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# Global well-posedness of the magnetic Hartree equation with non-Strichartz external fields

**Alessandro Michelangeli**[1,2]

[1] Institute of Mathematics, Ludwig Maximilian University of Munich, Theresienstr. 39, 80333 Munich, Germany
[2] International School for Advanced Studies SISSA, via Bonomea 265, 34136 Trieste, Italy

E-mail: michel@math.lmu.de

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**Abstract**
We study the magnetic Hartree equation with external fields to which magnetic Strichartz estimates are not necessarily applicable. We characterise the appropriate notion of energy space and in such a space we prove the global well-posedness of the associated initial value problem by means of energy methods only.

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## 1. Introduction and main result

In this work we study the Cauchy problem for the non-linear Schrödinger equation

$$\mathrm{i}\partial_t u \;=\; -(\nabla - \mathrm{i}\mathbf{A})^2 u + Vu + (W * |u|^2)\, u \tag{1}$$

in the unknown $u(t,x)$, with $(t,x) \in \mathbb{R}\times\mathbb{R}^d$, $d \in \mathbb{N}$, for given measurable functions $\mathbf{A} : \mathbb{R}^d \to \mathbb{R}^d$, $V : \mathbb{R}^d \to \mathbb{R}$, and $W : \mathbb{R}^d \to \mathbb{R}$. The non-linearity

$$\mathcal{N}(u) \;:=\; (W * |u|^2)\, u \tag{2}$$

is non-local and is customarily referred to as the Hartree non-linearity and the functions $\mathbf{A}$ and $V$ are usually interpreted respectively as an external magnetic and an external electric field. For this reason we refer to (1) as *(electro-)magnetic Hartree equation*.

Among the other reasons of interest towards the magnetic Hartree equation, let us mention that in $d = 3$ space dimensions (1) is the effective evolution equation for a quantum many-body system of identical bosons (each of which is characterised by the one-body orbital $u(t,\cdot) \in L^2(\mathbb{R}^3)$ at time $t$) which are coupled by a two-body interaction $W(x_j - x_k)$ between particle $j$ and particle $k$ and which are further subject to an external magnetic field $\mathbf{A}$ and an external electric field $V$ ($V$ has as well the natural interpretation of a confining potential for the bosons). The magnetic Hartree equation emerges from the many-body *linear* Schrödinger equation in a suitable limit of infinitely many particles when the many-body Hamiltonian is re-scaled with a mean-field scaling (see the review [17] and the references therein): the cubic non-linearity (2) is indeed the effect of the mean-field scaling and is interpreted as a self-interaction of the particle in the orbital $u$ (its density being $|u|^2$).

As we shall discuss in the first part of this Introduction, our motivation is to study the well-posedness of the initial value problem associated with (1) in a regime of $\mathbf{A}$'s and $V$'s for which standard existence + uniqueness techniques based on magnetic Strichartz estimates are *not* applicable. In fact, magnetic Strichartz estimates are well known for classes of $\mathbf{A}$'s and $V$'s under certain restrictions on their regularity or on their behaviour at spatial infinity. In this work we shall rather discuss the well-posedness without Strichartz estimates, by means of tools that are standard in a way (energy methods plus diamagnetism) but whose application in this context allows one to consider $\mathbf{A}$'s and $V$'s that escape the Strichartz-restrictions. Let us stress that our interest is towards a Hartree equation with *singular coefficients*; from the point of view of the criticality, (1) is in any case a (energy) *sub-critical* non-linear Schrödinger equation.

Let us consider the initial value problem

$$\begin{cases} \mathrm{i}\partial_t u &= -(\nabla - \mathrm{i}\mathbf{A})^2 u + Vu + (W * |u|^2)\, u \\ u(0) &= \varphi\,. \end{cases} \tag{3}$$

In fact, when $\mathbf{A} \equiv 0$, (3) has already been extensively studied and its main features (well-posedness and scattering) are well understood (see, e.g. [5] and the references therein). In particular, the following is known ( [5], corollary 6.1.2):

**Theorem 1.1.** *Let $d \in \mathbb{N}$. Under the conditions*

*(i)* $\mathbf{A} \equiv 0$
*(ii)* $V : \mathbb{R}^d \to \mathbb{R}$, $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ *for some* $p \geqslant 1$, $p > \frac{d}{2}$
*(iii)* $W : \mathbb{R}^d \to \mathbb{R}$, *even,* $W \in L^r(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ *for some* $r \geqslant 1$, $r > \frac{d}{4}$
*(iv)* $W_- \in L^q(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ *for some* $q \geqslant 1$, $q \geqslant \frac{d}{2}$ *(and* $q > 1$ *if* $d = 2$*)*

*the initial value problem (3) is globally well-posed in $H^1(\mathbb{R}^d)$, meaning that (3) has a unique solution $u \in C(\mathbb{R}, H^1(\mathbb{R}^d))$, which is also in $L^\infty(\mathbb{R}, H^1(\mathbb{R}^d))$ and which depends continuously on the initial value in the sense that if $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^d)$ as $n \to \infty$, then the corresponding solutions $u_n$ and $u$ to (3) with initial data, respectively, $\varphi_n$ and $\varphi$, are such that $u_n \to u$ in $L^\infty(\mathbb{R}, H^1(\mathbb{R}^d))$. For such a $u$, the charge and the energy (given by (4) below with $\mathbf{A} \equiv 0$) are constant for all $t \in \mathbb{R}$*

In the special case $W(x) = \lambda |x|^{-\alpha}$, theorem 1.1 (plus some further analysis in the case of small initial data, see [5], corollary 6.1.5 and remark 6.8.2) implies the following:

**Corollary 1.2.** *Let $d \in \mathbb{N}$. Under the conditions*

*(i)* $\mathbf{A} \equiv 0$
*(ii)* $V : \mathbb{R}^d \to \mathbb{R}$, $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ *for some* $p \geqslant 1$, $p > \frac{d}{2}$
*(iii)* $W(x) = \lambda |x|^{-\alpha}$ *for some* $\lambda \geqslant 0$ *and* $0 \leqslant \alpha < \min\{d, 4\}$, *or for some* $\lambda < 0$ *and* $0 \leqslant \alpha < 2$

*the initial value problem (3) is globally well-posed in $H^1(\mathbb{R}^d)$, with uniformly bounded norm $\|u(t)\|_{H^1}$. Global well-posedness holds true also if in (iii) one has $\lambda < 0$ and $2 \leqslant \alpha < \min\{d, 4\}$, provided that $\|\varphi\|_{H^1}$ is sufficiently small.*

The proof of theorem 1.1 is based on a very standard scheme, where the two fundamental tools are the Strichartz estimates for the solutions to the free Schrödinger equation $i\partial_t u = -\Delta u$, which are needed to establish the existence and uniqueness of the solution to (3), with $\mathbf{A} \equiv 0$, locally in time (see, e.g. [5], section 4.2), and the conservation of charge and energy for the solutions to (3), which is needed to establish the actual existence globally in time ( [5], section 3.4). Alternatively, local existence can be proved first in a weak sense by means of charge and energy conservation only: the solution turns out then to be strong once uniqueness is proved (see, e.g. [5], section 3.3).

Within a natural modification of this scheme one can also exploit the dispersive properties of $-\Delta + V$ as a whole and hence use the Strichartz estimates, when existing, for the linear evolution $\mathrm{e}^{-\mathrm{i}t(-\Delta+V)}\varphi$, as long as the non-linearity is sub-critical with respect to the energy space of the differential operator $-\Delta + V$. For instance, Strichartz estimates for the scale-covariance elliptic operator $-\Delta + a|x|^{-2}$ in $d \geqslant 2$ space dimensions are known for $a$'s up to the Hardy threshold, $a > -\frac{1}{4}(d-2)^2$ [3]. Alternatively, the usual time-decay dispersive estimate $\|\mathrm{e}^{-\mathrm{i}t(-\Delta+a|x|^{-2})}\varphi\|_\infty \leqslant |t|^{-d/2}\|\varphi\|_1$ (from which Strichartz estimates follow in a standard way) are known in the special cases $d = 3$ [9] and $d = 2$ [10]. These results, in turn, allow to establish a well-posedness result analogous to that of corollary 1.2 but with potentials that are more singular than what prescribed by the assumptions therein (see [16, 21] for an explicit result when $d \geqslant 3$).

When this well-posedness scheme is exported to the case $\mathbf{A} \neq 0$, charge and energy of the solutions $u(t, x)$ to (3) take the formal expression

$$\begin{aligned} \mathcal{Q}[u] &= \|u\|_2^2 \\ \mathcal{E}[u] &= \int_{\mathbb{R}^d} \Big( \tfrac{1}{2}\,|(\nabla - \mathrm{i}\mathbf{A})u|^2 + \tfrac{1}{2}\,V|u|^2 + \tfrac{1}{4}\,(W * |u|^2)|u|^2 \Big) \end{aligned} \tag{4}$$

and one searches for solutions that are regular enough so as (4) are conserved quantities. As for the needed 'magnetic' Strichartz estimates, a first approach is to regard (1) as a non-linear Schrödinger equation with free linear part given by $-(\nabla - \mathrm{i}\mathbf{A})^2 u$ and non-linearity $Vu + (W * |u|^2)u$, in which case one needs Strichartz estimates for the free magnetic evolution $\mathrm{e}^{-\mathrm{i}t(\nabla-\mathrm{i}\mathbf{A})^2}$, in place of the analogous estimates for the free evolution $\mathrm{e}^{\mathrm{i}t\Delta}$. In this case $\mathbf{A}$ must be chosen so that $-(\nabla - \mathrm{i}\mathbf{A})^2$ has a self-adjoint realisation on $L^2(\mathbb{R}^d)$: the natural energy space is then the form domain of such an operator, namely the magnetic Sobolev space $H^1_{\mathbf{A}}(\mathbb{R}^d)$ ( [14], section 7.20), and the potentials $V$ and $W$ have to ensure that the non-linearity is indeed a map from $H^1_{\mathbf{A}}(\mathbb{R}^d)$ into its dual with good locally Lipschitz features. A second natural approach is to regard (1) as consisting of a linear part $hu$, where $h$ is a self-adjoint realisation of the operator $-(\nabla - \mathrm{i}\mathbf{A})^2 + V$ on $L^2(\mathbb{R}^d)$, and a non-linearity $(W * |u|^2)u$: in this case one needs Strichartz estimates for the group $\mathrm{e}^{-\mathrm{i}th}$ and a requirement on $W$ so to make the non-linearity a suitable map from the form domain of $h$ into its dual.

In either approach above, in order for the non-linearity to be controlled by the electromagnetic kinetic energy and to have suitable locally Lipschitz properties in $L^p$-spaces controlled by the energy space—precisely as conditions (iii) and (iv) of theorem 1.1 make the corresponding non-linearity well-behaved with respect to $H^1(\mathbb{R}^d)$—one then combines the standard Sobolev embedding $H^1(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ and the diamagnetic inequality

$$|\nabla|f|| \leqslant |(\nabla - \mathrm{i}\mathbf{A})f| \qquad \begin{array}{l} \text{for a.e. } x \in \mathbb{R}^d \\ \forall f \in H^1_{\mathbf{A}}(\mathbb{R}^d) \end{array} \tag{5}$$

valid whenever $\mathbf{A} \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d)$ ( [14], theorem 7.21). This way one comes to a very natural, Strichartz-based generalisation of theorem 1.1 to the case $\mathbf{A} \neq 0$. Although we are not aware of a general formulation to quote from the literature, we cite at least theorem 9.1.5 of [5] for a special case in $d = 3$ dimensions and a constant magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$.

The magnetic Strichartz estimates needed in the well-posedness scheme described so far are known for certain classes of $\mathbf{A}$'s and $V$'s that are reviewed in appendix A. They include smooth electric and magnetic potentials $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$ and $\mathbf{A} \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ with a growth at spatial infinity that is at most quadratic for $V$ and linear for $\mathbf{A}$ (local-in-time Strichartz estimates [15, 23]). Alternatively, they also include rough $\mathbf{A}$'s and $V$'s up to the critical scaling $|\mathbf{A}(x)| \sim |x|^{-1}$ and $|V(x)| \sim |x|^{-2}$ (global-in-time Strichartz estimates): in particular, a general class of fields with sub-critical scaling both locally and at spatial infinity is covered in [7] for $d \geqslant 3$, special cases of critical-scaling $\mathbf{A}$-fields are treated in [9] when $d = 2$ (we have already mentioned above the case of critical-scaling $V$-fields when $d \geqslant 2$), and counterexamples are known to the validity of Strichartz estimates for certain $\mathbf{A}$-fields when $d \geqslant 3$ and certain $V$-fields when $d \geqslant 3$ which decay as $|x| \to \infty$ less than the critical behaviour [11, 12]. We refer to appendix A for details.

Alternative to the route discussed so far, our motivation in the present work is, as announced at the beginning, to study the well-posedness of the initial value problem (3) also in regimes of coefficients $\mathbf{A}$'s and $V$'s for which magnetic Strichartz estimates are not available or are known to fail. To this aim we only apply energy methods in the natural energy space associated with (1), namely:

(a) we set up the appropriate energy space that identifies, as a form domain, the self-adjoint realisation of the map $u \mapsto -(\nabla - \mathrm{i}\mathbf{A})^2 u + Vu$ corresponding to the linear part of the Hartree equation,
(b) we prove local existence and uniqueness by means of a contraction argument based on the fact that the Hartree non-linearity $\mathcal{N}(u) = (W * |u|^2)u$ is locally Lipschitz in the energy space (no conservation rules are used in this step),
(c) we prove that this solution is global by means of the charge and energy conservation.

