Results on the minimization of the Dirichlet functional among semicartesian parametrizations

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Abstract

We start to investigate the existence of conformal minimizers for the Dirichlet functional in the setting of *semicartesian parametrizations*, adapting to this context the results in [10] for the Plateau's problem. The final goal is to find area minimizing semicartesian parametrizations spanning a Jordan curve obtained as union of two graphs; this problem appeared in [5], in the study of the relaxed area functional for maps from the plane to the plane jumping on a line.

1 Introduction

Given two continuous maps $\gamma^{\pm} : [a,b] \subset \mathbb{R}_t \to \mathbb{R}^2_{(\xi,\eta)}$, let $\Gamma \subset \mathbb{R}^3 = \mathbb{R}_t \times \mathbb{R}^2_{(\xi,\eta)}$ be the curve obtained as the union of the graphs of $\gamma^{\pm}$, that is

$$\Gamma := \left\{(t,\xi,\eta) : t \in [a,b], (\xi,\eta) = \gamma^-(t)\right\} \cup \left\{(t,\xi,\eta) : t \in [a,b], (\xi,\eta) = \gamma^+(t)\right\};$$

let us suppose also that $\gamma^-(a) = \gamma^+(a)$, $\gamma^-(b) = \gamma^+(b)$, and $\gamma^-(t) \neq \gamma^+(t)$ for $t \in (a,b)$, so that Γ is a Jordan curve.

For a curve having this structure, the authors introduced in [5] the notion of *semicartesian parametrization spanning* Γ. A *semicartesian parametrization* is a pair $([[\sigma^-,\sigma^+]],\Phi)$, where $[[\sigma^-,\sigma^+]]$ is the planar domain defined as

$$[[\sigma^-,\sigma^+]] := \left\{(t,s) : t \in (a,b), s \in (\sigma^-(t),\sigma^+(t))\right\},$$

for some regular functions $\sigma^{\pm} : [a,b] \to \mathbb{R}_s$ satisfying $\sigma^-(a) = \sigma^+(a)$, $\sigma^-(b) = \sigma^+(b)$, and $\sigma^-(t) < \sigma^+(t)$ for $t \in (a,b)$, and $\Phi : [[\sigma^-,\sigma^+]] \to \mathbb{R}^3$ is such that its first component is the identity on the first parameter, namely

$$\Phi(t,s) = (t,\phi(t,s)), \quad (t,s) \in [[\sigma^-,\sigma^+]],$$

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for some $\phi : [[\sigma^-, \sigma^+]] \to \mathbb{R}^2$. If ϕ satisfies also the boundary condition

$$\phi(t, \sigma^{\pm}(t)) = \gamma^{\pm}(t),$$

then we say that (D, Φ) is a semicartesian parametrization *spanning* Γ. If this is the case, the semicartesian surface $\Phi(D)$ is such that its intersection with any plane of the form $\{t\} \times \mathbb{R}^2$, $t \in (a, b)$, is a curve (not necessarily simple) connecting the two distinct points $(t, \gamma^-(t))$ and $(t, \gamma^+(t))$. We underline that a semicartesian surface is a sort of intermediate object between a general disk-type surface and a graph of a scalar function defined on a planar domain.
In this notes we present some results addressing the problem of the existence of an area minimizing semicartesian parametrization spanning a given contour Γ, that is the existence of a solution of the following area minimizing problem

$$\begin{aligned} m(\Gamma) := \inf \Big\{ \int_{[[\sigma^-,\sigma^+]]} |\partial_t \Phi \wedge \partial_s \Phi| \, dt \, ds : ([[\sigma^-, \sigma^+]], \Phi) \text{ is a semicartesian} \\ \text{parametrization spanning } \Gamma \Big\}. \end{aligned} \tag{1.1}$$

The class of semicartesian surfaces has been introduced in [5] as a tool in the study of the relaxed area functional $\overline{\mathcal{A}}$ (*cf.* [1], [8], [9]) for a map $\mathbf{u} \in \mathrm{BV}(\Omega; \mathbb{R}^2)$ [3] jumping on a line $J \subset\subset \Omega$ and regular enough in $\Omega \setminus \overline{J}$, where $\Omega \subset \mathbb{R}^2$ is an open bounded set. Indeed there are results (see [5], [6]) showing that the singular contribution of $\overline{\mathcal{A}}$ for such a map can be estimated by $m(\Gamma)$, for a suitable Γ obtained as the union of the graphs of the traces of $\mathbf{u}$ on the two sides of J.
In [5] the authors solved problem (1.1) for curves Γ satisfying some further regularity assumptions: they proved that if Γ is analytic and union of two graphs joining in a non-degenerate way, then there exists an area-minimizing solution of the classical Plateau's problem for Γ admitting a semicartesian parametrization; this implies the existence of a solution of (1.1). Indeed, we recall that, given a rectifiable Jordan curve $\Gamma \subset \mathbb{R}^3$, the Plateau's problem consists in minimizing the area functional, among all immersions of the disk mapping the boundary monotonically onto Γ, that is solving

$$a(\Gamma) := \min_{X \in \mathcal{C}(\Gamma)} \int_B |\partial_u X \wedge \partial_v X| \, du \, dv, \tag{1.2}$$

where $B \subset \mathbb{R}^2_{(u,v)}$ denotes the unit disk and

$$\mathcal{C}(\Gamma) := \{X \in H^1(B; \mathbb{R}^3) : \ X_{|\partial B} \in \mathcal{C}(\partial B; \mathbb{R}^3) \text{ is a weakly monotonic parametrization of } \Gamma\}.$$

Since, given a semicartesian parametrization $([[\sigma^-, \sigma^+]], \Phi)$, it is possible to define an immersion $X \in \mathcal{C}(\Gamma)$ having the same area as $([[\sigma^-, \sigma^+]], \Phi)$(1), there holds $m(\Gamma) \geq a(\Gamma)$. Proving that a solution of the Plateau's problem can be represented by a semicartesian parametrization implies therefore that $m(\Gamma) = a(\Gamma)$, and that there exists a solution to (1.1).
In this work we try to solve directly problem (1.1) adapting, to our semicartesian context, the strategy of [10, Chapter 4] in the context of the Plateau's problem: in few words, it consists

(1)For example composing Φ with the conformal map sending the disk onto $[[\sigma^-, \sigma^+]]$, whose existence is guaranteed by the Riemann mapping theorem, see e.g. [14].

in finding a map $X_{\min} \in \mathcal{C}(\Gamma)$ that minimizes the *Dirichlet functional*, rather than the area functional, and next proving that it is indeed also a solution to (1.2). In this second step it turns out to be crucial, the fact that the map $X_{\min}$ is $\mathcal{C}^2$-regular and that it is conformal, namely it satisfies

$$\partial_u X_{\min} \cdot \partial_v X_{\min} = 0 \qquad \text{and} \qquad |\partial_u X_{\min}| = |\partial_v X_{\min}|. \tag{1.3}$$

In our semicartesian setting, however, the situation is different, mainly because both the domains and the maps are free to vary.
We consider curves Γ union of two graphs of maps $\gamma^\pm \in \mathcal{C}^{m,\alpha}([a,b];\mathbb{R}^2)$, where $(m,\alpha) = (0,1)$ or $(m,\alpha) = (1,\lambda)$ with $\lambda \in (0,1)$. We define some classes of semicartesian parametrizations spanning Γ depending on the properties of the maps $\sigma^\pm$ bounding the domain (see Section 2 for more details);

- $\mathrm{semicart}^{m,\alpha}(\Gamma)$ is the class of semicartesian parametrizations $([[\sigma^-,\sigma^+]],\Phi)$ spanning Γ such that $\sigma^\pm \in \mathcal{C}^{m,\alpha}([a,b])$;
- $\forall \mathrm{S} > 0$, $\mathrm{semicart}^{m,\alpha}_{\mathrm{S}}(\Gamma)$ is the class of semicartesian parametrizations $([[\sigma^-,\sigma^+]],\Phi) \in \mathrm{semicart}^{m,\alpha}(\Gamma)$ such that $||\sigma^\pm||_{m,\alpha} \leq \mathrm{S}$.

