&+ \int_{\mathbb{R}} \big[-\langle \rho * (\theta_0 a \cdot \nabla v), \rho * (\theta_0 \dot{v})\rangle_{L^2(\Omega_0)} - 2\langle \rho * (b\theta_0 \dot{v}), \rho * (\theta_0 \nabla \dot{v})\rangle_{L^2(\Omega_0)} \big]\, ds \\
&+ \int_{\mathbb{R}} \big[-2\langle \rho * (\theta_0 \dot{v} \operatorname{div} b), \rho * (\theta_0 \dot{v})\rangle_{L^2(\Omega_0)} + \langle \rho * (\theta_0 g), \rho * (\theta_0 \dot{v})\rangle_{L^2(\Omega_0)} \big]\, ds\,.
\end{aligned} \tag{2.59}
$$

Let us now perform the second passage to the limit: we let the index m associated to the convolution ρ_m tend to $+\infty$. Let us study separately the asymptotics of the terms appearing in (2.59). The left-hand side converges to

$$
\frac{1}{2}\|\dot{v}\|^2_{L^2(\Omega_0)}(t_0) - \frac{1}{2}\|\dot{v}\|^2_{L^2(\Omega_0)}(0) + \frac{1}{2}\langle \nabla v, B\nabla v\rangle_{L^2(\Omega_0)}(t_0) - \frac{1}{2}\langle \nabla v, B\nabla v\rangle_{L^2(\Omega_0)}(0)\,. \tag{2.60}
$$

Here we have used the weak continuity of $\dot{v}$ and ∇v (see Remark 2.7) and the fact that $\rho * \rho$ is still a smooth even mollifier with integral $1/2$. By the strong approximation property of the convolution and by the dominated convergence theorem, it is easy to check that in the right-hand side of (2.59) we have

$$\begin{aligned}
&\lim_{m\to+\infty}\int_{\mathbb{R}}\langle \dot B\rho*(\theta_0\nabla v),\rho*(\theta_0\nabla v)\rangle_{L^2(\Omega_0)}\,ds=\int_0^{t_0}\langle \dot B\nabla v,\nabla v\rangle_{L^2(\Omega_0)}\,ds\,,\\
&\lim_{m\to+\infty}\int_{\mathbb{R}}\langle \rho*(\theta_0 a\cdot\nabla v),\rho*(\theta_0\dot v)\rangle_{L^2(\Omega_0)}\,ds=\int_0^{t_0}\langle a\cdot\nabla v,\dot v\rangle_{L^2(\Omega_0)}\,ds\,,\\
&\lim_{m\to+\infty}\int_{\mathbb{R}}\langle \rho*(\theta_0\dot v\operatorname{div} b),\rho*(\theta_0\dot v)\rangle_{L^2(\Omega_0)}\,ds=\int_0^{t_0}\langle \operatorname{div} b,|\dot v|^2\rangle_{L^1(\Omega_0)}\,ds\,,\\
&\lim_{m\to+\infty}\int_{\mathbb{R}}\langle \rho*(\theta_0 g),\rho*(\theta_0\dot v)\rangle_{L^2(\Omega_0)}\,ds=\int_0^{t_0}\langle g,\dot v\rangle_{L^2(\Omega_0)}\,ds\,.
\end{aligned}\tag{2.61}$$

For the remaining two terms of (2.59) we claim that

$$\lim_{m\to+\infty}\int_{\mathbb{R}}\langle \rho*(b\theta_0\dot v),\rho*(\theta_0\nabla\dot v)\rangle_{L^2(\Omega_0)}\,ds=-\frac12\int_0^{t_0}\langle \operatorname{div} b,|\dot v|^2\rangle_{L^1(\Omega_0)}\,ds\,,\tag{2.62}$$

$$\lim_{m\to+\infty}\int_{\mathbb{R}}\langle B\rho*(\theta_0\nabla v)-\rho*(B\theta_0\nabla v),\dot\rho*(\theta_0\nabla v)\rangle_{L^2(\Omega_0)}\,ds=0\,.\tag{2.63}$$

Once proved the claim we are done: indeed, the results obtained in (2.60)-(2.63) combined with the equality (2.59) imply the statement (2.50).

Set for brevity

$$\zeta:=\rho*(b\theta_0\dot v)-b\,\rho*(\theta_0\dot v)\,,\quad \eta:=\rho*(\theta_0\dot v)\,.$$

Thus

$$\int_{\mathbb{R}}\langle \rho*(b\theta_0\dot v),\rho*(\theta_0\nabla\dot v)\rangle_{L^2(\Omega_0)}\,ds=\int_{\mathbb{R}}[\langle b\,\eta,\nabla\eta\rangle_{L^2(\Omega_0)}+\langle\zeta,\nabla\eta\rangle_{L^2(\Omega_0)}]\,ds\,.\tag{2.64}$$

Integrating by parts (recall that by (H8) and (1.2) b satisfies $b\cdot\nu=0$ on the boundary of Ω_0), it is easy to see that the first term in the right-hand side of (2.64) gives the desired limit in (2.62). Therefore it is enough to show that the second term vanishes as $m\to+\infty$. Exploiting the equality

$$\nabla\eta(s)=(\rho*(\theta_0\nabla\dot v))(s)=(\dot\rho*(\theta_0\nabla v))(s)+\rho(s-t_0)\nabla v(t_0)-\rho(s)\nabla v(0)$$

we may rewrite

$$\int_{\mathbb{R}}\langle\zeta,\nabla\eta\rangle_{L^2(\Omega_0)}\,ds=-\int_{\mathbb{R}}\langle\dot\zeta,\rho*(\theta_0\nabla v)\rangle_{L^2(\Omega_0)}+\int_{\mathbb{R}}\langle\zeta,\rho(s-t_0)\nabla v(t_0)-\rho(s)\nabla v(0)\rangle_{L^2(\Omega_0)}\,ds\,.$$

Note that, since ρ and θ_0 have compact support, for m big enough ζ and $\dot\zeta$ are identically zero out of the interval $I:=(-2T,2T)$. As $m\to+\infty$, it is easy to check that $\zeta\to0$ strongly in $L^2(I;L^2(\Omega_0;\mathbb{R}^d))$. Therefore it is enough to show that also the derivative $\dot\zeta$ converges (weakly) to zero in $L^2(I;L^2(\Omega_0;\mathbb{R}^d))$. Notice that by (H11) and (H12) we know that $b\in\mathrm{Lip}(\overline I;L^2(\Omega;\mathbb{R}^d))\subset\mathrm{Lip}(\overline I;L^2(\Omega;\mathbb{R}^d))$, so that $\dot b\in L^\infty(I;L^2(\Omega;\mathbb{R}^d))$ and there exists $L>0$ such that $\|\dot b(s,\cdot)\|_{L^\infty(\Omega)}\le L$ for a.e. $s\in I$. Therefore, for a.e. $t\in I$, we may write

$$\begin{aligned}
\dot\zeta(t)&=\Big(\dot\rho*(b\theta_0\dot v)-\dot b\,\rho*(\theta_0\dot v)-b\,\dot\rho*(\theta_0\dot v)\Big)(t)\\
&=\int_{\mathbb{R}}[\dot\rho(t-s)b(s)\theta_0(s)\dot v(s)-\dot b(t)\rho(t-s)\theta_0(s)\dot v(s)-b(t)\dot\rho(t-s)\theta_0(s)\dot v(s)]\,ds\\
&=\int_0^{t_0}\dot v(s)\dot\rho(t-s)\frac{[b(s)-b(t)]}{(s-t)}\,(s-t)\,ds-\int_0^{t_0}\dot b(t)\rho(t-s)\dot v(s)\,ds\,.
\end{aligned}$$

By the L^∞ boundedness of $\|\dot v\|_{L^2(\Omega_0)}$, the aforementioned properties of b, and the bounds

$$\int_{\mathbb{R}}|t\dot\rho(t)|\,dt\,,\ \int_{\mathbb{R}}\rho(t)\,dt<+\infty\,,$$

we deduce that $\dot\zeta$ is uniformly bounded in $L^2(I;L^2(\Omega_0;\mathbb{R}^d))$. In order to verify that the weak limit (which exists up to subsequences) is zero, let us study $\dot\zeta$ as an element of $L^2(I;H^{-1}_D(\Omega_0;\mathbb{R}^d))$, which, after an integration by parts in time, can be written as

$$\begin{aligned}\dot\zeta(t) &= -\rho(t-t_0)b(t_0)\dot v(t_0)+\rho(t)b(0)\dot v(0)+\rho(t-t_0)b(t)\dot v(t_0)-\rho(t)b(t)\dot v(0)\\ &\quad+\int_0^{t_0}\dot v(s)\rho(t-s)(\dot b(s)-\dot b(t))\,ds+\int_0^{t_0}\ddot v(s)\rho(t-s)(b(s)-b(t))\,ds\,.\end{aligned}\tag{2.65}$$