All the conditions on $\mathbf{A}$, $V$, and $W$ needed for such a programme are collected together for convenience here:

$$\begin{aligned}
& d \in \mathbb{N} \\
& \mathbf{A} \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d) \\
& V \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}) \\
& V_- \text{ is } \Delta\text{-form-bounded with relative bound } < 1 \\
& W \in L^{q_0}(\mathbb{R}^d, \mathbb{R}) + L^\infty(\mathbb{R}^d, \mathbb{R}) \\
& \nabla W \in L^{q_1}(\mathbb{R}^d, \mathbb{R}^d) + L^\infty(\mathbb{R}^d, \mathbb{R}^d) \\
& \text{for some } q_0 \geqslant \max\{1, \tfrac{d}{2}\} (q_0 > 1 \text{ if } d = 2) \\
& \text{and some } q_1 \geqslant \max\{1, \tfrac{d}{3}\} \\
& W \text{ is even}\,.
\end{aligned} \tag{6}$$

To start with part (a) of the above programme, let us introduce now the energy space $\mathcal{H}^1(\mathbb{R}^d)$ and collect its relevant properties in the following proposition.

**Definition 1.3.** Assume that $\mathbf{A}$ and $V$ satisfy conditions (6). The energy space for the magnetic Schrödinger operator with magnetic potential $\mathbf{A}$ and electric potential $V$ is

$$\mathcal{H}^1(\mathbb{R}^d) \;:=\; \left\{ f \in L^2(\mathbb{R}^d) \;\middle|\; \begin{matrix} (\nabla - \mathrm{i}\mathbf{A}) f \in L^2(\mathbb{R}^d) \\ V_+^{1/2} f \in L^2(\mathbb{R}^d) \end{matrix} \right\}. \tag{7}$$

$\mathcal{H}^1(\mathbb{R}^d)$ is equipped with the scalar product

$$\langle f,g\rangle'_{\mathcal{H}^1} \;:=\; \int_{\mathbb{R}^d}\Big(\overline{(\nabla-\mathrm{i}\mathbf{A})f}\cdot(\nabla-\mathrm{i}\mathbf{A})g \,+\, \overline{f}(V_+ +1)g\Big) \tag{8}$$

and with the associated norm

$$\|f\|'_{\mathcal{H}^1} \;:=\; \Big(\|(\nabla-\mathrm{i}\mathbf{A})f\|_2^2+\|\langle V_+^{1/2}\rangle f\|_2^2\Big)^{1/2}. \tag{9}$$

**Proposition 1.4.** *Assume that* $\mathbf{A}$ *and* $V$ *satisfy conditions (6). Then:*

*(i)* $(\mathcal{H}^1(\mathbb{R}^d),\langle\cdot,\cdot\rangle'_{\mathcal{H}^1})$ *is a Hilbert space and (up to an isomorphism) coincides with the completion of* $(C_0^\infty(\mathbb{R}^d),\|\cdot\|'_{\mathcal{H}^1})$.

*(ii)* $\mathcal{H}^1(\mathbb{R}^d)$ *is the form domain of a unique self-adjoint operator* $h$ *on* $L^2(\mathbb{R}^d)$ *whose quadratic form is*

$$\begin{aligned} h[f,g] &= \textstyle\int_{\mathbb{R}^d}\Big(\overline{(\nabla-\mathrm{i}\mathbf{A})f}\cdot(\nabla-\mathrm{i}\mathbf{A})g \,+\, \overline{f}\,V\,g\Big)\\ \mathcal{D}[h] &= \mathcal{H}^1(\mathbb{R}^d)\,. \end{aligned} \tag{10}$$

*One has*

$$\begin{aligned} \mathcal{D}(h) &= \Big\{f\in L^2(\mathbb{R}^d)\,|\,-(\nabla-\mathrm{i}\mathbf{A})^2f+Vf\in L^2(\mathbb{R}^d)\Big\}\\ hf &= -(\nabla-\mathrm{i}\mathbf{A})^2f+Vf. \end{aligned} \tag{11}$$

*In particular,* $C_0^\infty(\mathbb{R}^d)\subset\mathcal{D}(h)\subset\mathcal{H}^1(\mathbb{R}^d)$.

*(iii)* $h$ *is bounded below, and so is its quadratic form, with the same bound.*

*(iv) If* $-M\in\mathbb{R}$ *is any lower bound for* $h$*, then* $\mathcal{H}^1(\mathbb{R}^d)$ *is complete with respect to the norm*

$$\|f\|_{\mathcal{H}^1} \;:=\; \Big(\int_{\mathbb{R}^d}\big(|(\nabla-\mathrm{i}\mathbf{A})f|^2+V|f|^2+(1+M)|f|^2\big)\Big)^{1/2} \tag{12}$$

*and for* $|M|$ *large enough the two norms* $\|\cdot\|_{\mathcal{H}^1}$ *and* $\|\cdot\|'_{\mathcal{H}^1}$ *are equivalent:*

$$\|f\|_{\mathcal{H}^1} \;\approx\; \|f\|'_{\mathcal{H}^1} \qquad \forall f\in\mathcal{H}^1(\mathbb{R}^d)\,. \tag{13}$$

*(v) One has*

$$\|\mathrm{e}^{-\mathrm{i}th}f\|_{\mathcal{H}^1} \;=\; \|f\|_{\mathcal{H}^1} \qquad \forall f\in\mathcal{H}^1(\mathbb{R}^d)\,,\quad \forall t\in\mathbb{R}\,. \tag{14}$$

We shall prove proposition 1.4 within the more general discussion of section 2, where we carry on in parallel both the operator language and the quadratic form language. Let us only make here a few important remarks.

**Remark 1.5.** (i) Definition 1.3 and part (ii) of proposition 1.4 indicate that $\mathcal{H}^1(\mathbb{R}^d)$ is the natural space where to give meaning to the energy $\int_{\mathbb{R}^d}|(\nabla-\mathrm{i}\mathbf{A})f|^2+\int_{\mathbb{R}^d}V|f|^2$ of the magnetic Schrödinger operator. The underlying operator $h$ is in fact 'naturally unique' in the precise sense of proposition 2.3(iii) below, and what is fundamental for our energy arguments is that the $L^2$-unitary group associated to $h$ maps unitarily $\mathcal{H}^1(\mathbb{R}^d)$ into itself (part (v) of proposition 1.4), meaning that the solution to the linear ('non-interacting') part $\mathrm{i}\partial_t u=-(\nabla-\mathrm{i}\mathbf{A})^2u+Vu$ of the Hartree equation, whose initial datum has finite $\mathcal{H}^1$-energy, is unique and remains in the energy space at later times.

(ii) The boundedness from below of $h$ is precisely what conditions (6) on $V_-$ are designed for, and it is fundamental to introduce the natural energy norm $\|\cdot\|_{\mathcal{H}^1}$ in $\mathcal{H}^1(\mathbb{R}^d)$. In turn, the energy norm is central both to express the property that the Hartree non-linearity is locally Lipschitz in the energy space (part (b) of the above programme) and to extend the local solution globally in time (part (c)) by controlling the $\mathcal{H}^1$-norm in terms of the total energy and charge, two quantities that are conserved in time. The equivalent norm $\|\cdot\|'_{\mathcal{H}^1}$ is simply more manageable to prove the Lipschitz property. Whereas $\|\cdot\|'_{\mathcal{H}^1}$ makes sense irrespectively of $V_-$, we need suitable conditions on $V_-$ as in (6) in order to prove the norm equivalence (13).

(iii) Definition 1.3 and proposition 1.4 are consistent with these two special cases:

$$\begin{array}{c}\text{when } \mathbf{A}\equiv\mathbf{0},\ V\equiv 0\\ (\mathcal{H}^1(\mathbb{R}^d),\|\cdot\|_{\mathcal{H}^1})\ \cong\ (\mathcal{H}^1(\mathbb{R}^d),\|\cdot\|'_{\mathcal{H}^1})\ =\ (H^1(\mathbb{R}^d),\|\cdot\|_{H^1})\,,\end{array}\tag{15}$$

$$\begin{array}{c}\text{when } V\equiv 0\\ (\mathcal{H}^1(\mathbb{R}^d),\|\cdot\|_{\mathcal{H}^1})\ \cong\ (\mathcal{H}^1(\mathbb{R}^d),\|\cdot\|'_{\mathcal{H}^1})\ =\ (H^1_{\mathbf{A}}(\mathbb{R}^d),\|\cdot\|_{H^1_{\mathbf{A}}})\,.\end{array}\tag{16}$$

Introduced the energy space $\mathcal{H}^1(\mathbb{R}^d)$, we can now state our main result concerning the well-posedness of the Hartree equation.

**Theorem 1.6.** *Assume that* $\mathbf{A}$ *and* $V$ *satisfy conditions (6). Then:*

*(i) For any* $\varphi\in\mathcal{H}^1(\mathbb{R}^d)$ *there exists a unique solution to the initial value problem (3) in the space* $C(\mathbb{R},\mathcal{H}^1(\mathbb{R}^d))\cap C^1(\mathbb{R},\mathcal{H}^1(\mathbb{R}^d)^*)$ *with initial datum* $\varphi$ *at* $t=0$.

*(ii) For this solution one has*

$$\sup_{t\in\mathbb{R}}\|u(t)\|_{\mathcal{H}^1}\ <\ +\infty\,.\tag{17}$$

*(iii) For this solution one also has that the charge* $\mathcal{Q}[u]$ *and the energy* $\mathcal{E}[u]$ *defined by*

$$\begin{aligned}\mathcal{Q}[u]\ &:=\ \|u\|_2^2\\ \mathcal{E}[u]\ &:=\ \int_{\mathbb{R}^d}\Big(\tfrac{1}{2}\,|(\nabla-\mathrm{i}\mathbf{A})u|^2+\tfrac{1}{2}\,V|u|^2+\tfrac{1}{4}\,(W*|u|^2)|u|^2\Big)\end{aligned}\tag{18}$$

*are constant in* $t$.

*(iii) Let* $\varphi_n\to\varphi$ *in* $\mathcal{H}^1(\mathbb{R}^d)$ *as* $n\to\infty$ *and, correspondingly, according to part (i), let* $u_n\in C(\mathbb{R},\mathcal{H}^1(\mathbb{R}^d))\cap C^1(\mathbb{R},\mathcal{H}^1(\mathbb{R}^d)^*)$ *and* $u\in C(\mathbb{R},\mathcal{H}^1(\mathbb{R}^d))\cap C^1(\mathbb{R},\mathcal{H}^1(\mathbb{R}^d)^*)$ *be the unique solution to (3) with initial datum at* $t=0$ *given, respectively, by* $\varphi_n$ *and* $\varphi$*. Then* $u_n\to u$ *in* $L^\infty(\mathbb{R},\mathcal{H}^1(\mathbb{R}^d))$ *as* $n\to\infty$.

We shall split the proof of theorem 1.6 into a number of steps, namely proposition 4.1 in section 4 and proposition 5.1, corollary 5.2, and remark 5.3 in section 5.

A few important comments on theorem 1.6 are in order.

**Remark 1.7.** (i) As sketched in the first part of this Introduction, a result like theorem 1.6 can be obtained within the standard well-posedness scheme for semi-linear Schrödinger equations by means of magnetic Strichartz estimates. Our conditions (6), however, are considerably more general then those needed for the validity of magnetic Strichartz estimates (see appendix A for details).

(ii) When $\mathbf{A}\equiv 0$, the comparison with theorem 1.1 (that is based on Strichartz estimates for the free group $e^{it\Delta}$) shows that in order to prove global well-posedness with sole energy methods we need somewhat more restrictive conditions on $V_-$ and on $W$. More precisely, we have the same requirements on $W_-$ as in theorem 1.1, but additional restrictions on $\nabla W$.

(iii) Related to (ii), in the special case where $\mathbf{A}\equiv 0$ and $W(x)=\lambda|x|^{-\alpha}$, thus comparing theorem 1.6 with corollary 1.2, we see that we can cover the same range $\alpha\in[0,2)$ as in the approach via Strichartz estimates. The latter, in the case of *repulsive* non-linearity, i.e. $\lambda>0$, allows to cover the larger range $\alpha\in[0,\min\{d,4\})$, whereas the energy methods are clearly insensitive to the sign of $\lambda$.

(iv) Recently Cao [4] proved with energy methods theorem 1.6 (in fact, only part (i) explicitly) in the special case $W(x)=|x|^{-1}$ under implicit conditions on $\mathbf{A}$ and $V$ that ensure $-(\nabla-\mathrm{i}\mathbf{A})^2+V$ to be self-adjoint and positive, plus a further unnecessary restriction on the size of $V_-$. From this perspective our work generalises [4] and our proof highlights the fact that it is rather a suitable smallness of $V_-$ with respect to $-\Delta$, and hence the consequent semi-boundedness of the magnetic energy form, the correct assumption to make.

To conclude this Introduction, we collect here the notation we adopted, besides the standard one. For shortness we set $\nabla_{\mathbf{A}} \equiv (\nabla - \mathrm{i}\mathbf{A})$, $\langle z \rangle \equiv \sqrt{1+z^2}$, $V_\pm \equiv \max\{\pm V, 0\}$. Given an operator $T$ on a Hilbert space, $|T|$ and $\overline{T}$ are, respectively, the operator absolute value and the operator closure of $T$; $\mathcal{D}(T)$ is the domain of $T$. $T[\cdot,\cdot]$ is the quadratic form associated with a self-adjoint operator $T$, and $\mathcal{D}[T] \equiv \mathcal{D}(|T|^{1/2})$ is the its form domain; also, $T[f] \equiv T[f, f]$ (see, e.g. [18]). $\mathcal{H}^*$ denotes the topological dual of a Hilbert space $\mathcal{H}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H},\mathcal{H}^*}$ is the corresponding duality product.