We observe that this second class has better compactness properties than $\mathrm{semicart}^{m,\alpha}(\Gamma)$. In Theorem 2.7 we prove with classical tools the existence of a minimum for the Dirichlet functional in $\mathrm{semicart}^{m,\alpha}_{\mathrm{S}}(\Gamma)$ for every $\mathrm{S} > 0$, denoted by $([[\sigma^-_{\mathrm{S}},\sigma^+_{\mathrm{S}}]],\Phi_{\mathrm{S}})$.
What we would like to prove next is that, if S is large enough, this minimum belongs to a subset of $\mathrm{semicart}^{m,\alpha}_{\mathrm{S}}(\Gamma)$ that does not depend on S, and thus that it possible to find a minimum of the Dirichlet functional in the whole space $\mathrm{semicart}^{m,\alpha}(\Gamma)$; in this reasoning conformality should play an important role.[(2)]. However, due to regularity issues, we are not able to give a complete proof of this assertion, and we obtain only partial results.
In the case $(m,\alpha) = (1,\lambda)$, $\lambda \in (0,1)$, , we are able to prove that if the minimum $([[\sigma^-_{\mathrm{S}},\sigma^+_{\mathrm{S}}]],\Phi_{\mathrm{S}})$ in $\mathrm{semicart}^{1,\lambda}_{\mathrm{S}}(\Gamma)$ is such that $\sigma^\pm_{\mathrm{S}}$ satisfy the *strict* inequality

$$||\sigma^\pm_{\mathrm{S}}||_{1,\alpha} < \mathrm{S}, \tag{1.4}$$

then Φ_{S} is conformal (see Theorem 2.9). We miss the proof that (1.4) is valid for S large enough.
If $(m,\alpha) = (0,1)$, we are able to prove the existence of a minimum for the Dirichlet functional in $\mathrm{semicart}^{0,1}(\Gamma)$ only assuming the further hypothesis that there exists a constant M such that, for every $\mathrm{S} > M$,

$$\Phi_{\mathrm{S}} \in \mathcal{C}^{0,1}\left(\overline{[[\sigma^-_{\mathrm{S}},\sigma^+_{\mathrm{S}}]]};\mathbb{R}^3\right), \tag{1.5}$$

where $([[\sigma^-_{\mathrm{S}},\sigma^+_{\mathrm{S}}]],\Phi_{\mathrm{S}})$ minimizes the Dirichlet functional in $\mathrm{semicart}^{0,1}(\Gamma)$, see Theorem 2.11. Before obtaining this result, we also prove Theorem 2.10 that concerns the conformality of maps Φ_{S}. Let G be such that $||\gamma^\pm||_{0,1} \leq G$; we prove that if $S > G$ and $([[\sigma^-_{\mathrm{S}},\sigma^+_{\mathrm{S}}]],\Phi_{\mathrm{S}})$ satisfies (1.5), then $||\sigma^\pm_{\mathrm{S}}||_{0,1} \geq \mathrm{G}$. This fact implies that Φ_{S} is conformal and aslo that the minimum of the Dirichlet functional in $\mathrm{semicart}^{0,1}_{\mathrm{S}}(\Gamma)$ belongs to a subset independent of S.

(2) We observe that this strategy recall the one used to prove the existence of non-parametric minimal surfaces, see e.g. [12].

In Section 2 we give the definitions of the semicartesian setting and we state the results; in Section 3 we prove Theorem 2.7, that is the existence of a minimizing semicartesian parametrization in $\mathrm{semicart}^{m,\alpha}_{\mathrm{S}}(\Gamma)$; in Section 4 we prove Theorems 2.10 and 2.9, that is the conformality properties of the minimizers $([[\sigma^-_{\mathrm{S}}, \sigma^+_{\mathrm{S}}]]; \Phi_{\mathrm{S}})$; we conclude probing in Section 5 Theorem 2.11, namely the existence of a Dirichlet minimizing semicartesian parametrization in $\mathrm{semicart}^{0,1}(\Gamma)$, assuming (1.5).

2 Setting and main results

Let $[a, b] \subset \mathbb{R}_t$ be a bounded interval. We decompose $\mathbb{R}^3$ as $\mathbb{R}^3 = \mathbb{R}_t \times \mathbb{R}^2_{(\xi,\eta)} = \mathbb{R}^3_{(t,\xi,\eta)}$, where the subscript denotes the name of the variables. The following definition is a particular case of a definition proposed in [5].

In this paper (m, α) is a pair belonging to

$$\{(0, 1), (1, \lambda)\}, \text{ with } \lambda \in (0, 1).$$

We believe the case $(m, \alpha) = (0, 1)$ to be the most interesting case.

If $g : [a, b] \to \mathbb{R}^n$, with $n \in \{1, 2\}$, we define the seminorms

$$||g||_{m,\alpha} = \begin{cases} \mathrm{lip}(g) & \text{if } (m, \alpha) = (0, 1), \\ \mathrm{lip}(g) + \sup\left\{ \frac{|\dot g(t_1)-\dot g(t_2)|}{|t_1-t_2|^\lambda},\ t_1, t_2 \in [a, b],\ t_1 \neq t_2 \right\} & \text{if } (m, \alpha) = (1, \lambda). \end{cases}$$

Definition 2.1 (Union of two $\mathcal{C}^{m,\alpha}$ graphs on $[a, b]$). The set $\Gamma \subset \mathbb{R}^3_{(t,\xi,\eta)}$ is said to be the *union of the two $\mathcal{C}^{m,\alpha}$ graphs of $\gamma^\pm$ in* $[a, b]$, if

$$\Gamma = \Gamma^- \cup \Gamma^+, \qquad \text{with} \quad \Gamma^\pm := \{(t, \xi, \eta) : t \in [a, b], (\xi, \eta) = \gamma^\pm(t)\},$$

where the two maps $\gamma^\pm : [a, b] \to \mathbb{R}^2$ satisfy the following properties:

- $\gamma^\pm \in \mathcal{C}^{m,\alpha}([a, b]; \mathbb{R}^2)$,
- $\gamma^-(a) = \gamma^+(a)$ and $\gamma^-(b) = \gamma^+(b)$,
- $\gamma^-(t) \neq \gamma^+(t)$ for all $t \in (a, b)$.

If $\mathrm{G} > 0$ is such that

$$||\gamma^\pm||_{m,\alpha} \leq \mathrm{G}, \tag{2.1}$$

we say that Γ is union of two G-$\mathcal{C}^{m,\alpha}$ graphs.

Definition 2.2 ((m, α) admissible and strictly admissible domain). Let $\sigma^\pm : [a, b] \to \mathbb{R}_s$ be two functions satisfying the following properties:

- $\sigma^\pm \in \mathcal{C}^{m,\alpha}([a, b])$,
- $\sigma^-(a) = \sigma^+(a)$ and $\sigma^-(b) = \sigma^+(b)$,
- $\sigma^-(t) < \sigma^+(t)$ for all $t \in (a, b)$.

The open set

$$[[\sigma^-, \sigma^+]] := \{(t, s) : t \in (a, b), s \in (\sigma^-(t), \sigma^+(t))\} \subset \mathbb{R}_t \times \mathbb{R}_s$$

is called an (m, α) admissible domain. If $||\sigma^\pm||_{m,\alpha} \leq \mathrm{S}$ (respectively $||\sigma^\pm||_{m,\alpha} < \mathrm{S}$) for some positive constant S, then $[[\sigma^-, \sigma^+]]$ is called a (m, α) S-admissible (respectively strictly (m, α) S-admissible) domain.

Remark 2.3. The boundary $\partial[[\sigma^-, \sigma^+]]$ has at least two points where it is not differentiable, $(a, \sigma^-(a))$ and $(b, \sigma^-(b))$. Moreover, only if $(m, \alpha) = (0, 1)$ the regularity of $\partial[[\sigma^-, \sigma^+]]$ is the same as the regularity of σ^- and σ^+.

Definition 2.4 (**Semicartesian map spanning** Γ)**.** Let Γ be the union of two $\mathcal{C}^{m,\alpha}$ graphs of $\gamma^\pm$. Let $[[\sigma^-, \sigma^+]]$ be a (m, α) admissible domain. A *semicartesian map in* $[[\sigma^-, \sigma^+]]$ *spanning* Γ is a map $\Phi : [[\sigma^-, \sigma^+]] \to \mathbb{R}^3_{(t,\xi,\eta)}$ of the form

$$\Phi(t, s) = (t, \phi(t, s)), \qquad (t, s) \in [[\sigma^-, \sigma^+]],$$

where

- $\phi \in H^1([[\sigma^-, \sigma^+]]; \mathbb{R}^2_{(\xi,\eta)})$,
- $\phi(t, \sigma^\pm(t)) = \gamma^\pm(t)$ for almost all $t \in [a, b]$.

The pair $([[\sigma^-, \sigma^+]], \Phi)$ will be called a *semicartesian parametrization with* (m, α) *domain* (or just a *semicartesian parametrization* if there is no ambiguity) spanning Γ, and we set

$$\mathrm{semicart}^{m,\alpha}(\Gamma) = \{ \ ([[\sigma^-, \sigma^+]], \Phi) : \ ([[\sigma^-, \sigma^+]], \Phi) \text{ is a semicartesian parametrization with } (m, \alpha) \text{ admissible domain } \}$$

For $\mathrm{S} > 0$, we also set

$$\mathrm{semicart}^{m,\alpha}_{\mathrm{S}}(\Gamma) := \left\{ ([[\sigma^-, \sigma^+]], \Phi) \in \mathrm{semicart}^{m,\alpha}(\Gamma) : \ ||\sigma^\pm||_{m,\alpha} \leq \mathrm{S} \right\}.$$

Clearly

$$0 < \mathrm{S}_1 < \mathrm{S}_2 \quad \Longrightarrow \quad \mathrm{semicart}^{m,\alpha}_{\mathrm{S}_1}(\Gamma) \subseteq \mathrm{semicart}^{m,\alpha}_{\mathrm{S}_2}(\Gamma),$$

and

$$\mathrm{semicart}^{m,\alpha}(\Gamma) = \bigcup_{\mathrm{S}>0} \mathrm{semicart}^{m,\alpha}_{\mathrm{S}}(\Gamma). \tag{2.2}$$

Remark 2.5. We stress the the indices in the symbol $\mathrm{semicart}^{m,\alpha}(\Gamma)$ refer to the regularity of the functions $\sigma^\pm$ defining the domain and not to the regularity of the semicartesian map.