We want to show that $\dot\zeta$ converges strongly to 0 in $L^2(I;H^{-1}_D(\Omega_0;\mathbb{R}^d))$, as $m\to+\infty$. The only difficult term is the first integral on the right-hand side of the previous formula. Approximating $\dot b$ by convolution with respect to time we obtain a sequence $\beta_k\in C^0(\overline I;L^2(\Omega;\mathbb{R}^d))$ such that $\lim_k\beta_k(s)=\dot b(s)$ in $L^2(\Omega;\mathbb{R}^d)$ for a.e. $s\in I$, and such that $\|\beta_k(s,\cdot)\|_{L^\infty(\Omega)}\le L$ for every $s\in\overline I$. For a.e. $t\in I$ we write

$$\begin{aligned}\int_0^{t_0}\dot v(s)\rho(t-s)(\dot b(s)-\dot b(t))\,ds&=\int_0^{t_0}\dot v(s)\rho(t-s)(\dot b(s)-\beta_k(s))\,ds+\int_0^{t_0}\dot v(s)\rho(t-s)(\beta_k(s)-\beta_k(t))\,ds\\ &\quad+\int_0^{t_0}\dot v(s)\rho(t-s)(\beta_k(t)-\dot b(t))\,ds=:F_k(t)+G_k(t)+H_k(t)\,.\end{aligned}\tag{2.66}$$

By Fubini theorem, we obtain

$$\int_I\|F_k(t)\|^2_{L^2(\Omega)}\,dt\le 4T(\max\rho^2)\int_0^{t_0}\|\dot v(s)(\dot b(s)-\beta_k(s))\|^2_{L^2(\Omega)}\,ds\,.\tag{2.67}$$

Since $\dot v\in L^\infty(I;L^2(\Omega))$ and $\|\dot b(s)-\beta_k(s)\|_{L^\infty(\Omega)}\le 2L$, we have the inequality $\|\dot v(s)(\dot b(s)-\beta_k(s))\|^2_{L^2(\Omega)}\le 4L^2\|\dot v\|^2_{L^\infty(I;L^2(\Omega))}$. On the other hand, for a.e. $s\in I$ we have $\lim_k\dot v(s)(\dot b(s)-\beta_k(s))=0$ in $L^2(\Omega;\mathbb{R}^d)$ by the dominated convergence theorem in Ω, since $\dot b(s)-\beta_k(s)\to0$ in $L^2(\Omega;\mathbb{R}^d)$ as $k\to+\infty$ and is bounded in $L^\infty(\Omega;\mathbb{R}^d)$. Therefore, by the dominated convergence theorem in I, for m fixed the right-hand side of (2.67) tends to zero, consequently

$$\int_I\|F_k(t)\|^2_{L^2(\Omega)}\,dt\to0\quad\text{as }k\to+\infty\,.\tag{2.68}$$

To prove that

$$\int_I\|H_k(t)\|^2_{L^2(\Omega)}\,dt\to0\quad\text{as }k\to+\infty\tag{2.69}$$

it is enough to observe that $t\mapsto\int_0^{t_0}\dot v(s)\rho(t-s)\,ds$ is in $L^\infty(I;L^2(\Omega))$, while, for a.e. t, $\beta_k(t)\to\dot b(t)$ in $L^2(\Omega;\mathbb{R}^d)$ as $k\to+\infty$ and is bounded in $L^\infty(\Omega;\mathbb{R}^d)$.

By (2.68) and (2.69), for every $\varepsilon>0$ we can fix $k\in\mathbb{N}$ such that

$$\int_0^T\|F_k(t)\|^2_{L^2(\Omega)}\,dt+\int_0^T\|H_k(t)\|^2_{L^2(\Omega)}\,dt<\varepsilon\,.\tag{2.70}$$

In view of the continuity of β_k, we can determine m_0 such that for $\rho=\rho_m$ with $m\ge m_0$ we have

$$\Big\|\int_0^{t_0}\dot v(s)\rho(t-s)(\beta_k(s)-\beta_k(t))\,ds\Big\|_{L^2(\Omega)}<\varepsilon\,.\tag{2.71}$$

The inequalities (2.70) and (2.71) imply that the left-hand side of (2.66) tends to zero in $L^2(I;L^2(\Omega;\mathbb{R}^d))$, in the limit as $m\to+\infty$. Since the sum of the other terms in the right-hand side of (2.65) converges strongly to 0 in $L^2(I;H^{-1}_D(\Omega_0;\mathbb{R}^d))$, we conclude that $\dot\zeta$ converges strongly to 0 in $L^2(I;H^{-1}_D(\Omega_0;\mathbb{R}^d))$ as $m\to+\infty$.

For brevity, we omit the proof of claim (2.63): exploiting the Lipschitz regularity of B, the result follows in a similar way as already done for the claim (2.62).

This concludes the proof of the representation formula (2.50), and gives the desired continuity of $\mathcal{E}_B(v,\cdot)$ in $[0,T]$. □

We are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2. The proof consists in showing that generalized solutions of (1.25) (cf. Definition 2.5) are indeed weak solutions (cf. Definition 1.5). In view of Theorems 2.6 and 2.10, we know that problem (1.25) admits a unique generalized solution v. In order to prove that v is a weak solution, we need to check that it is more regular, more precisely that it satisfies (1.34)-(1.37).

Let us first consider the case in which w, and hence z, is zero. As pointed out in Remark 2.7, v belongs to $C^0([0,T];L^2(\Omega_0))\cap C_w([0,T];H^1_D(\Omega_0))$ and its derivative $\dot v$ belongs to $C_w([0,T];L^2(\Omega_0))$; in addition, thanks to Proposition 2.11, ∇v and $\dot v$ are (strongly) continuous from $[0,T]$ to $L^2(\Omega_0;\mathbb{R}^d)$ and to $L^2(\Omega_0)$, respectively. Therefore the properties (1.34)-(1.36) are readily verified. Eventually, since both $\dot v$ and $\ddot v$ are elements of $L^2((0,T);H^{-1}_D(\Omega_0))$, we infer that $\dot v\in W^{1,2}((0,T),H^{-1}_D(\Omega_0))$ which is contained into $AC([0,T];H^{-1}_D(\Omega_0))$, thanks to the reflexivity of $H^{-1}_D(\Omega_0)$. This gives (1.37) and concludes the proof. The general case, when $z\neq 0$, can be deduced by difference, exploiting the regularity of $v-z$ that we have just proved and the regularity (2.7) of z. □

3. Continuous dependence on the data

In the next theorem we exploit the energy equality (2.50) to derive the continuous dependence on the data, both for problem (1.6)-(1.9) and problem (1.25)-(1.28).

The manifold Γ introduced in (H3), the initial crack Γ_0, and the Dirichlet boundary datum w_D are kept fixed. We consider a sequence Γ^n_t of families of closed subsets of Γ, with $\Gamma^n_s\subset\Gamma^n_t$ for every $s\le t$, as well as a sequence f^n of source terms and a sequence $(u^{0,n},u^{1,n})$ of initial data. The convergence of the corresponding solutions will be obtained under the assumptions detailed in the following theorem.

Theorem 3.1. *Let Φ, $\Psi\colon[0,T]\times\overline{\Omega}\to\overline{\Omega}$ be two functions satisfying (H7)-(H12) and (2.1). Let $f\in L^2((0,T);L^2(\Omega))$, $u^0\in H^1_D(\Omega_0)+w(0)$, and $u^1\in L^2(\Omega_0)$. For every $n\in\mathbb{N}$, assume that there exist two functions Φ^n, $\Psi^n\colon[0,T]\times\overline{\Omega}\to\overline{\Omega}$ satisfying (H7)-(H12) and (2.1) with Γ_t replaced by Γ^n_t, and let $f^n\in L^2((0,T);L^2(\Omega))$, $u^{0,n}\in H^1_D(\Omega_0)+w(0)$, and $u^{1,n}\in L^2(\Omega_0)$. For every $n\in\mathbb{N}$, let u^n be the weak solution of problem (1.6) with growing crack Γ^n_t, forcing term f^n, initial position $u^{0,n}$, initial velocity $u^{1,n}$, and Dirichlet-Neumann boundary conditions as in (1.8) and (1.9) with Γ_t replaced by Γ^n_t. Similarly, let v^n be the weak solution of (1.25)-(1.28), where the coefficients (1.29)-(1.33) are constructed starting from $\Phi^n,\Psi^n,f^n,u^{0,n}$, and $u^{1,n}$. Assume that there exist two constants $C>0$ and $\delta_0>0$ such that the following inequalities hold for every $n\in\mathbb{N}$:*

$$\det D\Phi^n(t,\cdot)\ge\delta_0\quad\text{for every } t\in[0,T]\,,\tag{3.1}$$

$$\|\Phi^n(t,\cdot)-\Phi^n(s,\cdot)\|_{L^\infty(\Omega)},\ \|\partial_i\Phi^n(t,\cdot)-\partial_i\Phi^n(s,\cdot)\|_{L^\infty(\Omega)}\le C|t-s|\quad\text{for every } t,s\in[0,T]\,,\tag{3.2}$$

$$\|\dot\Phi^n(t,\cdot)-\dot\Phi^n(s,\cdot)\|_{L^\infty(\Omega)}\le C|t-s|\quad\text{for every } t,s\in[0,T]\,,\tag{3.3}$$

$$\|\partial^2_{ij}\Phi^n(t,\cdot)\|_{L^\infty(\Omega)}\le C\quad\text{for every } t\in[0,T]\,.\tag{3.4}$$