## 2. Energy space and energy norm for the magnetic Schrödinger operator

In this Section we introduce and discuss the notion of energy space for the magnetic Schrödinger operator $-(\nabla - \mathrm{i}\mathbf{A})^2 + V$ which is appropriate for the initial value problem (3), revisiting also a number of facts from the literature. In particular, we prove proposition 1.4.

Whether one identifies the energy space through an operator language or a quadratic form language, this space is meant to be the *form domain* of a self-adjoint realisation of the operator

$$h_0 \;=\; -(\nabla - \mathrm{i}\mathbf{A})^2 + V \qquad \text{with domain } \; \mathcal{D}(h_0) = C_0^\infty(\mathbb{R}^d) \tag{19}$$

acting on the Hilbert space $L^2(\mathbb{R}^d)$ for given real-valued measurable functions $\mathbf{A} : \mathbb{R}^d \to \mathbb{R}^d$ and $V : \mathbb{R}^d \to \mathbb{R}$.

In the theory of Schrödinger operators, $h_0$ has been studied extensively under fairly general conditions of regularity and integrability on the coefficients $\mathbf{A}$ and $V$ (as, e.g. in the Leinfelder and Simander theorems, [13]). One typically regards $h_0$ as a 'distortion' of the Laplacian, namely $h_0 = -\Delta + 2\mathrm{i}\,\mathbf{A}\cdot\nabla + \mathrm{i}\,\mathrm{div}\mathbf{A} + \mathbf{A}^2 + V$, so that for its essential self-adjointness it is enough to require a suitable local integrability and regularity of $\mathbf{A}$ and size restrictions on $V_- := \max(-V, 0)$ analogous to those needed for the essential self-adjointness of $-\Delta + V$, typically a local singularity for $V_-$ that is small with respect to $-\Delta$ and a not too high growth of $V_-$ at spatial infinity; in contrast, no restrictions are needed on $V_+ := \max(V, 0)$, except for an amount of natural local integrability of $V$.

In this work we need to further constrain (the self-adjoint realisation of) $h_0$ to be bounded below, in order to apply the energy methods with which we prove the well-posedness study the initial value problem (3). This boils down to prevent $V_-$ to grow at spatial infinity, still allowing for a local singularity for $V_-$ that is suitably small with respect to $-\Delta$.

Moreover, in order to set up a contraction argument in the energy space and to express the solution to (3) by means of a Duhamel formula (see (51) below), we do not need strictly speaking to require $h_0$ to be essentially self-adjoint; we only need that the natural energy form for $-(\nabla - \mathrm{i}\mathbf{A})^2 + V$ (in fact, the quadratic form $q_{h_0}^{\max}$ introduced in proposition 2.3(i)) realises uniquely a self-adjoint operator $h$, which is of course a self-adjoint extension of $h_0$. Thus, we only need conditions on $\mathbf{A}$ and $V$ for the energy quadratic form associated to the magnetic Schrödinger operator to be closed and bounded below, namely it is enough to work in the framework of quadratic forms instead of the slightly more restrictive framework of the magnetic Schrödinger operator itself.

For the sake of completeness, however, throughout this Section we carry on in parallel both the operator and the quadratic form language. We introduce these two classes of conditions:

- *Conditions (C.1)*:

$$\begin{aligned} &\mathbf{A} \in L^4_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d), \ \mathrm{div}\mathbf{A} \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \\ &V \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}) \\ &V_- \text{ is } \Delta\text{-bounded with relative bound } a < 1 \end{aligned}$$

- *Conditions (C.2)*:

$$\begin{aligned} &\mathbf{A} \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d) \\ &V \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}) \\ &V_- \text{ is form-bounded w.r.t. to } -\Delta \text{ with relative bound } a < 1. \end{aligned}$$

Clearly (C.1)⇒(C.2), so in the rest of this work we shall assume the more general class (C.2).

We organise our discussion in the following two Propositions.

**Proposition 2.1.** *Let* $\mathbf{A}$ *and* $V$ *satisfy conditions (C.1). Correspondingly, let* $h_0$ *be the operator defined in (19).*

*(i)* $h_0$ *is essentially self-adjoint and* $\overline{h_0}$ *is (self-adjoint and) bounded below.*
*(ii)* $h_0$, $\overline{h_0}$, *and the quadratic form of* $\overline{h_0}$ *have the same largest lower bound. Denoting by* $-M \in \mathbb{R}$ *a lower bound of* $h_0$*, the form domain* $\mathcal{D}[\overline{h_0}]$ *is the completion of* $C^\infty_0(\mathbb{R}^d)$ *with respect to the norm* $\|f\|_{h_0} := (\langle f, h_0 f\rangle + (1+M)\|f\|_2^2)^{1/2}$.
*(iii)* $\mathcal{D}[\overline{h_0}]$ *is complete with respect to the norm*

$$\|f\|_{\overline{h_0}} := (\overline{h_0}[f] + (1+M)\|f\|_2^2)^{1/2}.$$

*(iv) One has*

$$\|\mathrm{e}^{-\mathrm{i}t\overline{h_0}} f\|_{\overline{h_0}} = \|f\|_{\overline{h_0}} \qquad \forall t \in \mathbb{R}. \tag{20}$$

*(v) For* $|M|$ *sufficiently large, the norm* $\|\cdot\|_{h_0}$ *on* $C^\infty_0(\mathbb{R}^d)$ *is equivalent to the norm*

$$\|f\|'_{\mathcal{H}^1} = \big(\|\nabla_{\mathbf{A}} f\|_2^2 + \|\langle V_+^{1/2}\rangle f\|_2^2\big)^{1/2};$$

*(vi) The space* $\mathcal{D}[\overline{h_0}]$ *is the completion of* $(C^\infty_0(\mathbb{R}^d), \|\cdot\|'_{\mathcal{H}^1})$.

**Proof.** Part (i) follows from a Leinfelder–Simander theorem ([13], theorem 3), owing to the assumptions (C.1). As for part (ii), since $h_0$ is symmetric and bounded below, then the quadratic form $(f,g) \mapsto \langle f, h_0 g\rangle$, $f,g \in C^\infty_0(\mathbb{R}^d)$, is closable and by standard arguments (see, e.g. [18], proposition 10.3) its closure is the form $q$ whose domain $\mathcal{D}(q)$ is the completion of $(C^\infty_0(\mathbb{R}^d), \|\cdot\|_{h_0})$ and whose action is given by $q[f,g] = \lim_{n\to\infty}\langle f_n, h_0 g_n\rangle$ for $f,g \in \mathcal{D}(q)$ and for any two Cauchy sequences $(f_n)_n$ and $(g_n)_n$ in $(C^\infty_0(\mathbb{R}^d), \|\cdot\|_{h_0})$ such that $f_n \to f$ and $g_n \to g$ in $L^2$. From this, and from the fact that $\overline{h_0} + M$ is self-adjoint and positive, we now show that $\mathcal{D}(q) = \mathcal{D}((\overline{h_0}+M)^{1/2})$. Indeed, $f \in \mathcal{D}(q)$ if and only if for some Cauchy sequence $(f_n)_n$ in $(C^\infty_0(\mathbb{R}^d), \|\cdot\|_{h_0})$ one has $f_n \to f$ in $L^2$, which is equivalent to the fact that for some $(f_n)_n$ in $C^\infty_0(\mathbb{R}^d)$ one has $\|f_n - f\|_2 \to 0$ and $\langle f_n - f_m, (h_0+M) f_n - f_m\rangle \to 0$ as $n,m \to \infty$, i.e. the sequence $(f_n)_n$ $L^2$-converges to $f$ and it is Cauchy in $C^\infty_0(\mathbb{R}^d)$ with respect to the norm $\|\phi\|^{\overline{h_0}} := (\|\phi\|_2^2 + \|(\overline{h_0}+M)^{1/2}\phi\|_2^2)^{1/2}$. In this norm $\mathcal{D}((\overline{h_0}+M)^{1/2})$ is closed, because $\overline{h_0}+M$ is a self-adjoint and hence closed operator, so the last condition on $(f_n)_n$ is equivalent to the fact that $f_n \to f$ in $L^2(\mathbb{R}^d)$ and $f_n \to g \in \mathcal{D}((\overline{h_0}+M)^{1/2})$ in the $\|\,\|^{\overline{h_0}}$-norm. Conclusion: $f \in \mathcal{D}(q) \Leftrightarrow f = g \in \mathcal{D}((\overline{h_0}+M)^{1/2})$, and the claim $\mathcal{D}(q) = \mathcal{D}((\overline{h_0}+M)^{1/2})$ is proved. From this last fact we conclude (see, e.g. [18], proposition 10.5(i) for the straightforward argument) $\mathcal{D}(q) = \mathcal{D}(|\overline{h_0}+M|^{1/2}) = \mathcal{D}(|\overline{h_0}|^{1/2}) = \mathcal{D}[\overline{h_0}]$. Part (iii) is equivalent to the claim that the quadratic form of $\overline{h_0}$ is closed and bounded below, which follows straightforwardly (see, e.g. [18], proposition 10.5(ii)) from the fact that $\overline{h_0}$ is self-adjoint and bounded below. Part (iv) follows from the fact that the self-adjointness of $\overline{h_0}$ implies $\overline{h_0}[\mathrm{e}^{-\mathrm{i}t\overline{h_0}}f] = \overline{h_0}[f]$ and $\|\mathrm{e}^{-\mathrm{i}t\overline{h_0}}f\|_2 = \|f\|_2$ for all $f \in \mathcal{D}[\overline{h_0}]$. As for part (v), let $a \in (0,1)$ and $b > 0$ be the two constants in terms of which the $\Delta$-boundedness of $V_-$ is expressed, namely, $\|V_- f\|_2 \leqslant a\|\Delta f\|_2 + b\|f\|_2$ for $f \in H^2(\mathbb{R}^d)$. Then $V_-$ is also form-bounded with respect to $-\Delta$ with the same relative bound $a$, whence

$$\int_{\mathbb{R}^d} V|f|^2 \mathrm{d}x \;=\; \int_{\mathbb{R}^d}(V_+ - V_-)|f|^2\mathrm{d}x \;\geqslant\; \|V_+^{1/2}f\|_2^2 - a\|\nabla f\|_2^2 - b'\|f\|_2^2$$

for some $b' > 0$ and for all $f \in H^1(\mathbb{R}^d)$. Since $f \in H^1(\mathbb{R}^d) \Rightarrow |f| \in H^1(\mathbb{R}^d)$, one also has

$$\int_{\mathbb{R}^d} V|f|^2 \mathrm{d}x \;\geqslant\; \|V_+^{1/2}f\|_2^2 - a\|\nabla|f|\|_2^2 - b'\|f\|_2^2$$

for any $f \in H^1(\mathbb{R}^d)$. Last inequality holds true in particular for any $f \in C_0^\infty(\mathbb{R}^d)$, and $C_0^\infty(\mathbb{R}^d) \subset H^1_{\mathbf{A}}(\mathbb{R}^d)$ under the conditions (C.1), so that the diamagnetic inequality (5) yields

$$\int_{\mathbb{R}^d} V|f|^2 \mathrm{d}x \;\geqslant\; \|V_+^{1/2}f\|_2^2 - a\|\nabla_{\mathbf{A}} f\|_2^2 - b'\|f\|_2^2\,, \qquad f \in C_0^\infty(\mathbb{R}^d)\,.$$

Therefore

$$\begin{aligned} \|f\|_{h_0}^2 \;&=\; \textstyle\int_{\mathbb{R}^d} \overline{f}\,\big(-(\nabla - \mathrm{i}\mathbf{A})^2 + V + M + 1\big)\,f\,\mathrm{d}x \\ &\geqslant\; (1-a)\|\nabla_{\mathbf{A}}f\|_2^2 + \|V_+^{1/2}f\|_2^2 + (1+M-b')\|f\|_2^2 \end{aligned}$$

and hence, choosing $M > 1 - b'$,

$$\|f\|_{h_0}^2 \;\gtrsim\; \|\nabla_{\mathbf{A}}f\|_2^2 + \|\langle V_+^{1/2}\rangle f\|_2^2\,, \qquad f \in C_0^\infty(\mathbb{R}^d)\,. \tag{21}$$

On the other hand,

$$\begin{aligned} \|f\|_{h_0}^2 \;&\leqslant\; \|\nabla_{\mathbf{A}}f\|_2^2 + \|V_+^{1/2}f\|_2^2 + |1+M|\,\|f\|_2^2 \\ &\lesssim\; \|f\|_2^2 + \|\langle V_+^{1/2}\rangle f\|_2^2\,, \qquad f \in C_0^\infty(\mathbb{R}^d)\,. \end{aligned} \tag{22}$$

(21) and (22) yield (v). Part (vi) follows from (ii) and (v). □

**Remark 2.2.** In the special case $\mathbf{A} \equiv \mathbf{0}$, $V \equiv 0$, proposition 2.1 above states the well-known facts that $-\Delta$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, its operator closure (the negative Laplacian on $H^2(\mathbb{R}^d)$) is positive, with form domain given by $H^1(\mathbb{R}^d)$, and the last space is the closure of $C_0^\infty(\mathbb{R}^d)$ in the $H^1$-norm. Similarly, in the special case $V \equiv 0$ proposition 2.1 above states that $-(\nabla - \mathrm{i}\mathbf{A})^2$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, its operator closure has form domain given by $H^1_{\mathbf{A}}(\mathbb{R}^d)$, and the last space is the closure of $C_0^\infty(\mathbb{R}^d)$ in the $H^1_{\mathbf{A}}$-norm.