Lemma 2.6. *Let* Γ *be union of two* $\mathcal{C}^{m,\alpha}$ *graphs. Let* $\mathrm{S} > 0$. *Then* $\mathrm{semicart}^{m,\alpha}_{\mathrm{S}}(\Gamma) \neq \emptyset$.

Proof. We choose two functions $\sigma^\pm \in \mathcal{C}^{m,\alpha}([a, b])$ with $||\sigma^\pm||_{m,\alpha} \leq \mathrm{S}$, such that $[[\sigma^-, \sigma^+]]$ is a (m, α) admissible domain, satisfying the following growth conditions

$$\begin{aligned} &\sigma^\pm(t) = \mathcal{O}(t - a), \\ &\sigma^+(t) > 0 \text{ and } \sigma^-(t) < 0, \\ &\dot\sigma^+(t) > 0 \text{ and } \dot\sigma^-(t) < 0, \end{aligned}$$

for $t \in [a, a + h]$, $h > 0$ small, and also satisfying similar requirements near $t = b$ (if $(m, \alpha) = (0, 1)$, the requirements on $\dot\sigma^\pm$ are intended to hold almost everywhere).

We define on $[[\sigma^-, \sigma^+]]$ the *linear interpolating* map

$$(t,s) \in [[\sigma^-, \sigma^+]] \to \left(t, \frac{(\sigma^+(t)-s)\gamma^-(t) + (s-\sigma^-(t))\gamma^+(t)}{\sigma^+(t)-\sigma^-(t)}\right) =: (t, \phi(t,s)).$$

Then the second condition of Definition 2.4 is satisfied. Therefore, we need only to check that $\phi \in H^1([[\sigma^-, \sigma^+]]; \mathbb{R}^2_{(\xi,\eta)})$. We have $|\phi(t,s)| \le \|\gamma^-\|_{L^\infty([a,b];\mathbb{R}^2_{(\xi,\eta)})} + \|\gamma^+\|_{L^\infty([a,b];\mathbb{R}^2_{(\xi,\eta)})}$, so that $\phi \in L^2([[\sigma^-, \sigma^+]]; \mathbb{R}^2_{(\xi,\eta)})$. Also ϕ_s is in $L^2([[\sigma^-, \sigma^+]]; \mathbb{R}^2_{(\xi,\eta)})$, since a direct computation gives

$$\int_{[[\sigma^-,\sigma^+]]} |\phi_s(t,s)|^2 \, dt \, ds = \int_{[[\sigma^-,\sigma^+]]} \left| \frac{\gamma^+(t) - \gamma^-(t)}{\sigma^+(t) - \sigma^-(t)} \right|^2 dt \, ds. \tag{2.3}$$

Let $\mathrm{G} > 0$ be such that (2.1) holds. Then, both for $(m, \alpha) = (0, 1)$ and for $m \ge 1$, we have:

$$|\gamma^+(t) - \gamma^-(t)| \le 2\mathrm{G}(t-a), \qquad t \in [a, a+h]; \tag{2.4}$$

moreover

$$|\sigma^+(t) - \sigma^-(t)| = \mathcal{O}(t-a), \qquad t \in [a, a+h] \tag{2.5}$$

(and similarly near $t = b$), hence the integral on the right hand side of (2.3) is finite.
Now we compute the t-derivative: rearranging terms we have

$$\phi_t = \frac{\dot\gamma^-(\sigma^+ - s)}{\sigma^+ - \sigma^-} + \frac{\dot\gamma^+(s - \sigma^-)}{\sigma^+ - \sigma^-} + \frac{(\gamma^+ - \gamma^-)(\dot\sigma^+\sigma^- - \sigma^+\dot\sigma^-)}{(\sigma^+ - \sigma^-)^2} + \frac{s(\gamma^+ - \gamma^-)(\dot\sigma^+ - \dot\sigma^-)}{(\sigma^+ - \sigma^-)^2}. \tag{2.6}$$

We observe that, since $s \in (\sigma^-(t), \sigma^+(t))$,

$$\frac{|\dot\gamma^-(\sigma^+ - s)|}{\sigma^+ - \sigma^-} \le |\dot\gamma^-|, \qquad \frac{|\dot\gamma^+(s - \sigma^-)|}{\sigma^+ - \sigma^-} \le |\dot\gamma^+|,$$

thus the first two terms on the right hand side of (2.6) are in $L^2([[\sigma^-, \sigma^+]]; \mathbb{R}^2_{(\xi,\eta)})$. The same holds for the other two terms, recalling (2.4), (2.5) and observing that

$$|\dot\sigma^+\sigma^- - \sigma^+\dot\sigma^-| \le \max\{|\dot\sigma^+\sigma^-|, |\sigma^+\dot\sigma^-|\} = \mathcal{O}(t-a), \quad t \in [a, a+h]$$

and

$$|s| \le \sigma^+(t) - \sigma^-(t).$$

Therefore $\phi \in H^1([[\sigma^-, \sigma^+]]; \mathbb{R}^2_{(\xi,\eta)})$. □

Given a bounded open set $\Omega \subset \mathbb{R}^2$, and $f \in H^1(\Omega; \mathbb{R}^2)$, we denote by $\mathrm{Dir}(f, \Omega)$ the Dirichlet functional [10] of f on Ω, namely

$$\mathrm{Dir}(f, \Omega) := \frac{1}{2} \int_\Omega |\nabla f|^2 \, dt \, ds,$$

where $|\nabla f|^2$ is the sum of the squares of the entries of the matrix ∇f.
Given $([[\sigma^-, \sigma^+]], \Phi) \in \mathrm{semicart}^{m,\alpha}(\Gamma)$, we have that $|\nabla\Phi|^2 = 1 + |\partial_t\phi|^2 + |\partial_t\phi|^2$, where ϕ is such that $\Phi(t,s) = (t, \phi(t,s))$; hence

$$\mathrm{Dir}(\Phi, [[\sigma^-, \sigma^+]]) = \frac{1}{2} \int_a^b \int_{\sigma^-(t)}^{\sigma^+(t)} \left[1 + |\partial_t\phi|^2 + |\partial_s\phi|^2\right] ds \, dt.$$

The main results of this paper are the following.

Theorem 2.7 (**Existence of minimizers in $\mathrm{semicart}_{\mathrm{S}}^{m,\alpha}(\Gamma)$**)**.** *Let Γ be a union of two $\mathcal{C}^{m,\alpha}$ graphs on $[a,b]$ and $\mathrm{S}>0$. Then the problem*

$$\min\left\{\mathrm{Dir}(\Phi,[[\sigma^-,\sigma^+]]) : ([[\sigma^-,\sigma^+]],\Phi)\in \mathrm{semicart}_{\mathrm{S}}^{m,\alpha}(\Gamma),\ \sigma^-(a)=0\right\} \tag{2.7}$$

admits a solution $([[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]],\Phi_{\mathrm{S}})$, and Φ_{S} is harmonic in $[[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]]$ and continuous on $\overline{[[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]]}$.

Notice that from (2.2), we deduce that for every $\mathrm{S}>0$

$$\mathrm{Dir}(\Phi_{\mathrm{S}},[[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]]) \geq \inf\{\mathrm{Dir}(\Phi,[[\sigma^-,\sigma^+]]) :\ ([[\sigma^-,\sigma^+]],\Phi)\in \mathrm{semicart}^{m,\alpha}(\Gamma)\},$$

thus, in general, $([[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]],\Phi_{\mathrm{S}})$ does not minimize the Dirichlet functional in $\mathrm{semicart}^{m,\alpha}(\Gamma)$.

Remark 2.8. The condition $\sigma^-(a)=0$ in (2.7) is necessary to avoid the non-compactness of domains of a minimizing sequence due to translation invariance.

About the conformality of minima, we have different results depending on the value of (m,α).