Furthermore, assume that the following properties hold as $n\to+\infty$:

$$\dot\Phi^n(t)\to\dot\Phi(t)\quad\text{strongly in } L^2(\Omega;\mathbb{R}^d)\,,\ \text{for a.e. } t\in(0,T)\,,\tag{3.5}$$

$$\partial_i\dot\Phi^n(t)\to\partial_i\dot\Phi(t)\quad\text{strongly in } L^2(\Omega;\mathbb{R}^d)\,,\ \text{for a.e. } t\in(0,T)\,,\tag{3.6}$$

$$\partial^2_{ij}\Phi^n(t)\to\partial^2_{ij}\Phi(t)\quad\text{strongly in } L^2(\Omega;\mathbb{R}^d)\,,\ \text{for a.e. } t\in(0,T)\,,\tag{3.7}$$

$$\ddot\Phi^n(t)\to\ddot\Phi(t)\quad\text{strongly in } L^2(\Omega;\mathbb{R}^d)\,,\ \text{for a.e. } t\in(0,T)\,,\tag{3.8}$$

$$f^n\to f\quad\text{strongly in } L^2((0,T);L^2(\Omega)),\tag{3.9}$$

$$u^{0,n}\to u^0\quad\text{strongly in } H^1(\Omega_0)\,,\quad u^{1,n}\to u^1\quad\text{strongly in } L^2(\Omega_0)\,.\tag{3.10}$$

Finally, assume that (3.1)-(3.8) are valid also for the sequence Ψ^n with limit Ψ. Under these assumptions, for every $t\in[0,T]$ we have:

$$u^n(t)\to u(t)\text{ and }\dot u^n(t)\to\dot u(t)\text{ strongly in } L^2(\Omega),\ \widehat\nabla u^n(t)\to\widehat\nabla u(t)\text{ strongly in } L^2(\Omega;\mathbb{R}^d)\,,\tag{3.11}$$

$$v^n(t)\to v(t)\text{ strongly in } H^1(\Omega_0)\,,\ \text{and }\dot v^n(t)\to\dot v(t)\text{ strongly in } L^2(\Omega_0)\,,\tag{3.12}$$

where u and v are the weak solutions of problems (1.6)-(1.9) and (1.25)-(1.28), associated to the limit diffeomorphisms Φ and Ψ.

Remark 3.2. Notice that for every $n\in\mathbb{N}$, $\Phi^n(0,\cdot)=id$, therefore, assumptions (3.5) and (3.6) imply that

$$\Phi^n(t)\to\Phi(t)\,,\ \partial_i\Phi^n(t)\to\partial_i\Phi(t)\text{ strongly in } L^2(\Omega;\mathbb{R}^d)\tag{3.13}$$

for every $t\in[0,T]$, $i\in\{1,\dots,d\}$. Moreover, in view of (3.8) the convergence (3.5) is valid for every time.

Before proving the theorem, we go back to the two-dimensional Example 1.14 and we present a possible sequence of diffeomorphisms Φ^n and Ψ^n satisfying (3.1)-(3.8).

Example 3.3. Let $d = 2$. We consider a $C^{2,1}$ simple curve Γ in the planar domain Ω, injectively parametrized by arc-length through a function $\gamma\colon [0,\ell] \to \mathbb{R}^2$, with $\ell := |\Gamma|$. We assume that $\gamma(0), \gamma(\ell) \in \partial\Omega$, $\gamma(s) \in \Omega$ for every $s \in (0,\ell)$, and $\Gamma_0 = \gamma([0,s_0])$ with $s_0 \in (0,\ell)$. Let $M > 0$ and $0 < \delta < c_A/2$ be fixed, and let $\mathcal{S}_{M,\delta}$ be the class of functions with the following properties:

$$s \in C^{1,1}([0,T])\,,\ s(0) = s_0\,,\ 0 \le \dot s(t) \le (c_A - 2\delta)^{1/2} \ \text{ for every } t \in [0,T]\,,$$

$$\|\ddot s\|_\infty \le M\,, \quad \sup_{\substack{\varphi \in C^1_c((0,T))\\ \|\varphi\|_\infty \le 1}} \int_0^T \ddot s(t)\dot\varphi(t)\,dt \le M\,.$$

Let $s^n : [0,T] \to \mathbb{R}$ be a sequence of functions in $\mathcal{S}_{M,\delta}$. In view of Examples 1.14 and 2.1, for every $n \in \mathbb{N}$ the functions $\Phi^n(t,y) := \widehat\Phi(s^n(t),y)$ and $\Psi^n(t,y) := \widehat\Psi(s^n(t),y)$ defined according to (1.67) satisfy (H7)-(H12) and (2.3). Moreover, it is easy to verify that the sequences Φ^n and Ψ^n satisfy the uniform bounds (3.1)-(3.4). The set $\mathcal{S}_{M,\delta}$ is compact: indeed, by the Ascoli-Arzelà theorem and by the compact embedding of BV into L^1 (see e.g. [1, Theorem 3.23]), we infer that there exists $s \in \mathcal{S}_{M,\delta}$ such that, up to a subsequence n_k, as $k \to +\infty$ we have

$$s^{n_k}(t) \to s(t) \ \text{ and } \ \dot s^{n_k}(t) \to \dot s(t) \ \text{ for every } t \in [0,T]\,, \quad \ddot s^{n_k}(t) \to \ddot s(t) \ \text{ for a.e. } t \in (0,T)\,.$$

In particular, as $k \to +\infty$ we conclude that the sequences $\Phi^{n_k}(t,\cdot)$ and $\Psi^{n_k}(t,\cdot)$ satisfy (3.5)-(3.8) with $\Phi(t,\cdot) := \widehat\Phi(s(t),\cdot)$ and $\Psi(t,\cdot) := \widehat\Psi(s(t),\cdot)$.

We conclude the section with the proof of Theorem 3.1.

Proof of Theorem 3.1. The proof is divided into several steps: in the first step we prove that the statement for the sequence u^n follows from the statement for v^n; in Step 2 we show that we can restrict ourselves to the case of homogeneous Dirichlet boundary conditions; finally, in the subsequent steps, we prove the strong convergence of v^n towards v under the assumption $w_D = 0$.

Step 1. (3.12) *implies* (3.11). Fix $t \in [0,T]$ and assume that (3.12) is satisfied. We claim that as $n \to +\infty$

$$\widehat\nabla u^n(t) \rightharpoonup \widehat\nabla u(t) \ \text{ weakly in } L^2(\Omega;\mathbb{R}^d)\,, \quad u^n(t) \rightharpoonup u(t) \ \text{ and } \ \dot u^n(t) \rightharpoonup \dot u(t) \ \text{ weakly in } L^2(\Omega)\,, \tag{3.14}$$

$$\|\widehat\nabla u^n(t)\|_{L^2(\Omega)} \to \|\widehat\nabla u(t)\|_{L^2(\Omega)}\,, \quad \|u^n(t)\|_{L^2(\Omega)} \to \|u(t)\|_{L^2(\Omega)}\,, \quad \|\dot u^n(t)\|_{L^2(\Omega)} \to \|\dot u(t)\|_{L^2(\Omega)}\,. \tag{3.15}$$

Let Ω^n_t denote the set $\Omega \setminus \Gamma^n_t$. In view of the energy bound (2.36) valid for weak solutions and the bounds (3.1)-(3.4) on the coefficients, we infer that $\widehat\nabla u^n(t)$, $u^n(t)$, and $\dot u^n(t)$ are bounded in $L^2(\Omega;\mathbb{R}^d)$, $L^2(\Omega)$, and $L^2(\Omega)$, uniformly with respect to n and t. In particular, up to subsequences, they converge weakly in these spaces. To determine the weak limits, fix a smooth test function $\varphi \in C^\infty_c(\Omega\setminus\Gamma;\mathbb{R}^d)$. By the change of variable formula (1.65) we obtain that

$$\begin{aligned}
\langle \widehat\nabla u^n(t), \varphi\rangle_{L^2(\Omega)} &= \langle (D\Psi^n(t,\cdot))^T \widehat\nabla v^n(t,\Psi^n(t,\cdot)), \varphi\rangle_{L^2(\Omega)}\\
&= \langle (D\Psi^n(t,\Phi^n(t,\cdot)))^T \widehat\nabla v^n(t,\cdot), \varphi(\Phi^n(t,\cdot)) \det D\Phi^n(t,\cdot)\rangle_{L^2(\Omega)}\\
&\to \langle (D\Psi(t,\Phi(t,\cdot)))^T \widehat\nabla v(t,\cdot), \varphi(\Phi(t,\cdot)) \det D\Phi(t,\cdot)\rangle_{L^2(\Omega)} = \langle \widehat\nabla u(t), \varphi\rangle_{L^2(\Omega))}
\end{aligned}$$

as $n \to +\infty$. For the convergence we have used the assumptions on the diffeomorphisms Ψ^n and the strong convergence in $H^1(\Omega_0)$ of v^n towards v (see (3.12)). The same argument applies to $u^n(t)$ and $\dot u^n(t)$, which converge weakly in $L^2(\Omega)$ to $u(t)$ and $\dot u(t)$, respectively. Therefore (3.14) is proved. With the same techniques we derive (3.15): exploiting the strong convergences (3.12) and (3.13), we get

$$\begin{aligned}
\|\widehat\nabla u^n(t)\|^2_{L^2(\Omega)} &= \|(D\Psi^n(t,\cdot))^T \widehat\nabla v^n(t,\Psi^n(t,\cdot))\|^2_{L^2(\Omega)}\\
&= \int_\Omega |(D\Psi^n(t,\Phi^n(t,\cdot)))^T \widehat\nabla v^n(t,\cdot)|^2 \det D\Phi^n(t,\cdot)\,dy\\
&\to \int_\Omega |(D\Psi(t,\Phi(t,\cdot)))^T \widehat\nabla v(t,\cdot)|^2 \det D\Phi(t,\cdot)\,dy = \|\widehat\nabla u(t)\|^2_{L^2(\Omega)}\,.
\end{aligned}$$

Similarly we prove that $\|u^n(t)\|^2_{L^2(\Omega)}$ converges to $\|u(t)\|^2_{L^2(\Omega)}$ and $\|\dot u^n(t)\|^2_{L^2(\Omega)}$ to $\|\dot u(t)\|^2_{L^2(\Omega)}$. By combining (3.14) and (3.15) we get (3.11), observing that the limit does not depend on the subsequences.