**Proposition 2.3.** *Let* $\mathbf{A}$ *and* $V$ *satisfy conditions (C.2). Correspondingly, let* $h_0$ *be the operator defined in (19). Then:*

*(i) The quadratic form* $q_{h_0}^{\max}$ *defined by*

$$\begin{aligned} \mathcal{D}(q_{h_0}^{\max}) \quad &:= \quad \Big\{ f \in L^2(\mathbb{R}^d)\,|\,\nabla_{\mathbf{A}}f\,,\,V_+^{1/2}f \in L^2(\mathbb{R}^d)\Big\} \\ q_{h_0}^{\max}[f,g] \quad &:= \quad \textstyle\int_{\mathbb{R}^d}\big((\overline{\nabla_{\mathbf{A}}f})\cdot\nabla_{\mathbf{A}}g \,+\, \overline{f}\,Vg\big) \end{aligned}$$

*is densely defined, closed, and bounded below.*

*(ii) The quadratic form*

$$(f,g) \;\longmapsto\; \langle f, h_0 g\rangle \;=\; \int_{\mathbb{R}^d}\big((\overline{\nabla_{\mathbf{A}}f})\cdot\nabla_{\mathbf{A}}g \,+\, \overline{f}\,Vg\big)\,, \qquad f,g \in C_0^\infty(\mathbb{R}^d)\,,$$

*is densely defined, closable, and bounded below. Thus, denoting by* $q_{h_0}^{\min}$ *its form closure, one has* $q_{h_0}^{\min} = \overline{q_{h_0}^{\max}|_{C_0^\infty}}$.

*(iii)* $C_0^\infty(\mathbb{R}^d)$ *is a form core for* $q_{h_0}^{\max}$ *and therefore* $q_{h_0}^{\max} = q_{h_0}^{\min} =: q_{h_0}$.

*(iv) Denoting by* $-M \in \mathbb{R}$ *a lower bound for* $q_{h_0}$*, the form domain* $\mathcal{D}(q_{h_0})$ *is complete in the norm*

$$\|f\|_{q_{h_0}} \;:=\; (q_{h_0}[f] + (1+M)\|f\|_2^2)^{1/2}$$

*and* $C_0^\infty(\mathbb{R}^d)$ *is dense in* $\mathcal{D}(q_{h_0})$ *in such a norm.*

*(v) The operator* $h$ *associated with the form* $q_{h_0}$*, namely*

$$\begin{aligned} \mathcal{D}(h) \;&:=\; \left\{ f \in \mathcal{D}(q_{h_0}) \,\middle|\, \begin{array}{c} \exists\, g_f \in L^2(\mathbb{R}^d) \text{ with} \\ q_{h_0}[\eta, f] = \langle \eta, g_f \rangle \;\forall \eta \in \mathcal{D}(q_{h_0}) \end{array} \right\} \\ hf \;&:=\; g_f\,, \end{aligned}$$

*is self-adjoint and bounded below, with the same lowest bound as* $q_{h_0}$*, and in fact*

$$\begin{aligned} \mathcal{D}(h) \;&=\; \{ f \in L^2(\mathbb{R}^d) \,|\, -(\nabla - \mathrm{i}\mathbf{A})^2 f + Vf \in L^2(\mathbb{R}^d) \} \\ hf \;&=\; -(\nabla - \mathrm{i}\mathbf{A})^2 f + Vf. \end{aligned}$$

$h$ *is a self-adjoint extension of* $h_0$*,* $\mathcal{D}[h] = \mathcal{D}(q_{h_0})$*, and* $h[f,g] = q_{h_0}[f,g]$*.*

*(vi) One has*

$$\|\mathrm{e}^{-\mathrm{i}th} f\|_{q_{h_0}} \;=\; \|f\|_{q_{h_0}} \qquad \forall t \in \mathbb{R}\,.$$

*(vii) For* $|M|$ *sufficiently large the norm* $\|\cdot\|_{q_{h_0}}$ *on* $C_0^\infty(\mathbb{R}^d)$ *is equivalent to the norm*

$$\|f\|'_{\mathcal{H}^1} \;=\; \left( \|\nabla_{\mathbf{A}} f\|_2^2 + \|\langle V_+^{1/2} \rangle f\|_2^2 \right)^{1/2}. \tag{23}$$

*(viii) The space* $\mathcal{D}(q_{h_0})$ *is the completion of* $(C_0^\infty(\mathbb{R}^d), \|\cdot\|'_{\mathcal{H}^1})$*.*

**Proof.** Parts (i) and (ii) follow by standard quadratic-form-theoretic arguments, in fact (i) is a straightforward extension of lemma 1 in [13] to the case in which $V$ has a negative part that is form bounded, with relative bound strictly smaller than 1, with respect to the negative Laplacian. Similarly, part (iii) follows by a generalisation of a theorem of Simon [19] and of Leinfelder and Simander [13]: in fact it straightforward to adapt the proof of theorem 1 in [13] to the case in which $V$ has a negative part that is form bounded, with relative bound strictly smaller than 1, with respect to the negative Laplacian. According to this generalisation, one has that under conditions (C.2) the space $C_0^\infty(\mathbb{R}^d)$ is a form core for $q_{h_0}^{\max}$, i.e. $C_0^\infty(\mathbb{R}^d)$ is dense in $\mathcal{D}(q_{h_0}^{\max})$ in the $\|\;\|_{q_{h_0}}$-norm. Since by (i) $\mathcal{D}(q_{h_0}^{\max})$ is complete in the $\|\;\|_{q_{h_0}}$-norm and by (ii) $\mathcal{D}(q_{h_0}^{\min})$ is the completion of $C_0^\infty(\mathbb{R}^d)$ in the same norm, then $\mathcal{D}(q_{h_0}^{\max}) = \mathcal{D}(q_{h_0}^{\min})$. Part (iv) is just a convenient way to re-write what was proved so far, and part (v) is again a standard quadratic-form-theoretic argument (see, e.g. [18], theorem 10.7). Part (vi) follows from the fact that the self-adjointness of $h$ implies $h[\mathrm{e}^{-\mathrm{i}th} f] = h[f]$ and $\|\mathrm{e}^{-\mathrm{i}th} f\|_2 = \|f\|_2$ for all $f \in \mathcal{D}[h]$. The proof of (vii) and (viii) is precisely the same as that of proposition 2.1(iv) and (v): the form boundedness of $V$ is now given by conditions (C.2), and the $\|\;\|_{\overline{h_0}}$-norm and the $\|\;\|_{q_{h_0}}$-norm coincide on $C_0^\infty(\mathbb{R}^d)$. □

**Remark 2.4.** Under the more restrictive conditions (C.1), the self-adjoint operator $h$ of proposition 2.3(v) is nothing but the operator $\overline{h_0}$ of proposition 2.1(i) and hence the quadratic form $q_{h_0}[f,g]$ of proposition 2.3(i)–(iii) is precisely the quadratic form $\overline{h_0}[f,g]$ of proposition 2.1(ii).

**Corollary 2.5.** *Assume that* $\mathbf{A}$ *and* $V$ *satisfy conditions (C.1) or, more generally, (C.2) as specified below. Then the space* $\mathcal{H}^1(\mathbb{R}^d)$ *and the norms* $\|\cdot\|'_{\mathcal{H}^1}$ *and* $\|\cdot\|_{\mathcal{H}^1}$ *defined in the Introduction respectively by (7), (9), and (12) make sense. One has:*

$$\mathcal{H}^1(\mathbb{R}^d) \quad = \begin{cases} \mathcal{D}[\overline{h_0}] & \text{under conditions (C.1)} \\ \mathcal{D}[h] & \text{under conditions (C.2)} \end{cases} \tag{24}$$

*and*

$$\|f\|_{\mathcal{H}^1} \quad = \begin{cases} \|f\|_{\overline{h_0}} & \text{under conditions (C.1)} \\ \|f\|_{q_{h_0}} & \text{under conditions (C.2).} \end{cases} \tag{25}$$

**Proof.** Conditions (C.2) (and hence conditions (C.1) too) are clearly contained in our general assumptions (6), which proves the first part of the statement. Assuming (C.2), proposition 2.3(i) and (v) gives $\mathcal{H}^1(\mathbb{R}^d) = \mathcal{D}(q_{h_0}) = \mathcal{D}[h]$, hence the second equality in (24). Assuming further (C.1), one has $h = \overline{h_0}$ (remark 2.4), whence $\mathcal{H}^1(\mathbb{R}^d) = \mathcal{D}[h] = \mathcal{D}[\overline{h_0}]$, which is the first equality in (24). As for (25), assuming (C.2), proposition 2.3(i) and (iv) gives $\|f\|_{\mathcal{H}^1} = \|f\|_{q_{h_0}}$, hence the second equality in (25). Assuming further (C.1), one has $q_{h_0}[f,g] = \overline{h_0}[f,g]$ (remark 2.4) and hence proposition 2.1(iii) gives $\|f\|_{q_{h_0}} = \|f\|_{\overline{h_0}}$; thus, $\|f\|_{\mathcal{H}^1} = \|f\|_{\overline{h_0}}$, which is the first equality in (25). □

Based upon the discussion above, it is clear now how proposition 1.4 is established.

**Proof.**[Proof of proposition 1.4] Part (ii) and in particular identities (10) and (11) follow from (24) of corollary 2.5 and from proposition 2.3(i), (v). Part (iii) is another consequence of proposition 2.3(v). The first statement of part (iv) follows from (25) of corollary 2.5 and from proposition 2.3(iv). The second statement of part (iv) follows from (25) of corollary 2.5 and from proposition 2.3(vii). Now that the equivalence of norms (13) is established, part (i) follows from part (iv) and from proposition 2.3(iv). Part (v) follows from (25) of corollary 2.5 and from proposition 2.3(vi). □

We conclude this discussion with three further remarks.

**Remark 2.6 (Weaker conditions of self-adjointness).** The essential self-adjointness of $h_0$ is actually known also when $V_-$ has a mild growth at infinity and hence the notion of energy space makes sense for a larger class of potentials then those considered in (C.1). However, in those cases $\overline{h_0}$ is not bounded below and the natural norm on the form domain $\mathcal{D}[\overline{h_0}]$, namely $(\|f\|_2^2 + \||\overline{h_0}|^{1/2} f\|_2^2)^{1/2}$, is not equivalent any longer to the $\|\ \|_{\mathcal{H}^1}$-norm, which plays a crucial role in our energy space argument for the well-posedness of (3) ($\|f\|_{\mathcal{H}^1}$ involves only $V_+$, not $V_-$). More precisely: if $V_-$ is form-bounded with respect to $-\Delta$ with relative bound $a < 1$, this constraints the singularity of $V_-$ at $x = 0$ to be at most of the form $|x|^{-2+\delta}$, $\delta > 0$ (or also $\beta|x|^{-2}$, if $|\beta|$ is sufficiently small), and the behaviour of $V_-$ at infinity to be vanishing with at most the same power law. One can deal with a more general class of potentials $V = V_+ - V_-$, $V_- = V_-^{(1)} + V_-^{(2)}$ where $V \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, $V_-^{(1)}$ is $(-\Delta)$-form-bounded with relative bound $a < 1$ and $V_-^{(2)} \leqslant c\langle x\rangle^2$ for some $c > 0$, thus allowing for a certain amount of divergence of $V$ to $-\infty$, as $|x| \to \infty$: in this case, by a Leinfelder-Simander theorem ([13], theorem 4), the corresponding $h_0$ is still essentially self-adjoint, albeit in general unbounded below, and $\mathcal{D}[\overline{h_0}]$ is not controlled any longer by the $\|\ \|_{\mathcal{H}^1}$-norm.

**Remark 2.7 (Alternative conditions).** Conditions (C.1), (C.2) above have various variants in the literature, which suit for further specific analysis. The $\Delta$-boundedness of $V_-$ is often replaced by the requirement that $V_-$ be in the Stummel class (as, e.g. in [1], theorem 1.2), as well as the form boundedness of $V_-$ is replaced by the requirement that $V_-$ be in the Kato class (as, e.g. in [20], theorem B.13.2), for Stummel potential are infinitesimally $\Delta$-bounded and Kato potentials are infinitesimally form bounded with respect to $-\Delta$. (The Kato and Stummel classes are discussed, among others, in [2], section 4, in [20], section A.2, and in [6], section 1.2.) It is also customary to formulate the smallness of $V_-$ in terms of its so-called Kato norm $\|V_-\|_K = \|\,|\cdot|^{2-d} * |V_-|\,\|_\infty$: for instance it is straightforward to show (see, e.g. [7], lemma 2.1) that if $\|V_-\|_K < 4\pi^{d/2}/\Gamma(\frac{d}{2}-1)$, then $\int_{\mathbb{R}^d} V_-|f|^2 \mathrm{d}x \leqslant a\|\nabla f\|_2^2$ for some $a \in (0,1)$ (recall, however, that the finiteness of $\|f\|_K$, unless $f$ is spherically symmetric, is not enough to guarantee that $f$ belongs to the Kato class: [2], appendix 1).