Theorem 2.9 (**Conformality of strictly admissible minimizers in $\mathrm{semicart}_{\mathrm{S}}^{1,\lambda}(\Gamma)$**)**.** *Let $\mathrm{G}>0$ and let Γ be union of two G-$\mathcal{C}^{1,\lambda}$ graphs. Fix $\mathrm{S}>0$ and suppose that $([[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]],\Phi_{\mathrm{S}})$ is a solution of (2.7). Suppose moreover that $[[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]]$ is S-strictly admissible. Then Φ is conformal.*

If $(m,\alpha)=(0,1)$, in order to get the analogous result we have to require a priori that $\Phi_{\mathrm{S}}\in\mathcal{C}^{0,1}(\overline{[[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]]};\mathbb{R}^3)$, with $([[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]],\Phi_{\mathrm{S}})$ as in Theorem 2.7. In the case $(m,\alpha)=(1,\lambda)$, this technical hypothesis is implied by classical elliptic regularity results, as explained in Remark 3.1. Under this further condition, we are able to prove something more than in Theorem 2.9.

Theorem 2.10 (**Conformality of minimizers in $\mathrm{semicart}_{\mathrm{S}}^{0,1}(\Gamma)$**)**.** *Let $\mathrm{G}>0$ and let Γ be union of two G-Lipschitz graphs. Fix $\mathrm{S}>0$ and suppose that $([[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]],\Phi_{\mathrm{S}})$ is a solution of (2.7) with*

$$\Phi_{\mathrm{S}}\in\mathcal{C}^{0,1}\left(\overline{[[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]]};\mathbb{R}^3\right).$$

The following assertions hold:

(i) if $[[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]]$ is S-strictly admissible, then Φ_{S} is conformal;

(ii) if $\mathrm{S}>\mathrm{G}$, then $||\sigma_{\mathrm{S}}^{\pm}||_{0,1}\leq \mathrm{G}<\mathrm{S}$, that is $[[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]]$ is S-strictly admissible.

We observe that assertions (i) and (ii) imply that if $\mathrm{S}>\mathrm{G}$ and $([[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]],\Phi_{\mathrm{S}})\in\mathrm{semicart}_{\mathrm{S}}^{0,1}(\Gamma)$ is a solution of (2.7) such that $\phi\in\mathcal{C}^{0,1}\left(\overline{[[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]]};\mathbb{R}^2\right)$, then ϕ_{S} is conformal. Moreover, the fact that under the hypotheses of Theorem 2.10 the minimum belongs to $\mathrm{semicart}_{\mathrm{G}}^{0,1}(\Gamma)$ for every $\mathrm{S}>\mathrm{G}$, implies the following result of existence of a minimizer of the Dirichlet functional in $\mathrm{semicart}^{0,1}(\Gamma)$.

Theorem 2.11. *Let $\mathrm{G}>0$ and let Γ be union of two G-Lipschitz graphs. Suppose that there exists a $\mathrm{M}>0$ such that $\phi_{\mathrm{S}}\in\mathcal{C}^{0,1}\left(\overline{[[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]]};\mathbb{R}^2\right)$ for every $\mathrm{S}>\mathrm{M}$. Then the problem*

$$\min\left\{\mathrm{Dir}(\Phi,[[\sigma^-,\sigma^+]]) : ([[\sigma^-,\sigma^+]],\Phi)\in \mathrm{semicart}^{0,1}(\Gamma),\ \sigma^-(a)=0\right\} \tag{2.8}$$

has a solution $([[\sigma^-,\sigma^+]],\Phi])\in\mathrm{semicart}_{\mathrm{G}}^{0,1}(\Gamma)$ and Φ is harmonic and conformal on $[[\sigma^-,\sigma^+]]$, and continuous on $\overline{[[\sigma^-,\sigma^+]]}$.

Remark 2.12. Let $([[\sigma^-,\sigma^+]],\Phi)$ be a solution to (2.8), for every (m,α). Then Φ is harmonic in $[[\sigma^-,\sigma^+]]$. This can be seen, as in [10, section 2.1] by exploiting the equality

$$0=\frac{d}{d\varepsilon}\mathrm{Dir}(X_\varepsilon,[[\sigma^-,\sigma^+]])_{|\varepsilon=0}$$

where $X_\varepsilon(t,s):=\Phi(t,s)+\varepsilon X(t,s)$ for $X\in\mathcal{C}^\infty_{\mathrm{c}}(\Omega;\mathbb{R}^3)$ of the form $X(t,s)=(0,\psi(t,s))$.

In Section 3 we prove Theorem 2.7. In Section 4 we prove Theorem 2.10, dividing the statement and the proof into Proposition 4.1, Lemma 4.3, and Proposition 4.4, and we deduce also the proof of Theorem 2.9. In Section 5 we prove Theorem 2.11.

3 Existence in $\mathrm{semicart}^{m,\alpha}_{\mathrm{S}}(\Gamma)$

In this section we prove Theorem 2.7. For simplicity, in this section the minimum will be denoted by $([[\sigma^-,\sigma^+]],\Phi)$, in place of $([[\sigma^-_{\mathrm{S}},\sigma^+_{\mathrm{S}}]],\Phi_{\mathrm{S}})$.

Proof of Theorem 2.7. Let $\big(([[\sigma_h^-,\sigma_h^+]],\Phi_h)\big)\subset\mathrm{semicart}^{m,\alpha}_{\mathrm{S}}(\Gamma)$ be a minimizing sequence for problem (2.7), namely $\sigma_h^+(a)=0$ for every h and

$$\begin{aligned}&\lim_{h\to+\infty}\mathrm{Dir}(\Phi_h,[[\sigma_h^-,\sigma_h^+]])\\&=\inf\big\{\mathrm{Dir}(\Phi,[[\sigma^-,\sigma^+]]):([[\sigma^-,\sigma^+]],\Phi)\in\mathrm{semicart}^{m,\alpha}_{\mathrm{S}}(\Gamma):\sigma^+(a)=0\big\}<+\infty;\end{aligned}\tag{3.1}$$

we write $\Phi_h(t,s)=(t,\phi_h(t,s))$ for any $(t,s)\in[[\sigma_h^-,\sigma_h^+]]$ and for any h.
Since the sequences $(\sigma_h^\pm)$ are bounded in the $\mathcal{C}^{m,\alpha}$ seminorm and $\sigma_h^+(a)=0$ for every h, we get that $(\sigma_h^\pm)$ are equibounded and equicontinuous sequences. Thus, by Ascoli-Arzelà theorem, we have, up to subsequences, that

$$\sigma_h^\pm\to\sigma^\pm,\qquad\text{uniformly as } h\to+\infty.$$

Moreover, if $(m,\alpha)=(1,\lambda)$, also

$$\dot\sigma_h^\pm\to\dot\sigma^\pm,\qquad\text{uniformly as } h\to+\infty,$$

since $\mathcal{C}^{1,\lambda}([a,b])$ is compactly embedded into $\mathcal{C}^1([a,b])$ ([2, Theorem 1.34]). Thus, in any case, the limit functions satisfy:

$$\begin{aligned}&\sigma^\pm\in\mathcal{C}^{m,\alpha}([a,b]),\\&||\sigma^\pm||_{m,\alpha}\le \mathrm{S},\\&\sigma^-(a)=\sigma^+(a)=0,\\&\sigma^-(b)=\sigma^+(b)\end{aligned}$$

In order to guarantee that $[[\sigma^-,\sigma^+]]$ is (m,α) S-admissible, we need to check that

$$\sigma^-(t)<\sigma^+(t),\quad t\in(a,b).\tag{3.2}$$

Suppose by contradiction that there exists $t_0 \in (a,b)$ such that $\sigma^-(t_0) = \sigma^+(t_0)$. The uniform convergence of $(\sigma_h^\pm)$ to $(\sigma^\pm)$ implies in particular that

$$\lim_{h\to+\infty} (\sigma_h^-(t_0) - \sigma_h^+(t_0)) = 0.$$

Let us select $\delta > 0$ such that

$$a < t_0 - \delta < t_0 + \delta < b.$$

Using the Cauchy-Schwarz's inequality and the Dirichlet condition $\phi_h(t, \sigma_h^\pm(t)) = \gamma^\pm(t)$, we have

$$\begin{aligned}
\mathrm{Dir}(\Phi_h, [[\sigma_h^-, \sigma_h^+]]) \geq & \frac{1}{2} \int_{t_0-\delta}^{t_0+\delta} \left(\int_{\sigma_h^-(t)}^{\sigma_h^+(t)} |\partial_s \phi_h(t,s)|^2 \, ds \right) dt \\
\geq & \frac{1}{2} \int_{t_0-\delta}^{t_0+\delta} \frac{1}{|\sigma_h^+(t) - \sigma_h^-(t)|} \left(\int_{\sigma_h^-(t)}^{\sigma_h^+(t)} |\partial_s \phi_h(t,s)| \, ds \right)^2 dt \\
\geq & \frac{1}{2} \int_{t_0-\delta}^{t_0+\delta} \frac{|\gamma^+(t) - \gamma^-(t)|^2}{|\sigma_h^+(t) - \sigma_h^-(t)|} dt \\
\geq & \frac{C}{2} \int_{t_0-\delta}^{t_0+\delta} \frac{1}{|\sigma_h^+(t) - \sigma_h^-(t)|} dt \\
\geq & \frac{C}{2} \int_{t_0-\delta}^{t_0+\delta} \frac{1}{2\mathrm{S}|t - t_0| + |\sigma_h^+(t_0) - \sigma_h^-(t_0)|} dt,
\end{aligned}$$