Step 2. Preliminaries about (3.12). In order to deal with the sequence of problems (1.25)-(1.28) constructed starting from Φ^n, Ψ^n, f^n, $u^{0,n}$, and $u^{1,n}$, it is worth recasting assumptions (3.2)-(3.10) in terms of the corresponding coefficients (1.29)-(1.33), which will be denoted by B^n, a^n, b^n, g^n, $v^{0,n}$, and $v^{1,n}$. It is easy to check that for every $n \in \mathbb{N}$ we have

$$\|B^n(t,\cdot)\|_{L^\infty(\Omega_0)}, \|b^n(t,\cdot)\|_{L^\infty(\Omega_0)}, \|\operatorname{div} b^n(t,\cdot)\|_{L^\infty(\Omega_0)}, \|a^n(t,\cdot)\|_{L^\infty(\Omega_0)} \le C \quad \text{for a.e. } t \in (0,T)\,, \tag{3.16}$$

$$\|B^n(t,\cdot) - B^n(s,\cdot)\|_{L^\infty(\Omega_0)}, \|b^n(t,\cdot) - b^n(s,\cdot)\|_{L^\infty(\Omega_0)} \le C|t-s| \quad \text{for every } t,s \in [0,T]\,, \tag{3.17}$$

for some constant $C > 0$ independent of t and s. Moreover, in view of (3.1) and (2.1), we infer that the ellipticity constant of B^n is bounded from below by a positive constant independent of n, t, and y. Furthermore, exploiting Lemma 4.7, in the limit as $n \to +\infty$ we have

$$B^n(t) \to B(t) \text{ and } \dot B^n(t) \to \dot B(t) \text{ strongly in } L^2(\Omega_0;\mathbb{R}^{d\times d})\,, \text{ for a.e. } t \in (0,T)\,, \tag{3.18}$$

$$b^n(t) \to b(t) \text{ and } a^n(t) \to a(t) \text{ strongly in } L^2(\Omega_0;\mathbb{R}^{d})\,, \text{ for a.e. } t \in (0,T)\,, \tag{3.19}$$

$$\operatorname{div} b^n(t) \to \operatorname{div} b(t) \text{ strongly in } L^2(\Omega_0)\,, \text{ for a.e. } t \in (0,T)\,, \tag{3.20}$$

$$g^n \to g \text{ strongly in } L^2((0,T);L^2(\Omega_0))\,, \tag{3.21}$$

$$v^{0,n} \to v^0 \text{ strongly in } H^1(\Omega_0)\,,\ v^{1,n} \to v^1 \text{ strongly in } L^2(\Omega_0)\,. \tag{3.22}$$

Let us now set $z^n(t,y) := w(t,\Phi^n(t,y))$. By construction, the functions z^n satisfy Lemma 2.4. As already noticed in the proof of Theorem 2.6, the difference $v^n - z^n$ is the weak solution of problem (1.25)-(1.28) with coefficients B^n, a^n, b^n, initial data

$$\hat v^{0,n} := v^{0,n} - z^n(0) \in H^1_D(\Omega_0)\,, \quad \hat v^{1,n} := v^{1,n} - \dot z^n(0) \in L^2(\Omega_0)\,,$$

homogeneous Dirichlet-Neumann boundary conditions, and source term $\hat g^n := g^n - h^n$, where

$$h^n := \ddot z^n - \operatorname{div}(B^n \nabla z^n) + a^n \cdot \nabla z^n - 2b^n \cdot \nabla \dot z^n \in L^2((0,T);L^2(\Omega_0))\,.$$

By using Lemma 4.7, it is easy to check that as $n \to +\infty$

$$z^n(t) - z(t) \to 0 \text{ strongly in } H^1_D(\Omega_0)\,,\ \dot z^n(t) - \dot z(t) \to 0 \text{ strongly in } L^2(\Omega_0)\,,$$
$$h^n \to h \text{ strongly in } L^2((0,T);L^2(\Omega_0))\,,$$

for every $t \in [0,T]$, where z and h are defined according to (2.6) and (2.12). Note that for the convergence of h^n we have used the regularity (2.7) and the computations done in the proof of Lemma 2.4. Similarly, in view of (3.9) and (3.10), we derive

$$\hat g^n \to g - h \text{ strongly in } L^2((0,T);L^2(\Omega_0))\,,$$
$$\hat v^{0,n} \to v^0 - z(0) \text{ strongly in } H^1(\Omega_0)\,,\ v^{1,n} \to v^1 - \dot z(0) \text{ strongly in } L^2(\Omega_0)\,.$$

Therefore, (3.12) is ensured once we prove the strong convergence of v^n and $\dot v^n$ in the case of homogeneous Dirichlet-Neumann boundary conditions, namely when $w = 0$. This is done in the following steps.

Step 3. The perturbed problems. We assume $w = 0$. For every $\varepsilon > 0$, let v_ε be the solution of the perturbed problem (2.13) and let v^n_ε be the solution of the corresponding problem with coefficients B^n, a^n, b^n, g^n, $v^{0,n}$, $v^{1,n}$. We already know that, as $\varepsilon \to 0$, v_ε weakly converges to v (see (2.31)-(2.33) in the proof of Theorem 2.6). Here we claim that the convergence is strong, that is, for every $t \in [0,T]$

$$v_\varepsilon(t) \to v(t) \quad \text{strongly in } H^1_D(\Omega_0) \quad \text{and} \quad \dot v_\varepsilon(t) \to \dot v(t) \quad \text{strongly in } L^2(\Omega_0)\,, \quad \text{as } \varepsilon \to 0\,. \tag{3.23}$$

Moreover, we claim that there exists a sequence of parameters $\varepsilon_n > 0$, converging to 0 as $n \to +\infty$, such that for every $t \in [0,T]$

$$v^n_{\varepsilon_n}(t) - v_{\varepsilon_n}(t) \to 0 \quad \text{strongly in } H^1_D(\Omega_0)\,, \quad \dot v^n_{\varepsilon_n}(t) - \dot v_{\varepsilon_n}(t) \to 0 \quad \text{strongly in } L^2(\Omega_0)\,, \tag{3.24}$$

$$v^n_{\varepsilon_n}(t) - v^n(t) \to 0 \quad \text{strongly in } H^1_D(\Omega_0)\,, \quad \dot v^n_{\varepsilon_n}(t) - \dot v^n(t) \to 0 \quad \text{strongly in } L^2(\Omega_0)\,, \tag{3.25}$$

as $n \to +\infty$. Once we prove the claims we are done. Indeed, by the triangle inequality we have

$$\limsup_{n\to+\infty} \|v^n(t) - v(t)\|_{H^1_D(\Omega_0)}$$

$$\le \limsup_{n\to+\infty}(\|v^n(t)-v^n_{\varepsilon_n}(t)\|_{H^1_D(\Omega_0)}+\|v^n_{\varepsilon_n}(t)-v_{\varepsilon_n}(t)\|_{H^1_D(\Omega_0)}+\|v_{\varepsilon_n}(t)-v(t)\|_{H^1_D(\Omega_0)})=0$$

and the same holds true for $\|\dot v^n(t)-\dot v(t)\|_{L^2(\Omega_0)}$.

The claims (3.23)-(3.25) will be proved in Steps 4, 5, and 7, respectively.

Step 4. Strong convergence of v_ε. Assume $w=0$ and define $X^\varepsilon := v_\varepsilon - v$. By comparing the energy equalities (2.28) and (2.50) for v_ε and v, respectively, it is easy to see that

$$\begin{aligned}&\mathcal{E}_B(X^\varepsilon,t)+\varepsilon\int_0^t\|\dot v_\varepsilon(s)\|^2_{H^1_D(\Omega_0)}\,ds\\&=\int_0^t\Big[\frac12\langle\dot B\nabla X^\varepsilon,\nabla X^\varepsilon\rangle_{L^2(\Omega_0)}-\langle a\cdot\nabla X^\varepsilon,\dot X^\varepsilon\rangle_{L^2(\Omega_0)}-\langle \operatorname{div} b,|\dot X^\varepsilon|^2\rangle_{L^1(\Omega)}\Big]\,ds+R_\varepsilon(t)\,,\end{aligned}\tag{3.26}$$

where $\mathcal{E}_B$ is defined according to (2.49) and

$$\begin{aligned}R_\varepsilon(t):=&-\langle\dot v_\varepsilon(t),\dot v(t)\rangle_{L^2(\Omega_0)}-\langle B(t)\nabla v_\varepsilon(t),\nabla v(t)\rangle_{L^2(\Omega_0)}+\|v^1\|^2_{L^2(\Omega_0)}+\langle B(0)\nabla v^0,\nabla v^0\rangle_{L^2(\Omega_0)}\\&+\int_0^t\Big[\langle\dot B\nabla v_\varepsilon,\nabla v\rangle_{L^2(\Omega_0)}-\langle a\cdot\nabla v_\varepsilon,\dot v\rangle_{L^2(\Omega_0)}-\langle a\cdot\nabla v,\dot v_\varepsilon\rangle_{L^2(\Omega_0)}-2\langle\operatorname{div} b,\dot v_\varepsilon\dot v\rangle_{L^1(\Omega)}+\langle g,\dot v_\varepsilon+\dot v\rangle_{L^2(\Omega_0)}\Big]\,ds\,.\end{aligned}$$