**Remark 2.8 (Time-dependent potentials).** Under suitable conditions on the time derivatives of $\mathbf{A}(x,t)$, $V(x,t)$ it is as well possible to prove the essential self-adjointness of the corresponding time-dependent magnetic Schrödinger operator, typically requiring that $\mathbf{A}(x,t)$, $V(x,t)$ are perturbations of time frozen potentials $\mathbf{A}(x,0)$, $V(x,0)$ that satisfy the usual conditions like (C.1). For this segment of the literature we refer to [1] and the references therein.

## 3. Local Lipschitz property of the non-linearity in the energy space

In this Section we will establish the following property of the Hartree non-linearity.

**Proposition 3.1.** *Let $d \geqslant 1$, integer. Assume the following:*

$$\begin{aligned}
&\mathbf{A} \in L^2_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^d)\\
&V \in L^1_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R})\\
&V_- \text{ is } \Delta\text{-form-bounded with relative bound } < 1\\
&W \in L^{q_0}(\mathbb{R}^d,\mathbb{R}) + L^\infty(\mathbb{R}^d,\mathbb{R})\\
&\nabla W \in L^{q_1}(\mathbb{R}^d,\mathbb{R}^d) + L^\infty(\mathbb{R}^d,\mathbb{R}^d)\\
&\text{for some } q_0 \geqslant \max\{1,\tfrac{d}{2}\} (q_0 > 1 \text{ if } d=2)\\
&\text{and some } q_1 \geqslant \max\{1,\tfrac{d}{3}\}\,.
\end{aligned} \tag{26}$$

*Correspondingly, let $(\mathcal{H}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{H}^1})$ be the energy space as defined in (7) and (13), and let $\mathcal{N}(u) = (W * |u|^2)\,u$ be the associated Hartree non-linearity. Then, for any $u, v \in \mathcal{H}^1(\mathbb{R}^d)$,*

$$\|\,\mathcal{N}(u) - \mathcal{N}(v)\,\|_{\mathcal{H}^1} \leqslant C_W \left(\|u\|^2_{\mathcal{H}^1} + \|v\|^2_{\mathcal{H}^1}\right) \|u - v\|_{\mathcal{H}^1} \tag{27}$$

*for some constant $C_W = C_W(\|W\|_{q_0}, \|W\|_\infty, \|\nabla W\|_{q_1}, \|\nabla W\|_\infty) > 0$ independent of $u$ and $v$. In particular, the map $u \mapsto \mathcal{N}(u)$ maps $\mathcal{H}^1(\mathbb{R}^d)$ into itself and is bounded on bounded sets of $\mathcal{H}^1(\mathbb{R}^d)$.*

We shall prove proposition (3.1) by means of lemma 3.2 below and using the equivalence of norms (13).

**Lemma 3.2.** *Assume conditions (26). Then*

$$\begin{aligned}
\|\,(W * |u|^2)u - (W * |v|^2)v\,\|_{\mathcal{H}^1} \leqslant\; & C_W \left(\|u\|^2_{\mathcal{H}^1} + \|v\|^2_{\mathcal{H}^1}\right) \|u - v\|_{\mathcal{H}^1}\\
& \forall\, u, v \in C^\infty_0(\mathbb{R}^d)
\end{aligned} \tag{28}$$

*for some constant $C_W = C_W(\|W\|_{q_0}, \|W\|_\infty, \|\nabla W\|_{q_1}, \|\nabla W\|_\infty) > 0$ independent of $u$ and $v$.*

**Proof.** One has

$$\begin{aligned}
\left\|\nabla_{\mathbf{A}}\big((W * |u|^2)u - (W * |v|^2)v\big)\right\|_2 \leqslant\; & \|\big(\nabla\, W * (|u|^2 - |v|^2)\big)u\|_2\\
&+ \|\big(\nabla\, W * |v|^2\big)(u - v)\|_2\\
&+ \|W * (|u|^2 - |v|^2)\, \nabla_{\mathbf{A}} u\|_2\\
&+ \|W * |v|^2\, \nabla_{\mathbf{A}}(u - v)\|_2
\end{aligned} \tag{29}$$

and

$$\big\| \langle V_+\rangle^{1/2}\big((W*|u|^2)u-(W*|v|^2)v\big)\,\big\|_2 \;\leqslant\; \|W*(|u|^2-|v|^2)\,\langle V_+\rangle^{1/2}u\|_2 + \|W*|v|^2\,\langle V_+\rangle^{1/2}(u-v)\|_2\,. \tag{30}$$

The first (and analogously the second) term in the r.h.s. of (29) is estimated as follows. If $d\geqslant 3$ and $q_1\in[\frac{d}{2},\infty]$, or if $d=2$ and $q_1\in(1,\infty]$, then

$$\begin{aligned}\|\big(\nabla\;W*(|u|^2-|v|^2)\big)u\|_2 &\leqslant \|(\nabla W)*(|u|^2-|v|^2)\|_{2q_1}\|u\|_{\frac{2q_1}{q_1-1}}\\ &\lesssim \|\nabla W\|_{q_1}\,\||u|^2-|v|^2\|_{\frac{2q_1}{2q_1-1}}\,\|u\|_{\frac{2q_1}{q_1-1}}\\ &\lesssim \|\nabla W\|_{q_1}\,\|u-v\|_{\frac{4q_1}{2q_1-1}}\,(\|u\|_{\frac{4q_1}{2q_1-1}}+\|v\|_{\frac{4q_1}{2q_1-1}})\,\||u|\|_{\frac{2q_1}{q_1-1}}\\ &\lesssim \|\nabla W\|_{q_1}\,\||u-v|\|_{H^1}\,(\||u|\|_{H^1}+\||v|\|_{H^1})\,\||u|\|_{H^1}\\ &\lesssim \|\nabla W\|_{q_1}\,\|u-v\|_{H^1_{\mathbf{A}}}\,(\|u\|_{H^1_{\mathbf{A}}}+\|v\|_{H^1_{\mathbf{A}}})\,\|u\|_{H^1_{\mathbf{A}}}\end{aligned} \tag{31}$$

where we used Hölder's inequality in the first and third step, Young's inequality in the second step, Sobolev's inequality in the fourth step, and the diamagnetic inequality in the last step. With the same scheme, if $d=2$ and $q_1=1$, then

$$\begin{aligned}\|\big(\nabla\;W*(|u|^2-|v|^2)\big)u\|_2 &\leqslant \|(\nabla W)*(|u|^2-|v|^2)\|_4\|u\|_4\\ &\lesssim \|\nabla W\|_1\,\||u|^2-|v|^2\|_4\,\|u\|_4\\ &\lesssim \|\nabla W\|_1\,\|u-v\|_8\,(\|u\|_8+\|v\|_8)\,\|u\|_4\\ &\lesssim \|\nabla W\|_1\,\||u-v|\|_{H^1}\,(\||u|\|_{H^1}+\||v|\|_{H^1})\,\||u|\|_{H^1}\\ &\lesssim \|\nabla W\|_1\,\|u-v\|_{H^1_{\mathbf{A}}}\,(\|u\|_{H^1_{\mathbf{A}}}+\|v\|_{H^1_{\mathbf{A}}})\,\|u\|_{H^1_{\mathbf{A}}}\,,\end{aligned} \tag{32}$$

if $d=1$ and $q_1\in[1,\infty]$, then

$$\begin{aligned}\|\big(\nabla\;W*\;\;(|u|^2-|v|^2)\big)u\|_2 &\leqslant \|(\nabla W)*(|u|^2-|v|^2)\|_{q_1(q_1+1)}\|u\|_{\frac{2q_1^2+2q_1}{q_1^2+q-2}}\\ &\lesssim \|\nabla W\|_{q_1}\,\||u|^2-|v|^2\|_{\frac{q_1+1}{q_1}}\,\|u\|_{\frac{2q_1^2+2q_1}{q_1^2+q-2}}\\ &\lesssim \|\nabla W\|_{q_1}\,\|u-v\|_{\frac{2(q_1+1)}{q_1}}\,(\|u\|_{\frac{2(q_1+1)}{q_1}}+\|v\|_{\frac{2(q_1+1)}{q_1}})\,\|u\|_{\frac{2q_1^2+2q_1}{q_1^2+q-2}}\\ &\lesssim \|\nabla W\|_{q_1}\,\||u-v|\|_{H^1}\,(\||u|\|_{H^1}+\||v|\|_{H^1})\,\||u|\|_{H^1}\\ &\lesssim \|\nabla W\|_{q_1}\,\|u-v\|_{H^1_{\mathbf{A}}}\,(\|u\|_{H^1_{\mathbf{A}}}+\|v\|_{H^1_{\mathbf{A}}})\,\|u\|_{H^1_{\mathbf{A}}}\,,\end{aligned} \tag{33}$$

and if $d\geqslant 3$ and $q_1\in[\frac{d}{3},\frac{d}{2}]$, then

$$\begin{aligned}\|\big(\nabla\;W*(|u|^2-|v|^2)\big)u\|_2 &\leqslant \|(\nabla W)*(|u|^2-|v|^2)\|_{\frac{dq_1}{d-2q_1}}\,\|u\|_{\frac{2dq_1}{dq_1-2d+4q_1}}\\ &\lesssim \|\nabla W\|_{q_1}\,\||u|^2-|v|^2\|_{\frac{d}{d-2}}\,\|u\|_{\frac{2dq_1}{dq_1-2d+4q_1}}\\ &\lesssim \|\nabla W\|_{q_1}\,\|u-v\|_{\frac{2d}{d-2}}\,(\|u\|_{\frac{2d}{d-2}}+\|v\|_{\frac{2d}{d-2}})\,\|u\|_{\frac{2dq_1}{dq_1-2d+4q_1}}\\ &\lesssim \|\nabla W\|_{q_1}\,\|\nabla|u-v|\|_2\,(\|\nabla|u|\|_2+\|\nabla|v|\|_2)\,\||u|\|_{H^1}\\ &\lesssim \|\nabla W\|_{q_1}\,\|\nabla_{\mathbf{A}}(u-v)\|_2\,(\|\nabla_{\mathbf{A}}u\|_2+\|\nabla_{\mathbf{A}}v\|_2)\,\|u\|_{H^1_{\mathbf{A}}}\,.\end{aligned} \tag{34}$$

We thus deduce from (31)–(34) that

$$\|\big(\nabla\;W*(|u|^2-|v|^2)\big)u\|_2 \;\lesssim\; \|\nabla W\|_{q_1}\,\big(\|u\|^2_{\mathcal{H}^1}+\|v\|^2_{\mathcal{H}^1}\big)\,\|u-v\|_{\mathcal{H}^1} \qquad q_1\in[\,\max\{1,\tfrac{d}{3}\}\,,\,\infty]\,, \tag{35}$$

and, analogously for the second term in the r.h.s. of (29),

$$\|\big(\nabla\;W*|v|^2\big)(u-v)\|_2 \;\lesssim\; \|\nabla W\|_{q_1}\,\big(\|u\|^2_{\mathcal{H}^1}+\|v\|^2_{\mathcal{H}^1}\big)\,\|u-v\|_{\mathcal{H}^1} \qquad q_1\in[\,\max\{1,\tfrac{d}{3}\}\,,\,\infty]\,. \tag{36}$$

The third (and, analogously, the fourth) term in the r.h.s. of (29) is estimated as

$$\begin{aligned} \|W*(|u|^2 & -|v|^2)\,\nabla_{\mathbf{A}}u\|_2 \;\leqslant\; \|W*(|u|^2-|v|^2)\|_\infty\,\|\nabla_{\mathbf{A}}u\|_2 \\ & \lesssim\; \|W\|_{q_0}\,\||u|^2-|v|^2\|_{\frac{q_0}{q_0-1}}\,\|\nabla_{\mathbf{A}}u\|_2 \\ & \leqslant\; \|W\|_{q_0}\,\|u-v\|_{\frac{2q_0}{q_0-1}}\,(\|u\|_{\frac{2q_0}{q_0-1}}+\|v\|_{\frac{2q_0}{q_0-1}})\,\|\nabla_{\mathbf{A}}u\|_2 \\ & \lesssim\; \|W\|_{q_0}\,\||u-v|\|_{H^1}\,(\||u|\|_{H^1}+\||v|\|_{H^1})\,\|\nabla_{\mathbf{A}}u\|_2 \\ & \leqslant\; \|W\|_{q_0}\,\|u-v\|_{H^1_{\mathbf{A}}}\,(\|u\|_{H^1_{\mathbf{A}}}+\|v\|_{H^1_{\mathbf{A}}})\,\|\nabla_{\mathbf{A}}u\|_2 \\ & \qquad q_0\in[\tfrac{d}{2},\infty] \quad \text{if } d\geqslant 3 \\ & \qquad q_0\in(1,\infty] \quad \text{if } d=2 \\ & \qquad q_0\in[1,\infty] \quad \text{if } d=1\,, \end{aligned} \tag{37}$$

where in (37) we used Hölder's in the first and third step, Young's inequality in the second step, Sobolev's inequality in the fourth step, and the diamagnetic inequality in the last step. Thus,

$$\begin{aligned} \|W*(|u|^2-|v|^2)\,\nabla_{\mathbf{A}}u\|_2 \;\; & \lesssim\; \|W\|_{q_0}\,\big(\|u\|_{\mathcal{H}^1}^2+\|v\|_{\mathcal{H}^1}^2\big)\,\|u-v\|_{\mathcal{H}^1} \\ & \qquad q_0\in[\tfrac{d}{2},\infty] \quad \text{if } d\geqslant 3 \\ & \qquad q_0\in(1,\infty] \quad \text{if } d=2 \\ & \qquad q_0\in[1,\infty] \quad \text{if } d=1\,, \end{aligned} \tag{38}$$

and, analogously for the fourth term in the r.h.s. of (29),

$$\begin{aligned} \|W*|v|^2\,\nabla_{\mathbf{A}}(u-v)\|_2 \;\; & \lesssim\; \|W\|_{q_0}\,\big(\|u\|_{\mathcal{H}^1}^2+\|v\|_{\mathcal{H}^1}^2\big)\,\|u-v\|_{\mathcal{H}^1} \\ & \qquad q_0\in[\tfrac{d}{2},\infty] \quad \text{if } d\geqslant 3 \\ & \qquad q_0\in(1,\infty] \quad \text{if } d=2 \\ & \qquad q_0\in[1,\infty] \quad \text{if } d=1\,. \end{aligned} \tag{39}$$