where the constant $C = C(t_0, \delta) \in (0, +\infty)$ is found by using the third condition on $\gamma^\pm$ in Definition 2.1, and the last inequality follows from the bound $||\sigma_h^\pm||_{m,\alpha} \leq \mathrm{S}$.
Since the right-hand side blows up as $h \to +\infty$, while the left-hand side is uniformly bounded with respect to h, from (3.1), we get a contradiction, and this proves (3.2).
We conclude that $[[\sigma^-, \sigma^+]]$ is a (m, α) S-admissible domain.
Now we have to look for a limit semicartesian map Φ defined in $[[\sigma^-, \sigma^+]]$. Without loss of generality, following an argument similar to [10, Theorem 1, section 4.3], we can suppose that ϕ_h is harmonic in its domain $[[\sigma_h^-, \sigma_h^+]]$ for any $h \in \mathbb{N}$. Indeed, if not, we can replace it with the unique solution of the system

$$\begin{cases} \Delta \psi_h = 0 & \text{in } [[\sigma_h^-, \sigma_h^+]], \\ \psi_h(t, \sigma_h^\pm(t)) = \gamma^\pm(t) & t \in [a,b], \end{cases}$$

which minimizes the Dirichlet functional among all functions with the same trace. We observe also that $[[\sigma_h^-, \sigma_h^+]]$ satisfies the exterior cone condition, hence $\psi_h \in \mathcal{C}\left(\overline{[[\sigma_h^-, \sigma_h^+]]}; \mathbb{R}^2\right) \cap \mathcal{C}^2([[\sigma_h^-, \sigma_h^+]]; \mathbb{R}^2)$, see [11, Theorem 6.13 and Problem 6.3].
Now, let $R := (a,b) \times (-M, M)$ for M large enough such that $[[\sigma_h^-, \sigma_h^+]] \subseteq R$ for all h, and $[[\sigma^-, \sigma^+]] \subseteq R$. We extend ϕ_h, and consequently also Φ_h, defining

$$\begin{aligned} \phi_h(t,s) &:= \gamma^-(t), \quad (t,s) \in R \cap \{s < \sigma_h^-(t)\}, \\ \phi_h(t,s) &:= \gamma^+(t), \quad (t,s) \in R \cap \{s > \sigma_h^+(t)\}. \end{aligned} \tag{3.3}$$

Since $\mathrm{Dir}(\Phi_h, R) \le C$ for some costant $C > 0$ independent of h, using also a Poincaré's type inequality (see [10, Theorem 1, section 4.6, p. 277]), we get that (ϕ_h) is a bounded sequence in $H^1(R;\mathbb{R}^2)$. We can then extract a (not relabeled) subsequence converging weakly to some $\phi \in H^1(R;\mathbb{R}^2)$. We observe also that the convergence is pointwise, due to the regularity of ϕ_h and that the restriction of ϕ to $[[\sigma^-,\sigma^+]]$ is harmonic. We define $\Phi(t,s) := (t, \phi(t,s))$ for any $(t,s) \in R$. In order to verify that the pair $([[\sigma^-,\sigma^+]],\Phi)$ is a semicartesian parametrization, we have to guarantee that $\Phi(t,\sigma^\pm(t)) = (t,\gamma^\pm(t))$ (for almost all $t \in (a,b)$). This is evident since, if we pass to the limit in (3.3), we get

$$\phi(t,s) := \gamma^-(t), \quad (t,s) \in R \cap \{s < \sigma^-(t)\}.$$
$$\phi(t,s) := \gamma^+(t), \quad (t,s) \in R \cap \{s > \sigma^+(t)\}.$$

We recall that the Dirichlet functional $\mathrm{Dir}(\cdot, R)$ is lower semicontinuous with respect to the weak convergence in $H^1(R;\mathbb{R}^3)$ (see [10, Theorem 1, section 4.6]). Thus

$$\mathrm{Dir}(\Phi, R) \le \lim_{h\to+\infty} \mathrm{Dir}(\Phi_h, R).$$

Noticing also that, since $\gamma^\pm$ are at least Lipschitz and $\sigma_h^\pm$ are equiLipschitz, we have

$$\lim_{h\to+\infty} \mathrm{Dir}(\Phi_h, R \setminus [[\sigma_h^-,\sigma_h^+]]) = \mathrm{Dir}(\Phi, R \setminus [[\sigma^-,\sigma^+]]),$$

and thus

$$\mathrm{Dir}(\Phi, [[\sigma^-,\sigma^+]]) \le \lim_{h\to+\infty} \mathrm{Dir}(\Phi_h, [[\sigma_h^-,\sigma_h^+]]).$$

Hence, $([[\sigma^-,\sigma^+]],\Phi)$ is a minimizer for (2.7), and Φ is harmonic in $[[\sigma^-,\sigma^+]]$ and continuous(3) on $\overline{[[\sigma^-,\sigma^+]]}$. □

Some interesting comments on the regularity of Φ_{S} are in order.

Remark 3.1. If $(m,\alpha) = (1,\lambda)$ the solution $([[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]],\Phi_{\mathrm{S}})$ satisfies

$$\phi_{\mathrm{S}} \in \mathcal{C}^{1,\alpha}([[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]] \cup \{(t,\sigma_{\mathrm{S}}^\pm(t))\}_{t\in(a,b)};\mathbb{R}^2).$$

The result is obtained applying the following regularity result with $T = \{(t,\sigma_{\mathrm{S}}^\pm(t)) : t \in (a,b)\}$.

Theorem 3.2 (**[11, Corollary 8.36]**)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, open, and connected set. Let $T \subset \partial\Omega$ be a (possibly empty) $\mathcal{C}^{1,\lambda}$ boundary portion of $\partial\Omega$, and let $g \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$. Then the weak solution u of*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

belongs to $\mathcal{C}^{1,\lambda}(\Omega \cup T)$.

The same result does not hold if T and g are merely Lipschitz; namely if Ω is a Lipschitz domain, $g \in C^{0,1}(\overline{\Omega})$, then the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

(3)Still by [11, Theorem 6.13 and Problem 6.3].

is not necessarily Lipschitz on $\overline{\Omega}$. We recall, for the sake of completeness an example, that can be found in [13]. Let

$$\Omega := \{(r\cos\theta, r\sin\theta) : \, r > 0, \, 0 < \theta < \omega\} \cap U,$$

where U is an open and bounded domain with smooth boundary and strictly containing the origin of the plane, and ω is a fixed angle. For any k we define

$$u_k = \begin{cases} r^{k\pi/\omega} \sin(k\pi\theta/\omega) & \text{if } k\pi/\omega \text{ is not an integer,} \\ r^{k\pi/\omega}[\ln r \sin(k\pi\theta/\omega) + \theta\cos(k\pi\theta/\omega)] & \text{otherwise .} \end{cases}$$

The map u_k is harmonic and coincides with a smooth function on $\partial\Omega$, but it belongs to the Sobolev space $W^{l,p}$ if and only if $l - 2/p < k\pi/\omega$. Thus, if we choose $k = 1, \omega > \pi$, we find an example of an harmonic function on a Lipschiz open set, with Lipschitz boundary datum that is not Lipschitz regular up to the boundary.
We recall also that we cannot guarantee Lipschitzianity up to the boundary even for graph-type minimal surfaces on a Lipschitz domain Ω with Lipschitz boundary datum g, that is solutions of

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega; \end{cases}$$

see [12] for more (an example can be found also in [4]).
However, these observations do not exclude that, for $(m, \alpha) = (0, 1)$, Φ_S may belong to $\mathcal{C}^{0,1}\left(\overline{[[\sigma_{\mathrm{S}}^-, \sigma_{\mathrm{S}}^+]]}; \mathbb{R}^2\right)$, because the minimum problem in (2.7) is on the *pair* $([[\sigma^-, \sigma^+]], \Phi)$ and not only on the functions with a fixed domain. This larger degree of freedom makes possible to hope for a global Lipschitzianity of Φ (which the assumption we make in (1.5)).

4 Conformality

In this section we prove Theorems 2.9 and 2.10. Proposition 4.1 proves assertion *(i)* of Theorem 2.10, Lemma 4.3 shows that conformality implies that the domain is indeed $(0, 1)$ G-admissible, and Proposition 4.4 shows assertion *(ii)* of Theorem 2.10.
The proof of Theorem 2.9 follows almost the same lines of the proof of Proposition 4.1 and on some cases the computations simplify.