In view of the weak convergences of $v_\varepsilon(t)$, $\nabla v_\varepsilon(t)$, and $\dot v_\varepsilon(t)$, and the energy equality (2.50), we infer that $R_\varepsilon(t)\to 0$ as $\varepsilon\to 0$. On the other hand, the uniform bounds on $\dot B$, a, and $\operatorname{div} b$, and the uniform ellipticity of B, imply that the integral term in the right-hand side of (3.26) can bounded from above by $C\int_0^t\mathcal{E}_B(X^\varepsilon,s)\,ds$, for a suitable $C>0$ independent of t and ε. Therefore, using (3.26) and Fatou's Lemma, we infer that for every $t\in[0,T]$

$$\limsup_{\varepsilon\to0}\mathcal{E}_B(X^\varepsilon,t)\le\limsup_{\varepsilon\to0}\Big(R_\varepsilon(t)+C\int_0^t\mathcal{E}_B(X^\varepsilon,s)\,ds\Big)\le C\int_0^t\limsup_{\varepsilon\to0}\mathcal{E}_B(X^\varepsilon,s)\,ds\,.$$

Finally, by Gronwall's Lemma, we conclude that $\lim_\varepsilon\mathcal{E}_B(X^\varepsilon,t)$ exists and it equals zero. This proves (3.23).

Step 5. Strong convergence of $v^n_{\varepsilon_n}-v_{\varepsilon_n}$. The functions $v^n_{\varepsilon_n}$ and v_{ε_n} solve two problems with different coefficients but with the same viscosity $\varepsilon_n>0$, whose precise value will be fixed at the end of the step. By linearity, it easy to verify that the difference $X^n:=v^n_{\varepsilon_n}-v_{\varepsilon_n}$ solves

$$\begin{aligned}&\langle\ddot X^n(t),\psi\rangle_{H^1_D(\Omega_0)}+\langle B(t)\nabla X^n(t),\nabla\psi\rangle_{L^2(\Omega_0)}+\langle a(t)\cdot\nabla X^n(t),\psi\rangle_{L^2(\Omega_0)}-2\langle b(t)\cdot\nabla\dot X^n(t),\psi\rangle_{L^2(\Omega_0)}\\&+\varepsilon_n\langle\dot X^n(t),\psi\rangle_{L^2(\Omega_0)}+\varepsilon_n\langle\nabla\dot X^n(t),\nabla\psi\rangle_{L^2(\Omega_0)}=\langle q^n(t),\psi\rangle_{H^1_D(\Omega_0)}\end{aligned}\tag{3.27}$$

for a.e. $t\in(0,T)$ and every $\psi\in H^1_D(\Omega_0)$, with initial data $X^{0,n}=v^{0,n}-v^0$ and $X^{1,n}=v^{1,n}-v^1$. The right-hand side of (3.27) is defined as

$$\begin{aligned}\langle q^n,\psi\rangle_{H^1_D(\Omega_0)}:=&\langle(g^n-g)-(a^n-a)\cdot\nabla v^n_{\varepsilon_n}-2(\operatorname{div} b^n-\operatorname{div} b)\dot v^n_{\varepsilon_n},\psi\rangle_{L^2(\Omega_0)}\\&-\langle(B^n-B)\nabla v^n_{\varepsilon_n}+2(b^n-b)\dot v^n_{\varepsilon_n},\nabla\psi\rangle_{L^2(\Omega_0)}\,.\end{aligned}$$

In particular, the forcing term q^n is an element of $L^2((0,T);H^{-1}_D(\Omega_0))$. Thanks to the uniform bounds (3.16) and (3.17), the energy estimate (2.30) implies that $\dot v^n_{\varepsilon_n}$ and $\nabla v^n_{\varepsilon_n}$ are uniformly bounded in $L^\infty((0,T);L^2(\Omega_0))$ and $L^\infty((0,T);L^2(\Omega_0;\mathbb{R}^d))$, respectively. Note that these bounds do not depend on the sequence ε_n. Therefore, by (3.18)-(3.21) we get

$$q^n\to0\ \text{ in } L^2((0,T);H^{-1}_D(\Omega_0))\quad\text{as } n\to+\infty\,,\tag{3.28}$$

and the rate of this convergence is independent of the choice of ε_n. We now want to write an energy estimate for X^n. Notice that, since we are dealing with the perturbed problems, the time derivatives $\dot X^n$ belong to $L^2((0,T);H^1_D(\Omega_0))$, thus they can be used as test functions in (3.27): integrating by parts we get

$$\begin{aligned}&\mathcal{E}_B(X^n,t)+\varepsilon_n\int_0^t\|\dot X^n(s)\|^2_{H^1_D(\Omega_0)}\,ds=\mathcal{E}_B(X^n,0)\\&+\int_0^t\Big[\frac12\langle\dot B\nabla X^n,\nabla X^n\rangle_{L^2(\Omega_0)}-\langle a\cdot\nabla X^n,\dot X^n\rangle_{L^2(\Omega_0)}-\langle\operatorname{div} b,|\dot X^n|^2\rangle_{L^1(\Omega)}+\langle q^n,\dot X^n\rangle_{H^1_D(\Omega_0)}\Big]\,ds\,.\end{aligned}$$

As in Step 4, the uniform bounds on $\dot{B}$, a, and $\operatorname{div} b$, together with the uniform ellipticity of B, yield

$$\mathcal{E}_B(X^n,t)+\varepsilon_n\int_0^t\|\dot{X}^n(s)\|^2_{H^1_D(\Omega_0)}\,ds\le\mathcal{E}_B(X^n,0)+C\int_0^t\mathcal{E}_B(X^n,s)\,ds+\int_0^t|\langle q^n,\dot{X}^n\rangle_{H^1_D(\Omega_0)}|\,ds\,,$$

for a suitable constant $C>0$ independent of n and t. The presence of ε_n allows us to get rid of the last term: by writing

$$\int_0^t|\langle q^n,\dot{X}^n\rangle_{H^1_D(\Omega_0)}|\,ds\le\frac{1}{2}\|q^n\|_{L^2((0,T);H^{-1}_D(\Omega_0))}+\frac{1}{2}\|q^n\|_{L^2((0,T);H^{-1}_D(\Omega_0))}\int_0^t\|\dot{X}^n\|^2_{H^1_D(\Omega_0)}\,ds$$

and choosing ε_n such that

$$\varepsilon_n-\frac{1}{2}\|q^n\|_{L^2((0,T);H^{-1}_D(\Omega_0))}\ge0\ \forall n\,,\quad\varepsilon_n\to0\,,$$

we conclude that

$$\mathcal{E}_B(X^n,t)\le C_n+C\int_0^t\mathcal{E}_{\mathcal{B}}(X^n,s)\,ds\,,$$

with

$$C_n:=\mathcal{E}_B(X^n,0)+\frac{1}{2}\|q^n\|_{L^2((0,T);H^{-1}_D(\Omega_0))}\,.$$

In view of (3.22) and (3.28) we infer that $C_n\to0$. Therefore, again by Fatou and Granwall's Lemmas, we have $\lim_n\mathcal{E}_B(X^n,t)=0$, concluding the proof of (3.24).

Step 6. Weak convergence of v^n to v. Assume $w=0$. For every $n\in\mathbb{N}$, v^n satisfies

$$\begin{aligned}&\langle\ddot{v}^n(t),\psi\rangle_{H^1_D(\Omega_0)}+\langle B^n(t)\nabla v^n(t),\psi\rangle_{L^2(\Omega_0)}+\langle a^n(t)\cdot\nabla v^n(t),\psi\rangle_{L^2(\Omega_0)}+2\langle\dot{v}^n(t),\operatorname{div}(b^n(t)\psi)\rangle_{L^2(\Omega_0)}\\&=\langle g^n(t),\psi\rangle_{L^2(\Omega_0)}\end{aligned}\tag{3.29}$$

for a.e. $t\in(0,T)$ and for every $\psi\in H^1_D(\Omega_0)$. As already pointed out in (2.35), every weak solution v^n has bounded energy, namely there exists $C>0$ such that, for every $t\in[0,T]$,

$$\|\dot{v}^n(t)\|^2_{L^2(\Omega_0)}+\|v^n(t)\|^2_{H^1_D(\Omega_0)}\le C\,.$$

Thanks to (3.16), (3.17), (3.21), and (3.22), the constant C can be chosen independent of n. In particular, there exists $\xi\in L^2((0,T);H^1_D(\Omega_0))\cap H^1((0,T);L^2(\Omega_0))$ such that, up to a subsequence,

$$v^n\rightharpoonup\xi\ \text{ weakly in } L^2((0,T);H^1_D(\Omega_0))\ \text{ and }\ \dot{v}^n\rightharpoonup\dot{\xi}\ \text{ weakly in } L^2((0,T);L^2(\Omega_0))\,.\tag{3.30}$$