Therefore (29), (35), (36), (38), and (39) yield

$$\begin{aligned} \big\|\nabla_{\mathbf{A}}\big( & (W*|u|^2)u-(W*|v|^2)v\big)\big\|_2 \\ & \lesssim\; (\|W\|_{q_0}+\|\nabla W\|_{q_1})\,\big(\|u\|_{\mathcal{H}^1}^2+\|v\|_{\mathcal{H}^1}^2\big)\,\|u-v\|_{\mathcal{H}^1} \\ & \qquad q_0\in[\,\max\{1,\tfrac{d}{2}\}\,,\ \infty]\text{ if } d\neq 2 \\ & \qquad q_0\in(1,\infty]\text{ if } d=2 \\ & \qquad q_1\in[\,\max\{1,\tfrac{d}{3}\}\,,\ \infty]. \end{aligned} \tag{40}$$

The treatment of the r.h.s. of (30) is the same as for the third and fourth term of (29): one re-does (37) with $\langle V_+\rangle^{1/2}u$ instead of $\nabla_{\mathbf{A}}u$ in the l.h.s. Thus,

$$\begin{aligned} \big\|\langle V_+\rangle^{1/2}\big( & (W*|u|^2)u-(W*|v|^2)v\big)\big\|_2 \\ & \lesssim\; (\|W\|_{q_0}+\|\nabla W\|_{q_1})\,\big(\|u\|_{\mathcal{H}^1}^2+\|v\|_{\mathcal{H}^1}^2\big)\,\|u-v\|_{\mathcal{H}^1} \\ & \qquad q_0\in[\,\max\{1,\tfrac{d}{2}\}\,,\ \infty]\text{ if } d\neq 2 \\ & \qquad q_0\in(1,\infty]\text{ if } d=2 \\ & \qquad q_1\in[\,\max\{1,\tfrac{d}{3}\}\,,\ \infty]. \end{aligned} \tag{41}$$

Last, combining (40) and (41) leads to (28). □

We are now ready for the proof of our initial claim.

**Proof.**[Proof of proposition 3.1] It follows from lemma 3.2 that

$$\big\|\mathcal{N}(u)-\mathcal{N}(v)\big\|_{\mathcal{H}^1} \;\leqslant\; C\,\big(\|u\|_{\mathcal{H}^1}^2+\|v\|_{\mathcal{H}^1}^2\big)\,\|u-v\|_{\mathcal{H}^1} \qquad \forall\,u,v\in C_0^\infty(\mathbb{R}^d)\,. \tag{42}$$

What is left is a density argument, using the fact that $C_0^\infty(\mathbb{R}^d)$ is dense in $\mathcal{H}^1(\mathbb{R}^d)$. To this aim let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two sequences in $C_0^\infty(\mathbb{R}^d)$ with

$$u_n\longrightarrow u \qquad \text{and} \qquad v_n\longrightarrow v \qquad \text{in } \mathcal{H}^1(\mathbb{R}^d) \text{ as } n\to\infty\,. \tag{43}$$

By (42)

$$\| \mathcal{N}(u_n) - \mathcal{N}(v_n) \|_{\mathcal{H}^1} \leqslant C \left( \|u_n\|^2_{\mathcal{H}^1} + \|v_n\|^2_{\mathcal{H}^1} \right) \|u_n - v_n\|_{\mathcal{H}^1} \qquad \forall n \in \mathbb{N} \tag{44}$$

and by (43)

$$\left( \|u_n\|^2_{\mathcal{H}^1} + \|v_n\|^2_{\mathcal{H}^1} \right) \|u_n - v_n\|_{\mathcal{H}^1} \longrightarrow \left( \|u\|^2_{\mathcal{H}^1} + \|v\|^2_{\mathcal{H}^1} \right) \|u - v\|_{\mathcal{H}^1} \tag{45}$$

as $n \to \infty$. In turn, by (44) both sequences $(\mathcal{N}(u_n))_{n\in\mathbb{N}}$ and $(\mathcal{N}(v_n))_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathcal{H}^1(\mathbb{R}^d)$ and hence

$$\mathcal{N}(u_n) \longrightarrow \mathcal{N}_u \qquad \text{and} \qquad \mathcal{N}(v_n) \longrightarrow \mathcal{N}_v \qquad \text{in } \mathcal{H}^1(\mathbb{R}^d) \text{ as } n \to \infty \tag{46}$$

for some $\mathcal{N}_u, \mathcal{N}_v$. Each convergence in (43) and (46) holds as well in the weaker $L^2$-sense, for $\|f\|_2 \leqslant \left( \|\nabla_{\mathbf{A}} f\|_2^2 + \|\langle V_+\rangle^{1/2} f\|_2^2 \right)^{1/2} = \|f\|'_{\mathcal{H}^1} \leqslant C\|f\|_{\mathcal{H}^1}$; therefore, by the Riesz-Fischer theorem and up to extracting a further subsequence,

$$u_n \longrightarrow u \qquad \text{and} \qquad v_n \longrightarrow v \qquad \text{pointwise a.e. as } n \to \infty \tag{47}$$

and

$$\mathcal{N}(u_n) \longrightarrow \mathcal{N}_u \qquad \text{and} \qquad \mathcal{N}(v_n) \longrightarrow \mathcal{N}_v \qquad \text{pointwise a.e. as } n \to \infty\,. \tag{48}$$

By (47) one also has

$$\mathcal{N}(u_n) \longrightarrow \mathcal{N}(u) \qquad \text{and} \qquad \mathcal{N}(v_n) \longrightarrow \mathcal{N}(v) \qquad \text{pointwise a.e. as } n \to \infty \tag{49}$$

thus, comparing (48) and (49), $\mathcal{N}_u = \mathcal{N}(u)$ and $\mathcal{N}_v = \mathcal{N}(v)$ as elements in $\mathcal{H}^1(\mathbb{R}^d)$. With this information, (46) gives

$$\| \mathcal{N}(u_n) - \mathcal{N}(v_n) \|_{\mathcal{H}^1} \longrightarrow \| \mathcal{N}(u) - \mathcal{N}(v) \|_{\mathcal{H}^1} \tag{50}$$

as $n \to \infty$. Last, taking the limits (45) and (50) in (44) leads to (27) for arbitrary $u, v \in \mathcal{H}^1(\mathbb{R}^d)$. □

## 4. Local well-posedness

In this Section we establish the local well-posedness of the initial value problem (3) in the energy space $\mathcal{H}^1(\mathbb{R}^d)$, in the following sense.

**Proposition 4.1.** *Assume that* $\mathbf{A}$, $V$, *and* $W$ *satisfy conditions (26). Then:*

*(i) For any* $\varphi \in \mathcal{H}^1(\mathbb{R}^d)$ *there exist* $T_{\min}, T_{\max} \in [0, +\infty]$ *such that the initial value problem (3) has a unique solution in the space* $C((-T_{\min}, T_{\max}), \mathcal{H}^1(\mathbb{R}^d)) \cap C^1((-T_{\min}, T_{\max}), \mathcal{H}^1(\mathbb{R}^d)^*)$ *with initial datum* $\varphi$ *at* $t = 0$.

*(ii) If* $T_{\max} < +\infty$, *then* $\lim_{t\uparrow T_{\max}} \|u(t)\|_{\mathcal{H}^1} = +\infty$.
*If* $T_{\min} < +\infty$, *then* $\lim_{t\downarrow -T_{\min}} \|u(t)\|_{\mathcal{H}^1} = +\infty$.

*(iii) Let* $\varphi_n \to \varphi$ *in* $\mathcal{H}^1(\mathbb{R}^d)$ *as* $n \to \infty$ *and, correspondingly, according to part (i), let* $u_n \in C(I_n, \mathcal{H}^1(\mathbb{R}^d)) \cap C^1(I_n, \mathcal{H}^1(\mathbb{R}^d)^*)$ *and* $u \in C(I, \mathcal{H}^1(\mathbb{R}^d)) \cap C^1(I, \mathcal{H}^1(\mathbb{R}^d)^*)$ *be the unique maximal solution to (3) with initial datum at* $t = 0$ *given, respectively, by* $\varphi_n$ *and* $\varphi$. *If* $J \subset I$ *is a closed interval, then* $J \subset I_n$ *for* $n$ *large enough and* $u_n \to u$ *in* $C(J, \mathcal{H}^1(\mathbb{R}^d))$ *as* $n \to \infty$.

The proof of proposition 4.1 is based on a contraction argument in the energy space $\mathcal{H}^1(\mathbb{R}^d)$ which is possible owing to the local Lipschitz property of the Hartree non-linearity established in proposition 3.1. In the following corollary 4.5 and lemmas 4.6–4.8 we will give the detail of this argument.

**Remark 4.2.** Unlike the standard treatment of the Hartree equation without magnetic field (see, e.g. [5], theorems 3.3.5 and 3.3.9, and propositions 4.2.3 and 4.2.9), we need *not* establish charge and energy conservation to prove the existence of a local solution that is continuous (and not just $L^\infty$) in $t$; moreover, as remarked in the Introduction, magnetic Strichartz estimates are not required (and in fact would not hold, in general, under conditions (26)) for the proof of local uniqueness. Also, because of the fact that charge and energy conservation is *not* used, the potential $W$ is not required by (26) to be an *even* function—this is instead needed to establish proposition 5.1, identity (70).

We first of all highlight this classical fact.

**Lemma 4.3 (Duhamel's formula).** *Let $I \subset \mathbb{R}$ be an open interval such that $0 \in I$ and let $u \in C(I, \mathcal{H}^1(\mathbb{R}^d))$. Assume that $\mathbf{A}$, $V$, and $W$ satisfy conditions (26). Then $u$ is a solution to the initial value problem (3) if and only if*

$$u(t) \;=\; \mathrm{e}^{-\mathrm{i}th}\varphi - i\int_0^t \mathrm{e}^{-\mathrm{i}(t-s)h}\mathcal{N}(u(s))\,\mathrm{d}s \qquad \forall t\in I. \tag{51}$$

**Proof.** By proposition 3.1 the map $\mathcal{N}:\mathcal{H}^1(\mathbb{R}^d)\to\mathcal{H}^1(\mathbb{R}^d)$ is bounded on $\mathcal{H}^1$-bounded sets, hence (51) is a standard Duhamel's formula (see, e.g. [5], proposition 3.1.3). □

Let us now start the proof of proposition 4.1 (lemma 4.4, corollary 4.5 and lemmas 4.6–4.8 below). For each $T, M>0$ and $\varphi\in\mathcal{H}^1(\mathbb{R}^d)$ we introduce the space

$$X_{T,M,\varphi} \;:=\; \left\{ u\in C((-T,T),\mathcal{H}^1(\mathbb{R}^d)) \;\middle|\; \begin{array}{l} \|u\|_{L^\infty([-T,T],\mathcal{H}^1(\mathbb{R}^d))}\leqslant M \\ u(0,\cdot)=\varphi(\cdot) \end{array} \right\} \tag{52}$$

equipped with the Banach norm

$$\|u\|_X \;:=\; \|u\|_{L^\infty([-T,T],\mathcal{H}^1(\mathbb{R}^d))}\,. \tag{53}$$

Moreover, for every $u\in X_{T,M,\varphi}$ and any $t\in(-T,T)$ we introduce the map $u\longmapsto S_\varphi(u)$ defined by

$$S_\varphi(u) \;:=\; \mathrm{e}^{-\mathrm{i}th}\varphi - i\int_0^t \mathrm{e}^{-\mathrm{i}(t-s)h}\mathcal{N}(u(s))\,\mathrm{d}s\,. \tag{54}$$

**Lemma 4.4.** *Assume that $\mathbf{A}$, $V$, and $W$ satisfy conditions (26). For every $M>0$ there exists $T(M)>0$ with the following property: for every $\varphi\in\mathcal{H}^1(\mathbb{R}^d)$ such that $\|\varphi\|_{\mathcal{H}^1}\leqslant M$, the map $S_\varphi$ defined by (54) is a contraction in the space $X_{T(M),2M,\varphi}$ defined by (52).*