Proposition 4.1 (Conformality of strictly admissible and Lipschitz minimizers in $\mathbf{semicart}^{0,1}_{\mathbf{S}}(\boldsymbol{\Gamma})$). *Let* $\mathrm{G} > 0$ *and let* Γ *be union of two* G*-Lipschitz graphs. Fix* $\mathrm{S} > 0$ *and suppose that* $([[\sigma_{\mathrm{S}}^-, \sigma_{\mathrm{S}}^+]], \Phi_{\mathrm{S}})$ *is a solution of* (2.7) *with* $\Phi_{\mathrm{S}} \in \mathcal{C}^{0,1}\left(\overline{[[\sigma_{\mathrm{S}}^-, \sigma_{\mathrm{S}}^+]]}; \mathbb{R}^3\right)$*. Suppose moreover that* $[[\sigma_{\mathrm{S}}^-, \sigma_{\mathrm{S}}^+]]$ *is* S*-strictly admissible. Then* Φ_{S} *is conformal.*

Proof. Again for simplicity we denote the minimum by $([[\sigma^-, \sigma^+]], \Phi)$ and we also set $D = [[\sigma^-, \sigma^+]]$. Let $\mu \in \mathrm{Lip}(\mathbb{R}^2) \cap \mathcal{C}^1(D)$(4); following [10], with the difference that they consider

(4)Equivalently we can consider $\mu \in \mathrm{Lip}(D) \cap \mathcal{C}^1(D)$; indeed every Lipschitz function on a Lipschitz domain admits a Lipschitz extension.

$\mathcal{C}^1$ regular vector field with values in $\mathbb{R}^2$, we take an *internal* variation of the form

$$\begin{aligned} &T_\varepsilon : D \mapsto \mathbb{R}^2 \\ &(t,s) \in D \mapsto (t, \sigma_\varepsilon(t,s)), \qquad \text{with } \ \sigma_\varepsilon(t,s) := s - \varepsilon\mu(t,s), \end{aligned} \tag{4.1}$$

for $(t,s) \in D$ and $\varepsilon \in \mathbb{R}$. If $|\varepsilon| < \frac{1}{2||\mu_s||_{L^\infty(D)}}$, this map is invertible and the inverse $s_\varepsilon(t,\sigma)$ is Lipschitz, by [7, Theorem 1], on the deformed domain

$$D_\varepsilon := \{(t,\sigma) = (t, \sigma_\varepsilon(t,s)) : \ (t,s) \in D\}.$$

We observe that, since $\mu \in \mathrm{Lip}(D)$, the functions

$$\sigma_\varepsilon^\pm(t) := \sigma_\varepsilon(t, \sigma^\pm(t)) \tag{4.2}$$

are Lipschitz; moreover, the strict S-admissibility of D entails that there exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ the deformed domain D_ε is also S-admissible. We define the function Φ_ε on the deformed domain D_ε as

$$\Phi_\varepsilon(t,\sigma) := \Phi(t, s_\varepsilon(t,\sigma)), \quad (t,\sigma) \in D_\varepsilon,$$

and, for $|\varepsilon| < \varepsilon_0$, we compute

$$F(\varepsilon) := \mathrm{Dir}(\Phi_\varepsilon, D_\varepsilon) = \frac{1}{2}\int_{D_\varepsilon} |\nabla(\Phi\,(t, s_\varepsilon(t,\sigma)))|^2 \, dt\, d\sigma.$$

Since $\sigma_\varepsilon(t, s_\varepsilon(t,\sigma)) = \sigma$, using (4.1),

$$s_\varepsilon(t,\sigma) - \varepsilon\mu(t, s_\varepsilon(t,\sigma)) = \sigma;$$

differentiating with respect to t and σ (see [3, Theorem 3.101] for a chain rule for Lipschitz functions) we get for almost every $(\tau,\sigma) \in D_\varepsilon$,

$$\begin{aligned} s_{\varepsilon,t}(t,\sigma) &= \frac{\varepsilon\mu_t(t, s_\varepsilon(t,\sigma))}{1 - \varepsilon\mu_s(t, s_\varepsilon(t,\sigma))}, \\ s_{\varepsilon,\sigma}(t,\sigma) &= \frac{1}{1 - \varepsilon\mu_s(t, s_\varepsilon(t,\sigma))}. \end{aligned}$$

If $(t,\sigma) \in D \cap D_\varepsilon$, recalling that $\mu \in \mathcal{C}^1(D)$, we get also

$$\begin{aligned} s_{\varepsilon,t}(t,\sigma) &= \varepsilon\mu_t(t,\sigma) + o(\varepsilon) \qquad \text{as } \varepsilon \to 0, \\ s_{\varepsilon,\sigma}(t,\sigma) &= 1 + \varepsilon\mu_s(t,\sigma) + o(\varepsilon) \qquad \text{as } \varepsilon \to 0. \end{aligned} \tag{4.3}$$

We have:

$$\begin{aligned} \frac{d}{d\varepsilon}F(\varepsilon)_{|\varepsilon=0} &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[\mathrm{Dir}(\Phi_\varepsilon, D_\varepsilon) - \mathrm{Dir}(\Phi, D)\right] \\ &= \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\left\{\int_{D_\varepsilon} |\nabla\Phi_\varepsilon|^2 \, dt\, d\sigma - \int_D |\nabla\Phi|^2 \, dt\, ds\right\} \end{aligned} \tag{4.4}$$

For almost every $(t,\sigma) \in D_\varepsilon$, the derivatives of Φ_ε are given by:

$$\begin{aligned}\Phi_{\varepsilon,t}(t,\sigma) &= \Phi_t(t, s_\varepsilon(t,\sigma)) + \Phi_s(t, s_\varepsilon(t,\sigma)) s_{\varepsilon,t}(t,\sigma),\\ \Phi_{\varepsilon,\sigma}(t,\sigma) &= \Phi_s(t, s_\varepsilon(t,\sigma)) s_{\varepsilon,\sigma}(t,\sigma).\end{aligned}$$

Thus, recalling also that the absolute value of the determinant of the Jacobian of the change of variables T_ε is given by

$$\begin{vmatrix} 1 & 0 \\ -\varepsilon\mu_t & 1-\varepsilon\mu_s \end{vmatrix} = 1-\varepsilon\mu_s,$$

the limit in (4.4) becomes:

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_D \Big\{ \big[|\Phi_t|^2 + |\Phi_s|^2 (s^2_{\varepsilon,t} + s^2_{\varepsilon,\sigma}) + 2 s_{\varepsilon,t} \Phi_t\cdot\Phi_s\big](1-\varepsilon\mu_s) - |\Phi_t|^2 - |\Phi_s|^2 \Big\}\, dt\, ds, \tag{4.5}$$

Now, recalling that for $(t,\sigma) \in D_\varepsilon \cap D$ we have (4.3) for the derivatives of s_ε, it is convenient to compute the integral on $D = T_\varepsilon^{-1}(D_\varepsilon)$ as the contribution of the integral on $T_\varepsilon^{-1}(D_\varepsilon \cap D)$ and on $T_\varepsilon^{-1}(D_\varepsilon \setminus D)$ separately.
On $T_\varepsilon^{-1}(D_\varepsilon \cap D)$ we have

$$|\Phi_t|^2 + |\Phi_s|^2(s^2_{\varepsilon,t} + s^2_{\varepsilon,\sigma}) + 2 s_{\varepsilon,t}\Phi_t\cdot\Phi_s = 1 + |\phi_t|^2 + |\phi_s|^2 + 2\varepsilon[\mu_t \phi_t\cdot\phi_s + \mu_s|\phi_s|^2] + o(\varepsilon);$$

thus (4.5) on $T_\varepsilon^{-1}(D_\varepsilon \cap D)$ is:

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{T_\varepsilon^{-1}(D_\varepsilon\cap D)} &\big\{\big[1+|\phi_t|^2+|\phi_s|^2+2\varepsilon(\mu_t\phi_t\cdot\phi_s+\mu_s|\phi_s|^2)+o(\varepsilon)\big](1-\varepsilon\mu_s)\\ &\qquad\qquad - (1+|\phi_t|^2+|\phi_s|^2)\big\}\, dt\, ds\\ &= \lim_{\varepsilon\to 0}\int_{T_\varepsilon^{-1}(D_\varepsilon\cap D)} [(\mu_t\phi_t\cdot\phi_s + \mu_s|\phi_s|^2) - \mu_s(1+|\phi_t|^2+|\phi_s|^2) + o(\varepsilon)]\, dt\, ds\\ &= \int_D [A\mu_t + B\mu_s]\, dt\, ds,\end{aligned} \tag{4.6}$$

where

$$A = \phi_t\cdot\phi_s, \qquad B = \frac{1}{2}\left(|\phi_s|^2 - |\phi_t|^2 - 1\right) \tag{4.7}$$

are harmonic and bounded in D[(5)]. On the other hand, the integral in (4.5) on $T_\varepsilon^{-1}(D_\varepsilon \setminus D)$ is asymptotically negligible. Indeed we have:

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{T_\varepsilon^{-1}(D_\varepsilon\cap D)} \frac{1}{2\varepsilon}&\left\{|\phi_s|^2\left(\frac{\varepsilon^2\mu_t^2+1}{(1-\varepsilon\mu_s)^2} - 1\right) + 2\varepsilon\frac{\mu_t}{1-\varepsilon\mu_s}\phi_t\cdot\phi_s\right.\\ &\left. -\,\varepsilon\mu_s\left[|\phi_t|^2 + |\phi_s|^2\frac{\varepsilon^2\mu_t^2+1}{(1-\varepsilon\mu_s)^2} + 2\varepsilon\frac{\mu_t}{1-\varepsilon\mu_s}\phi_t\cdot\phi_s\right]\right\} dt\, ds = 0,\end{aligned}$$

since the integrand is bounded while the measure of the domain of integration tends to 0 as $\varepsilon \to 0$. Thus, using the minimality of (D,Φ) we have:

$$0 = \frac{d}{d\varepsilon}F(\varepsilon) = \int_D [A\mu_t + B\mu_s]\, dtds. \tag{4.8}$$

[(5)] Please note that our A, B correspond to a and b of [10, section 4.5] through $a = -2B$, $b = 2A$.

with (A, B) defined in (4.7). The vector field (A, B) is irrotational in D (and also divergence free), because

$$-B_t + A_s = -\phi_s \cdot \phi_{ts} + \phi_t \cdot \phi_{tt} + \phi_t \cdot \phi_{ss} + \phi_s \cdot \phi_{ts} = \phi_t \cdot \Delta\phi = 0$$

where the last equality follows from the harmonicity of ϕ (see (2.12)). Thus, since D is simply connected, there exists $f \in \mathcal{C}^1(D)$ such that

$$\nabla f = (A, B);$$

since (A, B) is bounded, we can extend f up to ∂D and $f \in \text{Lip}(D)$. Thus we can choose

$$\mu = f,$$

and get from (4.8) that

$$\int_D (A^2 + B^2)\, dtds = 0;$$

this implies $A = 0 = B$, that is, ϕ is conformal. $\square$

Remark 4.2. In [10], the authors are able to prove conformality of critical points of the Dirichlet functional in the class of the immersions of the disk mapping the boundary monotonically onto a given simple curve in $\mathbb{R}^3$ without any assumption on the boundary regularity. We cannot use the same techniques due to the greater rigidity of the semicartesian setting.

The following lemma provides a bound on the Lipschitz constant of the functions defining the domain of a semicartesian parametrization, under conformality hypothesis.

Lemma 4.3. *Let* $\mathrm{G} > 0$ *and* Γ *be union of two* G*-Lipschitz graphs. Let* $([[\rho^-, \rho^+]], \Psi) \in \text{semicart}^{0,1}_{\mathrm{S}}(\Gamma)$ *for some* $\mathrm{S} > 0$*. Suppose that* $\Psi \in \mathcal{C}^{0,1}\left(\overline{[[\rho^-, \rho^+]]}; \mathbb{R}^3\right)$ *and that it is conformal, that is for almost every* $(t, s) \in [[\rho^-, \rho^+]]$

$$\psi_t \cdot \psi_s = 0 \quad \text{and} \quad 1 + |\psi_t|^2 = |\psi_s|^2, \tag{4.9}$$

where ψ *is such that* $\Psi(t, s) = (t, \psi(t, s))$*. Then* $||\rho^\pm||_{0,1} \le \mathrm{G}$.

Proof. From the equality

$$\gamma^\pm(t) = \psi(t, \rho^\pm(t)),$$

using the conformality relations (4.9) and differentiating , we get

$$\dot\gamma^\pm(t) = \psi_t(t, \rho^\pm(t)) + \psi_s(t, \rho^\pm(t))\dot\rho^\pm(t), \quad \text{a.e. } t \in [a, b].$$

Hence, taking the squared norm and using the conformality relations,

$$\begin{aligned} |\dot\gamma^\pm(t)|^2 &= |\psi_t(t, \rho^\pm(t))|^2 + |\psi_s(t, \rho^\pm(t))|^2 \dot\rho^\pm(t)^2 \\ &= \dot\rho^\pm(t)^2 + |\psi_t(t, \dot\rho^\pm(t))|^2 (1 + \dot\rho^\pm(t)^2) \\ &\ge \dot\rho^\pm(t)^2, \end{aligned}$$

for almost every $t \in [a, b]$. $\square$

Proposition 4.1 and Lemma 4.3 imply that if a solution $([[\sigma^-,\sigma^+]],\Phi)$ of (2.7) is such that $\Phi \in \mathcal{C}^{0,1}\left([[\sigma^-,\sigma^+]];\mathbb{R}^3\right)$ and $||\sigma^\pm||_{0,1} < \mathrm{S}$, then

$$([[\sigma^-,\sigma^+]],\Phi) \in \mathrm{semicart}^{0,1}_{\mathrm{G}}(\Gamma).$$

In the next proposition, we prove that the same result holds without supposing a priori that $||\sigma^\pm||_{0,1} < \mathrm{S}$.

Proposition 4.4 (**Conformality of Lipschitz minimizers**). *Let* Γ *be union of two* G-*Lipschitz graphs. Let* $([[\sigma^-_{\mathrm{S}},\sigma^+_{\mathrm{S}}]],\Phi_{\mathrm{S}})$ *be a solution of* (2.7) *with* $\Phi_{\mathrm{S}} \in \mathcal{C}^{0,1}\left(\overline{[[\sigma^-_{\mathrm{S}},\sigma^+_{\mathrm{S}}]];\mathbb{R}^3}\right)$. *Then*

$$||\sigma^\pm||_{0,1} \leq \mathrm{G}.$$

In particular, $[[\sigma^-_{\mathrm{S}},\sigma^+_{\mathrm{S}}]]$ *is* S-*strictly admissible for any* $\mathrm{S} > \mathrm{G}$.

Proof. Again the minimum is denoted simply by $([[\sigma^-,\sigma^+],\Phi])$. Suppose by contradiction that one of the two Lipschitz functions σ^+,σ^- has a Lipschitz constant strictly larger than G. From Lemma 4.3 it follows that Φ is not conformal, hence

$$\int_{[[\sigma^-,\sigma^+]]} (A^2+B^2)\,dt\,ds > 0, \tag{4.10}$$

where A and B are defined in (4.7). In the following, we will employ the notation already used in the proof of Proposition 4.1. The aim is to find a deformation of $[[\sigma^-,\sigma^+]]$ that does not increase the Lipschitz constants of $\sigma^\pm$ and that strictly decreases the value of the Dirichlet functional.
A natural choice is the variation defined in (4.1), with

$$\mu = -f,$$

so that $\nabla\mu = (-A,-B)$. Contrary to Proposition 4.1, in this case the minimum is not a strictly admissible parametrization[(6)]; thus we have to check that the parametrizations $(D_\varepsilon,\Phi_\varepsilon)$ obtained by inner variation with the field $\mu = -f$ are admissible at least for $\varepsilon > 0$ small enough. If this is the case, from (4.10), we would reach the inequality

$$\frac{d}{d\varepsilon}F(\varepsilon)_{|\varepsilon=0^+} = -\int_{[[\sigma^-,\sigma^+]]} (A^2+B^2)\,dt\,ds < 0,$$

and thus a contradiction.
Thus the proof is reduced to prove that if $|\dot\sigma^+(t)| \geq M > \mathrm{G}$, then $|\dot\sigma^+_\varepsilon(t)| < |\dot\sigma^+(t)|$ (with $\sigma^\pm_\varepsilon$ defined as in (4.2)), or similar relations for $|\dot\sigma^-|$ and $|\dot\sigma^-_\varepsilon|$, guaranteeing that $D_\varepsilon = [[\sigma^-_\varepsilon,\sigma^+_\varepsilon]]$ is admissible for $\varepsilon > 0$ small enough. For simplicity we consider the case $\dot\sigma^+(t) > M$.
We have already observed that, due to the Lipschitzianity of ϕ, also f is Lipschitz. Thus the following computations are meaningful almost everywhere in $[a,b]$, where $\gamma^+(\cdot)$, $\sigma^+(\cdot)$ and $f(\cdot,\sigma^\pm(\cdot))$ are differentiable. Recalling (4.1), we get:

$$\dot\sigma^+_\varepsilon(t) = \dot\sigma^+(t) - \varepsilon\nabla\mu(t,\sigma^+(t))\cdot(1,\dot\sigma^+(t)) + o(\varepsilon).$$

[(6)]With (strictly) admissible parametrization we indicate a semicartesian parametrization whose domain is (strictly) admissible.