By combining the strong convergences (3.18)-(3.22) with the weak convergences (3.30), passing to the limit as $n\to+\infty$ in (3.29), we infer that $\ddot{\xi}\in L^2((0,T);H^{-1}_D(\Omega_0))$ and that ξ is a generalized solution of the limit problem (2.11), with initial conditions v^0 and v^1. By Theorem 2.10 such solution is unique, therefore $\xi=v$. Since the result does not depend on the subsequence, we conclude that the whole sequence v^n satisfies

$$v^n\rightharpoonup v\ \text{ weakly in } L^2((0,T);H^1_D(\Omega_0))\ \text{ and }\ \dot{v}^n\rightharpoonup\dot{v}\ \text{ weakly in } L^2((0,T);L^2(\Omega_0))\,.$$

Step 7. Strong convergence of $v^n_{\varepsilon_n}-v^n$. Assume $w=0$ and define $X^n:=v^n_{\varepsilon_n}-v^n$, with ε_n as in Step 6. Following the same procedure as in Step 4, we get

$$\begin{aligned}&\mathcal{E}_{B^n}(X^n,t)+\varepsilon_n\int_0^t\|\dot{v}^n_{\varepsilon_n}\|^2_{H^1_D(\Omega_0)}\,ds\\&=\int_0^t\Big[\frac{1}{2}\langle\dot{B}^n\nabla X^n,\nabla X^n\rangle_{L^2(\Omega_0)}-\langle a^n\cdot\nabla X^n,\dot{X}^n\rangle_{L^2(\Omega_0)}-\langle\operatorname{div}b^n,|\dot{X}^n|^2\rangle_{L^1(\Omega)}\Big]\,ds+R_n(t)\,,\end{aligned}$$

with

$$\begin{aligned}R_n(t):=&-\langle\dot{v}^n_{\varepsilon_n}(t),\dot{v}^n(t)\rangle_{L^2(\Omega_0)}-\langle B^n(t)\nabla v^n_{\varepsilon_n},\nabla v^n(t)\rangle_{L^2(\Omega_0)}+\|v^{1,n}\|^2_{L^2(\Omega_0)}+\langle B^n(0)\nabla v^{0,n},\nabla v^{0,n}\rangle_{L^2(\Omega_0)}\\&+\int_0^t\Big[\langle\dot{B}^n\nabla v^n_{\varepsilon_n},\nabla v^n\rangle_{L^2(\Omega_0)}-\langle a^n\cdot\nabla v^n_{\varepsilon_n},\dot{v}^n\rangle_{L^2(\Omega_0)}-\langle a^n\cdot\nabla v^n,\dot{v}^n_{\varepsilon_n}\rangle_{L^2(\Omega_0)}\Big]\,ds\\&+\int_0^t\big[-2\langle\operatorname{div}b^n,\dot{v}^n_{\varepsilon_n}\dot{v}^n\rangle_{L^1(\Omega)}+\langle g^n,\dot{v}^n_{\varepsilon_n}+\dot{v}^n\rangle_{L^2(\Omega_0)}\big]\,ds\,.\end{aligned}\tag{3.31}$$

Exploiting the uniform bounds (3.16) and (3.17), and the fact that each B^n satisfies the ellipticity condition (2.2) with the same constant, as in Step 4 we infer that

$$\mathcal{E}_{B^n}(X^n,t)+\varepsilon_n\int_0^t\|\dot v^n_{\varepsilon_n}\|^2_{H^1_D(\Omega_0)}\,ds\le C\int_0^t\mathcal{E}_{B^n}(X^n,s)\,ds+R_n(t)\,,\tag{3.32}$$

for some $C>0$ independent of n and t.

Let us show that $R_n(t)\to 0$ as $n\to+\infty$. In view of the strong convergences (3.22) and (3.5) (whose validity at every time is discussed in Remark 3.2), we infer that, as $n\to+\infty$,

$$\langle B^n(0)\nabla v^{0,n},\nabla v^{0,n}\rangle_{L^2(\Omega_0)}=\langle (A(0)-\dot\Phi^n(0)\otimes\dot\Phi^n(0))\nabla v^{0,n},\nabla v^{0,n}\rangle_{L^2(\Omega_0)}\to\langle B(0)\nabla v^0,\nabla v^0\rangle_{L^2(\Omega_0)}\,.\tag{3.33}$$

In view of Steps 4 and 5, $v^n_{\varepsilon_n}(t)$ converges strongly to $v(t)$ for every t, while, by Step 6, $v^n(t)$ converges weakly to $v(t)$. Thus

$$\langle \dot v^n_{\varepsilon_n}(t),\dot v^n(t)\rangle_{L^2(\Omega_0)}+\langle B^n(t)\nabla v^n_{\varepsilon_n}(t),\nabla v^n(t)\rangle_{L^2(\Omega_0)}\to\|\dot v(t)\|^2_{L^2(\Omega_0)}+\langle B(t)\nabla v(t),\nabla v(t)\rangle_{L^2(\Omega_0)}\,.\tag{3.34}$$

In view of (3.16)-(3.21), by dominated convergence in Ω and then in $(0,T)$, we can pass to the limit in the integral terms of R_n, and we get

$$\begin{aligned}&\int_0^t\Big[\langle\dot B^n\nabla v^n_{\varepsilon_n},\nabla v^n\rangle_{L^2(\Omega_0)}-\langle a^n\cdot\nabla v^n_{\varepsilon_n},v^n\rangle_{L^2(\Omega_0)}-\langle a^n\cdot\nabla v^n,v^n_{\varepsilon_n}\rangle_{L^2(\Omega_0)}\Big]\,ds\\&+\int_0^t\big[-2\langle\operatorname{div}b^n,\dot v^n_{\varepsilon_n}\dot v^n\rangle_{L^1(\Omega)}+\langle g^n,\dot v^n_{\varepsilon_n}+\dot v^n\rangle_{L^2(\Omega_0)}\big]\,ds\\&\to\int_0^t\Big[\langle\dot B\nabla v,\nabla v\rangle_{L^2(\Omega_0)}-2\langle a\cdot\nabla v,v\rangle_{L^2(\Omega_0)}-2\langle\operatorname{div}b,|\dot v|^2\rangle_{L^1(\Omega)}+2\langle g,\dot v\rangle_{L^2(\Omega_0)}\Big]\,ds\,.\end{aligned}\tag{3.35}$$

By combining (3.31) and (3.33)-(3.35) with the energy equality (2.50), we conclude that $R_n(t)\to 0$ as $n\to+\infty$ for every time. We now apply Fatou and Gronwall's Lemmas to (3.32), as in Step 5, and we obtain that $\lim_n\mathcal{E}_{B^n}(X^n,t)=0$. This convergence gives (3.25), since the ellipticity condition (2.2) for B^n holds with the same constant for every n. □

4. Appendix

For the benefit of the reader, we recall an existence result for evolution problems of second order in time, whose proof can be found in [6]. Let $\mathcal{B}(t;\cdot,\cdot),\mathcal{A}_1(t;\cdot,\cdot),\mathcal{A}_2(t;\cdot,\cdot)$ be three families of continuous bilinear forms over $H^1_D(\Omega_0)\times H^1_D(\Omega_0)$, with t varying in $[0,T]$, satisfying the following properties, where $\dot{\mathcal{B}}(\cdot;\eta,\xi)$ denotes the derivative of $\mathcal{B}(\cdot;\eta,\xi)$:

(i) for every $t\in[0,T]$ the form $\mathcal{B}(t;\cdot,\cdot)$ is symmetric;
(ii) there exists $c_0>0$ such that $\mathcal{B}(t;\eta,\eta)\ge c_0\|\eta\|^2_{H^1_D(\Omega_0)}$ for every $t\in[0,T]$, for every $\eta\in H^1_D(\Omega_0)$;
(iii) for every $\eta,\xi\in H^1_D(\Omega_0)$ the function $t\mapsto\mathcal{B}(t;\eta,\xi)$ is continuously differentiable in $[0,T]$;
(iv) there exists $c_1>0$ such that $|\dot{\mathcal{B}}(t;\eta,\xi)|\le c_1\|\eta\|_{H^1_D(\Omega_0)}\|\xi\|_{H^1_D(\Omega_0)}$ for every $t\in[0,T]$, for every $\eta,\xi\in H^1_D(\Omega_0)$;
(v) for every $\eta,\xi\in H^1_D(\Omega_0)$ the function $t\mapsto\mathcal{A}_1(t;\eta,\xi)$ is continuous in $[0,T]$;
(vi) there exists $c_2>0$ such that $|\mathcal{A}_1(t;\eta,\xi)|\le c_2\|\eta\|_{H^1_D(\Omega_0)}\|\xi\|_{L^2(\Omega_0)}$ for every $t\in[0,T]$, for every $\eta,\xi\in H^1_D(\Omega_0)$;
(vii) for every $\eta,\xi\in H^1_D(\Omega_0)$ the function $t\mapsto\mathcal{A}_2(t;\eta,\xi)$ is continuous in $[0,T]$;
(viii) there exists $c_3>0$ such that $|\mathcal{A}_2(t;\eta,\xi)|\le c_3\|\eta\|_{H^1_D(\Omega_0)}\|\xi\|_{L^2(\Omega_0)}$ for every $t\in[0,T]$, for every $\eta,\xi\in H^1_D(\Omega_0)$.