**Proof.** We choose

$$T(M) \;=\; \frac{1}{2}\,(8C_W M^2)^{-1}\,, \tag{55}$$

where $C_W$ is the constant of inequality (27) of proposition 3.1. Let $u\in X_{T(M),2M,\varphi}$, arbitrary. Then, for $t\in[0,T(M))$, we deduce from (14), (27), and (55)

$$\begin{aligned} \|S_\varphi(u)\|_{\mathcal{H}^1} &\leqslant \|\mathrm{e}^{-\mathrm{i}th}\varphi\|_{\mathcal{H}^1} + \textstyle\int_0^t \|\mathrm{e}^{-\mathrm{i}(t-s)h}\mathcal{N}(u(s))\|_{\mathcal{H}^1}\,\mathrm{d}s \\ &= \|\varphi\|_{\mathcal{H}^1} + \textstyle\int_0^t \|\mathcal{N}(u(s))\|_{\mathcal{H}^1}\,\mathrm{d}s \\ &\leqslant \|\varphi\|_{\mathcal{H}^1} + C_W\|u\|^3_{L^\infty([-t,t],\mathcal{H}^1)}T(M) \\ &\leqslant M + C_W\,(2M)^3\,T(M) \\ &\leqslant 2M \end{aligned} \tag{56}$$

and the same for $t \in (-T(M), 0]$, which proves that $S_\varphi(X_{T(M),2M,\varphi}) \subset X_{T(M),2M,\varphi}$. Moreover, given any two $u, v \in X_{T(M),2M,\varphi}$, (14), (27), and (55) give, for $t \in [0, T(M))$,

$$\begin{aligned} \|S_\varphi(u) - S_\varphi(v)\|_{\mathcal{H}^1} &\leqslant \int_0^t \left\| \mathrm{e}^{-\mathrm{i}(t-s)h} \left(\mathcal{N}(u(s)) - \mathcal{N}(v(s))\right) \right\|_{\mathcal{H}^1} \mathrm{d}s \\ &= \int_0^t \|(\mathcal{N}(u(s)) - \mathcal{N}(v(s)))\|_{\mathcal{H}^1} \mathrm{d}s \\ &\leqslant C_W \int_0^t \left(\|u(s)\|^2_{\mathcal{H}^1} + \|v(s)\|^2_{\mathcal{H}^1}\right) \|u(s) - v(s)\|_{\mathcal{H}^1} \mathrm{d}s \\ &\leqslant C_W\ T(M) \left( \|u\|^2_X + \|v\|^2_X \right) \|u - v\|_X \\ &\leqslant 8\, C_W M^2\, T(M)\, \|u - v\|_X \\ &= \tfrac{1}{2} \|u - v\|_X \end{aligned} \tag{57}$$

and the same for $t \in (-T(M), 0]$. Thus,

$$\|S_\varphi(u) - S_\varphi(v)\|_X \leqslant \frac{1}{2} \|u - v\|_X \qquad \forall u, v \in X_{T(M),2M,\varphi} \tag{58}$$

which completes the proof. □

**Corollary 4.5 (Local existence and uniqueness).** *Assume that* $\mathbf{A}$*,* $V$*, and* $W$ *satisfy conditions (26). Let* $\varphi \in \mathcal{H}^1(\mathbb{R}^d)$*. For every* $M > 0$ *there exists* $T(M) > 0$ *with the following property: for every* $\varphi \in \mathcal{H}^1(\mathbb{R}^d)$ *such that* $\|\varphi\|_{\mathcal{H}^1} \leqslant M$*, there exists a unique* $u \in C((-T(M), T(M)), \mathcal{H}^1(\mathbb{R}^d))$ *that solves the initial value problem (3). In addition,*

$$\|u\|_{L^\infty([-T(M),T(M)],\mathcal{H}^1(\mathbb{R}^d))} \leqslant 2M. \tag{59}$$

**Proof.** By lemma 4.4 and by Banach's fixed-point Theorem, there exists $T(M) > 0$ and there exists a unique $u \in X_{T(M),2M,\varphi}$ such that $S_\varphi(u) = u$. By lemma 4.3 this is equivalent to say that there exists $T(M) > 0$ and there exists a unique $u \in C((-T(M), T(M)), \mathcal{H}^1(\mathbb{R}^d))$ that solves the initial value problem (3). The fact that $u \in X_{T(M),2M,\varphi}$ obviously implies (59). □

**Lemma 4.6 (Maximality).** *Assume that* $\mathbf{A}$*,* $V$*, and* $W$ *satisfy conditions (26). Given* $\varphi \in \mathcal{H}^1(\mathbb{R}^d)$*, there exist* $T_{\min}, T_{\max} > 0$ *such that the initial value problem (3) has a unique solution in the space* $C((-T_{\min}, T_{\max}), \mathcal{H}^1(\mathbb{R}^d)) \cap C^1((-T_{\min}, T_{\max}), \mathcal{H}^1(\mathbb{R}^d)^*)$*.*

**Proof.** By corollary 4.5 the following definitions are well-posed:

$$\begin{aligned} T_{\max} &:= \sup\{T > 0 \,|\, \text{there exists a solution of (3) in } C([0, T_{\max}), \mathcal{H}^1(\mathbb{R}^d))\} \\ T_{\min} &:= \sup\{T > 0 \,|\, \text{there exists a solution of (3) in } C((-T_{\min}, 0], \mathcal{H}^1(\mathbb{R}^d))\} . \end{aligned} \tag{60}$$

Therefore for any $T \in [0, T_{\max})$, $T' \in (-T_{\min}, 0]$ there exists a solution to the initial value problem (3) in the space $C([-T', T], \mathcal{H}^1(\mathbb{R}^d)) \cap C^1([-T', T], \mathcal{H}^1(\mathbb{R}^d)^*)$. This solution is actually unique, for any two $u, v \in C([-T', T], \mathcal{H}^1(\mathbb{R}^d))$ that solve (3) must satisfy, owing to (51) and for any $t \in [0, T]$,

$$\begin{aligned} \|u(t) - v(t)\|_{\mathcal{H}^1} &\leqslant \int_0^t \left\| \mathrm{e}^{-\mathrm{i}(t-s)h} \left(\mathcal{N}(u(s)) - \mathcal{N}(v(s))\right) \right\|_{\mathcal{H}^1} \mathrm{d}s \\ &= \int_0^t \|(\mathcal{N}(u(s)) - \mathcal{N}(v(s)))\|_{\mathcal{H}^1} \mathrm{d}s \\ &\leqslant C_W \int_0^t \left(\|u(s)\|^2_{\mathcal{H}^1} + \|v(s)\|^2_{\mathcal{H}^1}\right) \|u(s) - v(s)\|_{\mathcal{H}^1} \mathrm{d}s \\ &\leqslant C_W \left( \|u\|^2_{L^\infty([-T,T],\mathcal{H}^1)} + \|v\|^2_{L^\infty([-T,T],\mathcal{H}^1)} \right) \int_0^t \|u(s) - v(s)\|_{\mathcal{H}^1} \mathrm{d}s \end{aligned} \tag{61}$$

(where we used (14) and (27)), and therefore $u(t) = v(t)$ on $[0, T]$ because of Gronwall's inequality; with an analogous argument $u(t) = v(t)$ also on $[-T', 0]$. □

**Lemma 4.7 (Blow-up alternative).** *Assume that* **A**, $V$*, and* $W$ *satisfy conditions (26). Given* $\varphi \in \mathcal{H}^1(\mathbb{R}^d)$*, let* $u$ *be the unique maximal solution to the initial value problem (3) in* $C((-T_{\min}, T_{\max}), \mathcal{H}^1(\mathbb{R}^d)) \cap C^1((-T_{\min}, T_{\max}), \mathcal{H}^1(\mathbb{R}^d)^*)$*, as given by lemma 4.6. If* $T_{\max} < +\infty$*, then* $\lim_{t\uparrow T_{\max}} \|u(t)\|_{\mathcal{H}^1} = +\infty$ *(respectively, if* $T_{\min} < +\infty$*, then* $\lim_{t\downarrow -T_{\min}} \|u(t)\|_{\mathcal{H}^1} = +\infty$*).*

**Proof.** Suppose that $T_{\max} < +\infty$ and assume that there exists $M > 0$ and a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $t_n \uparrow T_{\max}$ as $n \to \infty$ and $\|u(t_n)\|_{\mathcal{H}^1} \leqslant M$ for all $n$. Correspondingly, let $T(M)$ be the local existence time for a solution to (3) given by corollary 4.5 when the initial datum has $\mathcal{H}^1$-norm below $M$. Then there exists $k \in \mathbb{N}$ such that $t_k + T(M) > T_{\max}$. Let $v$ be the unique solution to (3) in $C((-T(M), T(M)), \mathcal{H}^1(\mathbb{R}^d))$ with initial datum $u(t_k)$. Then $v(\cdot - t_k)$ solves (3) in $C((t_k - T(M), t_k + T(M)), \mathcal{H}^1(\mathbb{R}^d))$, it coincides with $u(t_k)$ at $t = t_k$, and in fact owing to the same argument of (61) it can be glued to $u$ so to obtain a solution to (3) with initial datum $\varphi$ at $t = 0$ in the space $C((-T_{\min}, t_k + T(M)), \mathcal{H}^1(\mathbb{R}^d))$. Since the interval $(-T_{\min}, t_k + T(M))$ contains $(-T_{\min}, T_{\max})$, this conclusion violates the maximality of $T_{\max}$. Hence, necessarily, $\lim_{t\uparrow T_{\max}} \|u(t)\|_{\mathcal{H}^1} = +\infty$. The argument for $T_{\min}$ is completely analogous. □

**Lemma 4.8 (Continuous dependence on the initial data).** *Assume that* **A**, $V$*, and* $W$ *satisfy conditions (26). For each* $n \in \mathbb{N}$ *let* $\varphi_n, \varphi \in \mathcal{H}^1(\mathbb{R}^d)$ *with* $\varphi_n \to \varphi$ *in* $\mathcal{H}^1(\mathbb{R}^d)$ *as* $n \to \infty$*. Correspondingly, let* $u_n \in C(I_n, \mathcal{H}^1(\mathbb{R}^d)) \cap C^1(I_n, \mathcal{H}^1(\mathbb{R}^d)^*)$ *and* $u \in C(I, \mathcal{H}^1(\mathbb{R}^d)) \cap C^1(I, \mathcal{H}^1(\mathbb{R}^d)^*)$ *be the unique maximal solution to (3) with initial datum at* $t = 0$ *given, respectively, by* $\varphi_n$ *and* $\varphi$*. If* $J \subset I$ *is a closed interval, then* $J \subset I_n$ *for* $n$ *large enough and* $u_n \to u$ *in* $C(J, \mathcal{H}^1(\mathbb{R}^d))$ *as* $n \to \infty$*.*

**Proof.** It is enough to assume that $J \ni 0$. Let $M := 2\sup\{\,\|u(t)\|_{\mathcal{H}^1}\,|\,t \in J\,\}$ and, correspondingly, let $T(M) > 0$ be the local existence time for the solution to (3) in the space $C((-T(M), T(M)), \mathcal{H}^1(\mathbb{R}^d))$ with initial datum at $t = 0$ whose $\mathcal{H}^1$-norm is below $M$, as given in corollary 4.5. Since $\|\varphi_n\|_{\mathcal{H}^1} \to \|\varphi\|_{\mathcal{H}^1}$ as $n \to \infty$, then $\|\varphi_n\|_{\mathcal{H}^1} \leqslant M$ for $n$ large enough, and hence $u_n$ is defined at least on $[-T(M), T(M)]$. This means, in particular, that $[-T(M), T(M)] \in I \cap I_n$ for $n$ large enough. By means of Duhamel's formula (51), for any $t \in [-T(M), T(M)]$ one has

$$u_n(t) - u(t) \;=\; \mathrm{e}^{-\mathrm{i}th}(\varphi_n - \varphi) + S_\varphi(u_n) - S_\varphi(u) \tag{62}$$

where $S_\varphi$ is the map defined by (54). Therefore, if $t \in [0, T(M)]$ (if $t \in [-T(M), 0]$ one gets the same estimate), we deduce from (13), (57), and (62) that

$$\begin{aligned} \|u_n(t) - u(t)\|_{\mathcal{H}^1} &\leqslant \|\varphi_n - \varphi\|_{\mathcal{H}^1} + \|S_\varphi(u_n) - S_\varphi(u)\|_{\mathcal{H}^1} \\ &\leqslant \|\varphi_n - \varphi\|_{\mathcal{H}^1} + \tfrac{1}{2}\|u_n - u\|_{L^\infty([-T(M),T(M)],\mathcal{H}^1)} \end{aligned}$$

(recall that $T(M)$ is actually given by (55)). Hence,

$$\|u_n - u\|_{L^\infty([-T(M),T(M)],\mathcal{H}^1)} \;\leqslant\; 2\,\|\varphi_n - \varphi\|_{\mathcal{H}^1} \;\longrightarrow\; 0 \qquad \text{as } n \to \infty \tag{63}$$

thus proving that $u_n \to u$ in $C([-T(M), T(M)], \mathcal{H}^1(\mathbb{R}^d))$ as $n \to \infty$. Since $T(M)$ depends only on $M$, we may repeat the argument above to cover the whole interval $J$, thus concluding that $J \subset I_n$ for $n$ large enough and $u_n \to u$ in $C(J, \mathcal{H}^1(\mathbb{R}^d))$. □

## 5. Conservation laws and global well-posedness

In this last Section we conclude the proof of theorem 1.6 by establishing the conservation of charge and energy (proposition 5.1) and, consequently, the validity globally in time of the local solution determined in proposition 4.1 (corollary 5.2).