Let us prove that

$$\nabla\mu(t,\sigma^+(t))\cdot\tau^+(t)>0,$$

where $\tau^+(t)=(1,\dot\sigma^+)$ is the tangent vector to the graph of σ^+. Recalling that $\nabla\mu=(-A,-B)$, we get:

$$\nabla\mu\cdot\tau^+(t)=-A-\dot\sigma^+B=-\phi_t\cdot\phi_s-\frac{1}{2}\dot\sigma^+(|\phi_s|^2-|\phi_t|^2-1).$$

Differentiating $\phi(t,\sigma^+(t))=\gamma^+(t)$, we get $\phi_t=\dot\gamma^+-\dot\sigma^+\phi_s$; substituting it in the previous equation, we get

$$\begin{aligned}\nabla\mu\cdot\tau^+(t)=&-(\dot\gamma^+-\dot\sigma^+\phi_s)\cdot\phi_s-\frac{1}{2}\dot\sigma^+\left(|\phi_s|^2-(\dot\gamma^+-\dot\sigma^+\phi_s)\cdot(\dot\gamma^+-\dot\sigma^+\phi_s)-1\right)\\=&-\dot\gamma^+\cdot\phi_s+\frac{1}{2}\dot\sigma^+|\phi_s|^2+\frac{1}{2}\dot\sigma^+|\dot\gamma^+|^2+\frac{1}{2}(\dot\sigma^+)^3|\phi_s|^2-(\dot\sigma^+)^2\dot\gamma^+\cdot\phi_s+\frac{1}{2}\dot\sigma^+\\=&I+II+III+IV+V+VI.\end{aligned}$$

The first term can be bounded in absolute value using the Young inequality by

$$|I|=|\dot\gamma^+\cdot\phi_s|\le\frac{1}{2}(\dot\sigma^+)^{-1}|\dot\gamma^+|^2+\frac{1}{2}\dot\sigma^+|\phi_s|^2,$$

(recall that $\dot\sigma^+\ge M>0$). Similarly the fifth term can be bounded as

$$|V|=|(\dot\sigma^+)^2\dot\gamma^+\cdot\phi_s|\le\frac{1}{2}\dot\sigma^+|\dot\gamma^+|^2+\frac{1}{2}(\dot\sigma^+)^3|\phi_s|^2$$

so that

$$I+II\ge-\frac{1}{2}(\dot\sigma^+)^{-1}|\dot\gamma^+|^2$$

and

$$III+IV+V\ge 0.$$

Hence we obtain

$$\nabla\mu\cdot\tau^+(t)\ge-\frac{1}{2}(\dot\sigma^+)^{-1}|\dot\gamma^+|^2+\frac{1}{2}\dot\sigma^+\ge\frac{1}{2}(\dot\sigma^+)^{-1}\left((\dot\sigma^+)^2-|\dot\gamma^+|^2\right)>0,$$

where the last strict inequality holds since $\dot\sigma^+>M\ge|\dot\gamma^+|$.
Thus we have proven that the parametrizations $(D_\varepsilon,\Phi_\varepsilon)$ obtained through inner variation with the scalar field $\mu=-f$ are admissible, for $\varepsilon>0$ small enough, and thus we have found a contradiction. Hence every solution $([[\sigma^-,\sigma^+]],\Phi)$ of (2.7), for $(m,\alpha)=(0,1)$ such that $\phi\in\mathcal{C}^{0,1}\left(\overline{[[\sigma^-,\sigma^+]]};\mathbb{R}^2\right)$ satisfies

$$||\sigma^\pm||_{0,1}\le \mathrm{G};$$

moreover, for $\mathrm{S}>\mathrm{G}$, the solution is a strictly S−admissible parametrization and, applying Proposition 4.1, it is also conformal. Thus Theorem 2.10 is proven. □

Now we prove Theorem 2.9.

Proof of Theorem 2.9. The proof of Theorem 2.9 closely follows that of Proposition 4.1, but it is technically simpler due to the higher regularity assumptions. Indeed, from Remark 3.1 we have $\phi \in \mathcal{C}^{1,\lambda}(\widetilde{D};\mathbb{R}^2)$, where we $\widetilde{D}$ denotes the closure of the domain $[[\sigma^-, \sigma^+]]$ without the two angular points $(a, 0)$ and $(b, \sigma^-(b))$, that is

$$\widetilde{D} := [[\sigma^-, \sigma^+]] \cup \Big(\bigcup_{t\in(a,b)} (t, \sigma^\pm(t)) \Big).$$

We notice that in the case $(0, 1)$ we have not to pay attention to the two angular points, indeed the regularity of the functions $\sigma^\pm$ was the same as the regularity of $\partial[[\sigma^-, \sigma^+]]$. Anyway, since the inner deformation we are using is in some sense one-dimensional, we can overcome this difficulty by using scalar vector fields $\mu \in C^1((a, b) \times \mathbb{R})$ to deform the domain $D = [[\sigma^-, \sigma^+]]$, as in (4.1). Using the same notation as before, with this regularity assumption on μ and recalling that (D, Φ) is strictly $\mathcal{C}^{m,\alpha}$ S-admissible, we observe that $(D_\varepsilon, \Phi_\varepsilon)$ is $\mathcal{C}^{m,\alpha}$ S-admissible if $|\varepsilon|$ is small enough.
Denoting again by s_ε the inverse function of σ_ε, that now is a $\mathcal{C}^1$ function, we have that (4.3) holds true in the whole deformed domain D_ε; thus the computation in (4.6) is done in $T_\varepsilon^{-1}(D_\varepsilon) = D$, and we get again (4.8), with A and B defined in (4.7). Now, since $\phi \in \mathcal{C}^{1,\alpha}(\widetilde{D};\mathbb{R}^2)$, both A and B belong to $\mathcal{C}^{0,\alpha}(\widetilde{D})$ and thus the potential f such that $\nabla f = (A, B)$ can be chosen as scalar field, providing the desired conformality relations for ϕ. □

Remark 4.5. At this point it is difficult to give the analogous of Proposition 4.4 in the $(1, \lambda)$ case, because one should control not only the Lipschitz constant of $\sigma^\pm$, but also the $(0, \alpha)$ seminorm of the derivatives $\dot{\sigma}^\pm$.

5 Existence in $\mathrm{semicart}^{0,1}(\Gamma)$

Let us prove Theorem 2.11.

Proof of Theorem 2.11. Let us fix $\mathrm{S} > \mathrm{M} > \mathrm{G}$ (M is as in the statement of this Theorem) and let $([[\sigma_\mathrm{S}^-, \sigma_\mathrm{S}^+]], \Phi_\mathrm{S}) \in \mathrm{semicart}_\mathrm{S}^{0,1}(\Gamma)$ be the solution provided by Theorem 2.7; from the hypotheses and applying Theorem 2.10 we get that $([[\sigma_\mathrm{S}^-, \sigma_\mathrm{S}^+]], \Phi_\mathrm{S}) \in \mathrm{semicart}_\mathrm{G}^{0,1}(\Gamma)$. Let us suppose by contradiction that this semicartesian parametrization does not minimize the Dirichlet functional in $\mathrm{semicart}^{0,1}(\Gamma)$, that is let us suppose that there exists a semicartesian parametrization $([[\sigma_\mathrm{N}^-, \sigma_\mathrm{N}^+]], \Phi_\mathrm{N})$ belonging to $\mathrm{semicart}_\mathrm{N}^{0,1}(\Gamma)$ for some $\mathrm{N} > \mathrm{S}$ such that

$$\mathrm{Dir}([[\sigma_\mathrm{S}^-, \sigma_\mathrm{S}^+]], \Phi_\mathrm{S}) > \mathrm{Dir}([[\sigma_\mathrm{N}^-, \sigma_\mathrm{N}^+]], \Phi_\mathrm{N}).$$

From our hypotheses and Theorem 2.10, there exists also a parametrization

$$([[\sigma_{\mathrm{N,min}}^-, \sigma_{\mathrm{N,min}}^+]], \Phi_{\mathrm{N,min}}) \in \mathrm{semicart}_\mathrm{G}^{0,1}(\Gamma)$$

minimizing the Dirichlet functional in $\mathrm{semicart}_\mathrm{N}^{0,1}(\Gamma)$, and thus, in particular, in $\mathrm{semicart}_\mathrm{S}^{0,1}(\Gamma)$. Thus we get the contradictory inequalities chain

$$\mathrm{Dir}([[\sigma_\mathrm{S}^-, \sigma_\mathrm{S}^+]], \Phi_\mathrm{S}) > \mathrm{Dir}([[\sigma_\mathrm{N}^-, \sigma_\mathrm{N}^+]], \Phi_\mathrm{N}) \geq \mathrm{Dir}([[\sigma_{\mathrm{N,min}}^-, \sigma_{\mathrm{N,min}}^+]], \Phi_{\mathrm{N,min}}) \geq \mathrm{Dir}([[\sigma_\mathrm{S}^-, \sigma_\mathrm{S}^+]], \Phi_\mathrm{S}).$$

□

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