Theorem 4.1. *Let $k>0$, $v^0\in H^1_D(\Omega_0)$, $v^1\in L^2(\Omega_0)$, $g\in L^2((0,T);L^2(\Omega_0))$, and let $\mathcal{B}(t;\cdot,\cdot)$, $\mathcal{A}_1(t;\cdot,\cdot)$, $\mathcal{A}_2(t;\cdot,\cdot)$, $t\in[0,T]$, be three families of continuous bilinear forms over $H^1_D(\Omega_0)\times H^1_D(\Omega_0)$ satisfying the assumptions (i)-(viii) above. Then there exists $v\in H^1((0,T);H^1_D(\Omega_0))$ with $\ddot v\in L^2((0,T);H^{-1}_D(\Omega_0))$ such that, for a.e. $t\in(0,T)$ and every $\psi\in H^1_D(\Omega_0)$,*

$$\begin{aligned}&\langle\ddot v(t),\psi\rangle_{H^1_D(\Omega_0)}+\mathcal{B}(t;v(t),\psi)+\mathcal{A}_1(t;v(t),\psi)+\mathcal{A}_2(t;\dot v(t),\psi)\\&+k\langle\dot v(t),\psi\rangle_{L^2(\Omega_0)}+k\langle\nabla\dot v(t),\nabla\psi\rangle_{L^2(\Omega_0)}=\langle g(t),\psi\rangle_{L^2(\Omega_0)}\,,\end{aligned}\tag{4.1}$$

with initial conditions $v(0) = v^0$ *and* $\dot{v}(0) = v^1$.

Proof. See [6, Chapitre XVIII, §5, Théorème 1]. □

In the following lemmas we investigate some regularity properties of functions defined in Ω (or $[0,T] \times \Omega$), when composed with suitable diffeomorphisms of the domain into itself. Let us specify the class of diffeomorphisms under study.

Definition 4.2. We say that $\Lambda\colon [0,T] \times \overline{\Omega} \to \mathbb{R}^d$ is admissible if it belongs to the space $C^1([0,T] \times \overline{\Omega}; \mathbb{R}^d)$ and, for every $t \in [0,T]$, the function $\Lambda(t,\cdot)$ is a C^2 diffeomorphism of $\overline{\Omega}$ into itself such that $\Lambda(t,\Omega) = \Omega$ and $\Lambda(t,\Gamma) = \Gamma$.

Note that, according to (H7)-(H9), both Φ and Ψ are admissible in Definition 4.2.

Lemma 4.3. *Let* Λ *be as in Definition 4.2 and let* $f \in L^2(\Omega; \mathbb{R}^m)$. *Then* $f(\Lambda(t,\cdot))$ *is continuous from* $[0,T]$ *to* $L^2(\Omega; \mathbb{R}^m)$.

Proof. For every $\varepsilon > 0$ let $f_\varepsilon \in C_c^\infty(\Omega; \mathbb{R}^m)$ be such that $\|f - f_\varepsilon\|_{L^2(\Omega)} \le \varepsilon$. By the assumptions on Λ, via a change of variables it is easy to see that for every $t \in [0,T]$ we still have $\|f(\Lambda(t,\cdot)) - f_\varepsilon(\Lambda(t,\cdot))\|_{L^2(\Omega)} \le C\varepsilon$, for some constant C independent of t and ε. Moreover the composition $f_\varepsilon(\Lambda(t,\cdot))$ is continuous from $[0,T]$ to $L^2(\Omega; \mathbb{R}^m)$. Let $t_n \to t$ as $n \to +\infty$. By the triangle inequality, we have

$$\begin{aligned}\|f(\Lambda(t_n,\cdot)) - f(\Lambda(t,\cdot))\|_{L^2(\Omega)} &\le \|f(\Lambda(t_n,\cdot)) - f_\varepsilon(\Lambda(t_n,\cdot))\|_{L^2(\Omega)} + \|f(\Lambda(t,\cdot)) - f_\varepsilon(\Lambda(t,\cdot))\|_{L^2(\Omega)} \\ &\quad + \|f_\varepsilon(\Lambda(t_n,\cdot)) - f_\varepsilon(\Lambda(t,\cdot))\|_{L^2(\Omega)} \le 2C\varepsilon + \|f_\varepsilon(\Lambda(t_n,\cdot)) - f_\varepsilon(\Lambda(t,\cdot))\|_{L^2(\Omega)}\,.\end{aligned}$$

Passing to the limit first as $n \to +\infty$ and then as $\varepsilon \to 0$ we infer that $\|f(\Lambda(t_n,\cdot)) - f(\Lambda(t,\cdot))\|_{L^2(\Omega)} \to 0$ as $t_n \to t$, namely the desired L^2 continuity of $f(\Lambda(t,\cdot))$. □

Lemma 4.4. *Let* Λ *be as in Definition 4.2 and let* $f \in C^0([0,T]; L^2(\Omega; \mathbb{R}^m))$. *Then* $f(t,\Lambda(t,\cdot))$ *is continuous from* $[0,T]$ *to* $L^2(\Omega; \mathbb{R}^m)$.

Proof. Let $t_n \to t$ as $n \to +\infty$. By the triangle inequality we have

$$\|f(t_n,\Lambda(t_n,\cdot)) - f(t,\Lambda(t,\cdot))\|_{L^2(\Omega)} \le \|f(t_n,\Lambda(t_n,\cdot)) - f(t,\Lambda(t_n,\cdot))\|_{L^2(\Omega)} + \|f(t,\Lambda(t_n,\cdot)) - f(t,\Lambda(t,\cdot))\|_{L^2(\Omega)}\,.$$

Via a change of variables, we infer that

$$\|f(t_n,\Lambda(t_n,\cdot)) - f(t,\Lambda(t_n,\cdot))\|_{L^2(\Omega)} \le C\|f(t_n,\cdot) - f(t,\cdot)\|_{L^2(\Omega)}\,,$$

for some constant $C > 0$ independent of n and t. Since t is fixed, we can apply Lemma 4.3 to $f(t,\cdot)$, obtaining that $\|f(t,\Lambda(t_n,\cdot)) - f(t,\Lambda(t,\cdot))\|_{L^2(\Omega)} \to 0$ too, as $n \to +\infty$. This concludes the proof. □

Lemma 4.5. *Let* Λ *be as in Definition 4.2. There exists a constant* $C > 0$ *such that*

$$\|f(\Lambda(t,\cdot)) - f(\Lambda(s,\cdot))\|_{L^2(\Omega)} \le C\|\widehat{\nabla} f\|_{L^2(\Omega)}|t-s|$$

for every $f \in H^1(\Omega \setminus \Gamma)$ *and for every* $0 \le t \le s \le T$.

Proof. It is enough to prove the statement in $H^1(\Omega^\pm)$, where $\Omega^\pm$ are defined in (H4). We consider only the case of Ω^+. For every $\varepsilon > 0$ let $f_\varepsilon \in C^1(\overline{\Omega}^+)$ be such that $\|f - f_\varepsilon\|_{H^1(\Omega^+)} \le \varepsilon$. Then

$$f_\varepsilon(\Lambda(t,y)) - f_\varepsilon(\Lambda(s,y)) = \int_s^t \nabla f_\varepsilon(\Lambda(\tau,y)) \cdot \dot{\Lambda}(\tau,y)\,d\tau\,,$$

and hence

$$\|f_\varepsilon(\Lambda(t,\cdot)) - f_\varepsilon(\Lambda(s,\cdot))\|^2_{L^2(\Omega^+)} \le \sup_\tau \|\dot{\Lambda}\|^2_{L^\infty(\Omega^+)} \int_{\Omega^+} \Big(\int_s^t |\nabla f_\varepsilon(\Lambda(\tau,y))|\,d\tau \Big)^2 dy\,. \tag{4.2}$$

By applying the Hölder inequality, changing the order of integration, and performing the change of variables $x = \Lambda(\tau,y)$, we deduce the estimates

$$\begin{aligned}\int_{\Omega^+} \Big(\int_s^t |\nabla f_\varepsilon(\Lambda(\tau,y))|\,d\tau \Big)^2 &\le |t-s| \int_{\Omega^+} \int_s^t |\nabla f_\varepsilon(\Lambda(\tau,y))|^2\,d\tau\,dy \\ &= |t-s| \int_s^t \int_{\Omega^+} |\nabla f_\varepsilon(x)|^2 \det D(\Lambda^{-1})(\tau,x)\,dx\,d\tau\end{aligned}$$

$$\leq |t-s|^2 \sup_\tau \| \det D(\Lambda^{-1})(\tau,\cdot)\|_{L^\infty(\Omega^+)} \|\nabla f_\varepsilon\|^2_{L^2(\Omega^+)} \,. \tag{4.3}$$

Arguing as in Lemma 4.3, by the triangle inequality, in view of (4.2) combined with (4.3), we obtain

$$\begin{aligned}\|f(\Lambda(t,\cdot)) - f(\Lambda(s,\cdot))\|_{L^2(\Omega^+)} &\leq C\varepsilon + \|f_\varepsilon(\Lambda(t,\cdot)) - f_\varepsilon(\Lambda(s,\cdot))\|_{L^2(\Omega^+)} \\ &\leq C(\varepsilon + \|\nabla f_\varepsilon\|_{L^2(\Omega^+)}|t-s|) \leq C(\varepsilon + \|\nabla f\|_{L^2(\Omega^+)}|t-s|)\,,\end{aligned}$$

for some constant $C > 0$ independent of ε, t, and s. Finally, letting $\varepsilon \to 0$, we conclude the proof. □

Lemma 4.6. *Let Λ be as in Definition 4.2 and let $t \in [0,T]$. Then for every $f \in H^1(\Omega \setminus \Gamma)$*

$$\frac{1}{h}[f(\Lambda(t+h,\cdot)) - f(\Lambda(t,\cdot))] \to \widehat{\nabla} f(\Lambda(t,\cdot)) \cdot \dot\Lambda(t,\cdot) \quad \textit{strongly in } L^2(\Omega) \textit{ as } h \to 0\,.$$

Proof. It is enough to prove the strong convergence in $L^2(\Omega^\pm)$, where $\Omega^\pm$ are defined in (H4). We consider only the case of Ω^+.

Let T_h and L be the linear operators from $H^1(\Omega^+)$ to $L^2(\Omega^+)$ defined by

$$T_h(f) := \frac{1}{h}[f(\Lambda(t+h,\cdot)) - f(\Lambda(t,\cdot))] \quad \text{and} \quad L(f) := \nabla f(\Lambda(t,\cdot)) \cdot \dot\Lambda(t,\cdot)\,.$$

By a change of variables it is easy to see that these operators are continuous. If $f \in C^1(\overline{\Omega}^+)$ we have

$$T_h(f)(y) = \frac{1}{h}\int_0^h \nabla f(\Lambda(t+\tau,y)) \cdot \dot\Lambda(t+\tau,y)\,d\tau\,, \tag{4.4}$$

thus

$$\|T_h(f)\|_{L^2(\Omega^+)} \leq C\|f\|_{H^1(\Omega^+)}\,,$$

where $C > 0$ is a constant independent of f. By density, this inequality is valid for every $f \in H^1(\Omega^+)$.

For $\varepsilon > 0$ fixed, let $f_\varepsilon \in C^1(\overline{\Omega}^+)$ be such that $\|f - f_\varepsilon\|_{H^1(\Omega^+)} < \varepsilon$. Then

$$\begin{aligned}\|T_h(f) - L(f)\|_{L^2(\Omega^+)} &\leq \|T_h(f) - T_h(f_\varepsilon)\|_{L^2(\Omega^+)} + \|T_h(f_\varepsilon) - L(f_\varepsilon)\|_{L^2(\Omega^+)} + \|L(f) - L(f_\varepsilon)\|_{L^2(\Omega^+)} \\ &\leq 2C\varepsilon + \|T_h(f_\varepsilon) - L(f_\varepsilon)\|_{L^2(\Omega^+)}\,.\end{aligned}$$

In view of formula (4.4) we have $\|T_h(f_\varepsilon) - L(f_\varepsilon)\|_{L^2(\Omega^+)} \to 0$ as $h \to 0$, and hence

$$\limsup_h \|T_h(f) - L(f)\|_{L^2(\Omega^+)} \leq 2C\varepsilon\,.$$

By the arbitrariness of $\varepsilon > 0$, the proof is concluded.

□

Lemma 4.7. *Let Λ and Λ^n, $n \in \mathbb{N}$, be diffeomorphisms as in Definition 4.2, and let f and f_n, $n \in \mathbb{N}$, be elements of $L^2(\Omega;\mathbb{R}^m)$. Assume that there exist $\delta_1 > \delta_0 > 0$ such that $\delta_0 < \|\det D\Lambda^n(t,\cdot)\|_{L^\infty(\Omega)} < \delta_1$ for every $t \in [0,T]$, $n \in \mathbb{N}$, and that, as $n \to +\infty$,*

$$f^n \to f \textit{ strongly in } L^2(\Omega;\mathbb{R}^m)\,, \quad \Lambda^n(t,\cdot) \to \Lambda(t,\cdot) \textit{ strongly in } L^2(\Omega;\mathbb{R}^d) \textit{ for every } t \in [0,T]\,.$$

Then, as $n \to +\infty$,

$$f^n(\Lambda^n(t,\cdot)) \to f(\Lambda(t,\cdot)) \textit{ strongly in } L^2(\Omega;\mathbb{R}^m) \textit{ for every } t \in [0,T]\,.$$

Proof. For every $\varepsilon > 0$ let $f_\varepsilon \in C^\infty_c(\Omega;\mathbb{R}^m)$ be such that $\|f - f_\varepsilon\|_{L^2(\Omega)} \leq \varepsilon$. By the triangle inequality we may write

$$\begin{aligned}\|f^n(\Lambda^n(t,\cdot)) - f(\Lambda(t,\cdot))\|_{L^2(\Omega)} &\leq \|f^n(\Lambda^n(t,\cdot)) - f(\Lambda^n(t,\cdot))\|_{L^2(\Omega)} + \|f(\Lambda^n(t,\cdot)) - f_\varepsilon(\Lambda^n(t,\cdot))\|_{L^2(\Omega)} \\ &\quad + \|f_\varepsilon(\Lambda(t,\cdot)) - f(\Lambda(t,\cdot))\|_{L^2(\Omega)} + \|f_\varepsilon(\Lambda^n(t,\cdot)) - f_\varepsilon(\Lambda(t,\cdot))\|_{L^2(\Omega)}\,.\end{aligned}$$

We can bound the right-hand side as follows: performing a change of variables in the first three terms and exploiting the regularity of f_ε in the last term, we get

$$\begin{aligned}\|f(\Lambda^n(t,\cdot)) - f(\Lambda^n(t,\cdot))\|_{L^2(\Omega)} &\leq C(\|f^n - f\|_{L^2(\Omega)} + \|f - f_\varepsilon\|_{L^2(\Omega)}) \\ &\quad + \|\nabla f_\varepsilon\|_{L^\infty(\Omega)}\|\Lambda^n(t,\cdot) - \Lambda(t,\cdot)\|_{L^2(\Omega)}\,,\end{aligned}$$

for some constant $C > 0$ independent of n and ε. Passing to the limit as $n \to +\infty$ and then as $\varepsilon \to 0$, by the strong L^2 convergences of $\Lambda^n(t,\cdot)$, f^n, and f_ε, we conclude the proof. □

We conclude this Appendix by proving Lemma 2.4 on the regularity of the composite function $z(t,\cdot) := w(t,\Phi(t,\cdot))$.

Proof of Lemma 2.4. By the chain rule in Sobolev spaces, since every $\Phi(t,\cdot)$ is a C^2 diffeomorphism of $\overline{\Omega}$ into itself, we infer that the composite function $z(t,\cdot) := w(t,\Phi(t,\cdot))$ belongs to $H^2(\Omega_0)$ for a.e. $t \in (0,T)$, and its distributional derivatives read

$$\nabla z(t,\cdot) = D\Phi(t,\cdot)^T \nabla w(t,\Phi(t,\cdot)) \,, \quad \partial^2_{ij} z(t,\cdot) = \partial^2_{km} w(t,\Phi(t,\cdot))\partial_i \Phi_k(t,\cdot)\partial_j\Phi_m(t,\cdot) + \partial^2_{ij}\Phi_k(t,\cdot)\partial_k w(t,\Phi(t,\cdot)) \,.$$

Exploiting these explicit expressions for the distributional derivatives of z, the regularity assumptions (H7), (H11), (H12), and (1.10), we infer that also z belongs to $L^2((0,T);H^2(\Omega_0))$. Arguing as in the proof of Lemma 1.8, exploiting the regularity of w and $\dot w$, we infer that $z \in \mathrm{Lip}([0,T];L^2(\Omega_0))$ and the distributional time derivative reads $\dot z(t,\cdot) = \dot w(t,\Phi(t,\cdot)) + \nabla w(t,\Phi(t,\cdot))\cdot\dot\Phi(t,\cdot)$. By the chain rule, we infer that $\dot z(t,\cdot) \in H^1(\Omega_0)$ for a.e. $t \in (0,T)$ and, by direct computation, that $\dot z \in L^2((0,T);H^1(\Omega_0))$. Let us now pass to the second partial time derivative. Exploiting the Lipschitz continuity of $\dot\Phi$ (see (H11)) and the absolute continuity of $\dot w$ and ∇w from $[0,T]$ to $L^2(\Omega)$ and $L^2(\Omega;\mathbb{R}^d)$, respectively, it is easy to prove that $\dot z \in AC([0,T];H^{-1}(\Omega_0))$. In particular $\ddot z \in L^1((0,T);H^{-1}(\Omega_0))$ and, for a.e. $t \in (0,T)$, the action of $\ddot z(t)$ against any test function $\psi \in H^1(\Omega_0)$ can be deduced by the identity

$$\langle \ddot z(t), \psi\rangle_{H^1(\Omega_0)} = \lim_{h\to 0}\frac{1}{h}\langle \dot z(t+h) - \dot z(t), \psi\rangle_{L^2(\Omega)} \,.$$

Computing the last limit we infer that for a.e. $t \in (0,T)$

$$\ddot z(t,\cdot) = \ddot w(t,\Phi(t,\cdot)) + 2\nabla\dot w(t,\Phi(t,\cdot))\cdot\dot\Phi(t,\cdot) + \nabla w(t,\Phi(t,\cdot))\cdot\ddot\Phi(t,\cdot) + \partial^2_{hk} w(t,\Phi(t,\cdot))\dot\Phi_h(t,\cdot)\dot\Phi_k(t,\cdot) \,.$$

Therefore, in view of (H11), (H12), and (1.10), we derive $\ddot z \in L^2((0,T);L^2(\Omega))$, concluding the proof of (2.7). Eventually, the boundary conditions (2.8), (2.9), and (2.10) follow by (1.13) and by combining the hypothesis (1.12) with definition (1.29), assumption (H8) and property (1.2). □

Acknowledgements. This material is based on work supported by the Italian Ministry of Education, University, and Research under the Project "Calculus of Variations" (PRIN 2010-11) and by the European Research Council under Grant No. 290888 "Quasistatic and Dynamic Evolution Problems in Plasticity and Fracture". The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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