**Proposition 5.1 (Conservation of charge and density).** *Let* $\mathbf{A}$, $V$, *and* $W$ *satisfy conditions (6) and let* $\varphi \in \mathcal{H}^1(\mathbb{R}^d)$. *Correspondingly, let* $u \in C((-T_{\min}, T_{\max}), \mathcal{H}^1(\mathbb{R}^d)) \cap C^1((-T_{\min}, T_{\max}), \mathcal{H}^1(\mathbb{R}^d)^*)$ *be the unique maximal solution to the initial value problem (3) with initial datum* $\varphi$, *as guaranteed by proposition 4.1, and let* $\mathcal{Q}[u]$ *and* $\mathcal{E}[u]$ *be, respectively, the associated charge and density, as defined by (18). Then*

$$\mathcal{Q}[u(t)] \;=\; \mathcal{Q}[\varphi] \qquad \text{and} \qquad \mathcal{E}[u(t)] \;=\; \mathcal{E}[\varphi] \qquad \forall t \in (-T_{\min}, T_{\max})\,. \tag{64}$$

**Proof.** The first identity in (64) follows by

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 \;&=\; 2\,\mathfrak{Re}\langle u, \partial_t u\rangle_{\mathcal{H}^1,\mathcal{H}^{1*}} \\ &=\; 2\,\mathfrak{Re}\big\langle u, \mathrm{i}\,(\nabla - \mathrm{i}\mathbf{A})^2 u - \mathrm{i}\,(W * |u|^2)u\big\rangle_{\mathcal{H}^1,\mathcal{H}^{1*}} \\ &=\; 2\,\mathfrak{Re}\,\mathrm{i}\int_{\mathbb{R}^d}\big(|\nabla_{\mathbf{A}}u|^2 - (W*|u|^2)|u|^2\big) \;=\; 0\,, \end{aligned} \tag{65}$$

where we used (1) in the second step. As for the energy, we first use (2), (10), and (18) to re-write it as

$$\mathcal{E}[u] \;=\; \frac{1}{2}\,h[u] + \frac{1}{4}\,\langle u, \mathcal{N}(u)\rangle_{\mathcal{H}^1,\mathcal{H}^{1*}}\,. \tag{66}$$

One has

$$\begin{aligned} \mathrm{i}\,\|\partial_t u\|_2^2 \;&=\; \langle \partial_t u, \mathrm{i}\,\partial_t u\rangle_{\mathcal{H}^{1*},\mathcal{H}^{1*}} \\ &=\; \langle \partial_t u, hu\rangle_{\mathcal{H}^{1*},\mathcal{H}^{1*}} + \langle \partial_t u, \mathcal{N}(u)\rangle_{\mathcal{H}^{1*},\mathcal{H}^{1*}} \\ &=\; h[\partial_t u, u] + \langle \partial_t u, \mathcal{N}(u)\rangle_{\mathcal{H}^{1*},\mathcal{H}^{1*}}\,, \end{aligned} \tag{67}$$

where we used (1), (2), and (11) in the second step, and (10) in the third step. As a consequence,

$$\mathfrak{Re}\,\big(h[\partial_t u, u] + \langle \partial_t u, \mathcal{N}(u)\rangle_{\mathcal{H}^{1*},\mathcal{H}^{1*}}\big) \;=\; 0\,. \tag{68}$$

On the other hand, from (10) one has

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\,h[u] \;=\; \tfrac{\mathrm{d}}{\mathrm{d}t}\,h[u,u] \;=\; 2\,\mathfrak{Re}\,h[\partial_t u, u]\,. \tag{69}$$

Owing to the last condition in (6), one also has

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}t}\,\langle u, \mathcal{N}(u)\rangle_{\mathcal{H}^1,\mathcal{H}^{1*}} \\ &\quad=\; 2\,\mathfrak{Re}\,\langle \partial_t u, \mathcal{N}(u)\rangle_{\mathcal{H}^{1*},\mathcal{H}^{1*}} \;+\; 2\,\mathfrak{Re}\int_{\mathbb{R}^d}\mathrm{d}x\,|u(x)|^2\int_{\mathbb{R}^d}\mathrm{d}y\,W(x-y)\,\overline{\partial_t u(y)}\,u(y) \\ &\quad=\; 2\,\mathfrak{Re}\,\langle \partial_t u, \mathcal{N}(u)\rangle_{\mathcal{H}^{1*},\mathcal{H}^{1*}} \;+\; 2\,\mathfrak{Re}\int_{\mathbb{R}^d}\mathrm{d}y\,\overline{\partial_t u(y)}\Big(\int_{\mathbb{R}^d}\mathrm{d}x\,W(y-x)\,|u(x)|^2\Big)\,u(y) \\ &\quad=\; 4\,\mathfrak{Re}\,\langle \partial_t u, \mathcal{N}(u)\rangle_{\mathcal{H}^{1*},\mathcal{H}^{1*}}\,. \end{aligned} \tag{70}$$

Therefore, differentiating (66) in time yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{E}[u] \;=\; \mathfrak{Re}\,\big(h[\partial_t u, u] + \langle \partial_t u, \mathcal{N}(u)\rangle_{\mathcal{H}^{1*},\mathcal{H}^{1*}}\big) \;=\; 0\,, \tag{71}$$

where we used (69) and (70) in the first step and (68) in the second step. □

**Corollary 5.2.** *One has*

$$\sup_{t\in\mathbb{R}}\|u(t)\|_{\mathcal{H}^1} \;<\; +\infty\,. \tag{72}$$

*and hence* $T_{\min} = T_{\mathrm{maz}} = +\infty$.

**Proof.** (72) follows by

$$\|u\|^2_{\mathcal{H}^1} \;\leqslant\; 2\,\mathcal{E}[u] \,+\, |1+M|\,\mathcal{Q}[u] \;=\; 2\,\mathcal{E}[\varphi] \,+\, |1+M|\,\mathcal{Q}[\varphi] \tag{73}$$

which is a consequence of (12) and (18) (first step) and of (64) (second step). Then $T_{\min} = T_{\mathrm{maz}} = +\infty$, because of the blow-up alternative of proposition 4.1(ii). □

**Remark 5.3.** When $T_{\min} = T_{\mathrm{maz}} = +\infty$, and hence $u \in C(\mathbb{R}, \mathcal{H}^1(\mathbb{R}^d))$, the continuous dependence on the initial data stated in proposition 4.1(iii) holds in $L^\infty(\mathbb{R}, \mathcal{H}^1(\mathbb{R}^d))$.

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## Appendix A. Magnetic Strichartz estimates

In this appendix we review briefly the Strichartz estimates for the magnetic Schrödinger operator which are currently available in the literature. They are inequalities of the form

$$\|\mathrm{e}^{-\mathrm{i}th} f\|_{L^q(I,L^r(\mathbb{R}^d))} \;\leqslant\; C_{r,I}\,\|f\|_{L^2(\mathbb{R}^d)} \qquad \forall f \in L^2(\mathbb{R}^d) \tag{A.1}$$

for some constant $C_{r,I} > 0$ and for all the admissible couples $(q, r)$, i.e.

$$\frac{2}{q} = \frac{d}{2} - \frac{d}{r}\,, \qquad \begin{array}{ll} r \in \left[2, \frac{2d}{d-2}\right] & \text{if } d \geqslant 3 \\ r \in [2, \infty) & \text{if } d = 2 \\ r \in [2, \infty] & \text{if } d = 1, \end{array} \tag{A.2}$$

where $I \subset \mathbb{R}$ is an open interval with $0 \in \overline{I}$, $(t, x) \in I \times \mathbb{R}^d$, $h$ is a self-adjoint realisation of $-(\nabla - \mathrm{i}\mathbf{A})^2 + V$. We will tacitly exclude the endpoint case $(q, r) = (2, \frac{2d}{d-2})$ from the following discussion, for some of the results quoted below do not cover it.

Within the ample literature on this topic one can distinguish, roughly speaking, between *two regimes* of electromagnetic potentials, depending on whether (A.1)–(A.2) is required globally or only locally in time.

*Global-in-time electromagnetic Strichartz estimates*

When $I = \mathbb{R}$, there are natural obstructions for (A.1)–(A.2), essentially due to possible confinement properties of $h$, such as the occurrence of eigenvalues or resonances, which allow for the existence of non-dispersive components in $\mathrm{e}^{-\mathrm{i}th} f$. Restrictions on $\mathbf{A}$ and $V$ that impose an amount of decay at spatial infinity and a bound on the local singularities are thus needed to derive (A.1)–(A.2).

More precisely, the known scheme to prove (A.1)–(A.2) globally in time is to treat $-(\nabla - \mathrm{i}\mathbf{A})^2 + V$ perturbatively with respect to $-\Delta$ under conditions that preserve the dispersion properties of $-\Delta$, typically by excluding zero-energy resonances for $h$ or by imposing suitable repulsivity conditions on $V$ and non-trapping conditions on $\mathbf{A}$, namely imposing enough smallness to those obstructions to the dispersion such as the positive part $(\partial_r V)_+$ of $\partial_r V(x) := \frac{x}{|x|} \cdot \nabla V$ and the trapping component $B_\tau(x) := \frac{x}{|x|} B(x)$ of the $d \times d$ antisymmetric matrix $B = (B_{ij})$, $B_{jk} = \partial_j A_k - \partial_k A_j$, that represents the strength tensor of the magnetic field. (When $d = 3$ the independent entries of $(B_{jk})$ can be organised in the actual magnetic field vector $\mathbf{B} = \nabla \times \mathbf{A}$, and $B_\tau = \frac{x}{|x|} \times \mathbf{B}$.) The amount of regularity required for $\mathbf{A}$ and $V$ is virtually only the one needed for such smallness conditions to make sense; the perturbative scheme only accommodates potentials with a certain amount of local singularity and of decay at spatial infinity.

Within this scheme, D'Ancona, Fanelli, Vega, and Visciglia [7] proved (A.1)–(A.2) globally in time for $d \geqslant 3$, in fact covering also the endpoint case $(q, r) = (2, \frac{2d}{d-2})$ for $d \geqslant 4$, under conditions that, practically speaking, correspond to

$$\begin{aligned} |\mathbf{A}(x)| &\lesssim \begin{cases} |x|^{-(1+\delta_A)} \\ |x|^{-(1-\delta_A)} \end{cases} & |B_\tau(x)| &\lesssim \begin{cases} |x|^{-(2+\delta_B)} & \text{as } |x| \to \infty \\ |x|^{-(2-\delta_B)} & \text{as } |x| \to 0 \end{cases} \\ |V(x)| &\lesssim \begin{cases} |x|^{-(2+\delta)} \\ |x|^{-(2-\delta)} \end{cases} & |(\partial_r V(x))_+| &\lesssim \begin{cases} |x|^{-(3+\delta_V)} & \text{as } |x| \to \infty \\ |x|^{-(3-\delta_V)} & \text{as } |x| \to 0 \end{cases} \\ \|V_-\|_K &\leqslant 4\pi^{d/2}/\Gamma(\tfrac{d}{2}-1) \end{aligned} \tag{A.3}$$

for some $\delta$, $\delta_A > 0$, and some $\delta_B$, $\delta_V > 0$ for $d = 3$, $\delta_B = \delta_V = 0$ for $d \geqslant 4$, where $\|f\|_K := \| \, |x|^{2-d} * |f| \, \|_\infty$ is the Kato norm. In an analogous perspective Erdoğan, Goldberg, and Schlag [8] proved (A.1)–(A.2) globally in time for $d \geqslant 3$ requiring

$$\begin{aligned} \mathbf{A} &\in C^0(\mathbb{R}^d, \mathbb{R}^d) \\ |\mathbf{A}(x)| &\lesssim \langle x \rangle^{-(1+\delta_A)} \\ \langle x \rangle^{1+\delta'_A} |\mathbf{A}(x)| &\in \dot{W}^{\frac{1}{2}, 2d}(\mathbb{R}^d, \mathbb{R}^d) \\ |V(x)| &\leqslant \langle x \rangle^{-(2+\delta_V)} \\ h & \text{ has no zero-energy resonance} \\ h & \text{ has only continous spectrum} \end{aligned} \tag{A.4}$$

for some $\delta_A > \delta'_A > 0$ and $\delta_V > 0$.

The critical scalings $|\mathbf{A}(x)| = |x|^{-1}$ and $|V(x)| = |x|^{-2}$ are not covered by the above results in $d \geqslant 3$ space dimensions. In the purely electric case $\mathbf{A} \equiv \mathbf{0}$, critically scaling potentials $V$'s are known which still give rise to global-in-time Strichartz estimates. This is the case of the Strichartz estimates for the scale-covariance elliptic operator $h = -\Delta + a|x|^{-2}$ in $d \geqslant 2$ dimensions, proved by Burq, Planchon, and Stalker [3] for $a$'s up to the Hardy threshold, $a > -\frac{1}{4}(d-2)^2$.

An alternative approach towards (global-in-time) Strichartz estimates at scaling-criticality is to exploit the dispersive properties of $h = -(\nabla - \mathrm{i}\mathbf{A})^2 + V$, that is, to establish the time-decay dispersive estimate $\|\mathrm{e}^{-\mathrm{i}th}\varphi\|_\infty \leqslant |t|^{-d/2}\|\varphi\|_1$ (from which Strichartz estimates follow in a standard way). For instance, in the purely electric case the dispersive estimate $\|\mathrm{e}^{-\mathrm{i}t(-\Delta+a|x|^{-2})}\varphi\|_\infty \leqslant |t|^{-d/2}\|\varphi\|_1$ is known for $a > -\frac{1}{4}(d-2)^2$, proved by Fanelli, Felli, Fontelos, and Primo, in the special cases $d = 3$ [9] and $d = 2$ [10]. In the purely magnetic case $V \equiv 0$, the dispersive estimate $\|\mathrm{e}^{-\mathrm{i}t(\nabla-\mathrm{i}\mathbf{A})^2}\varphi\|_\infty \leqslant |t|^{-d/2}\|\varphi\|_1$ is known, proved again in [9], in the special case of the Aharonov-Bohm magnetic fields in $d = 2$ dimensions, namely $\mathbf{A}(x_1, x_2) = (-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2})$, $x = (x_1, x_2) \in \mathbb{R}^2$. In fact, the work [9] established the estimate $\|\mathrm{e}^{-\mathrm{i}th}\varphi\|_\infty \leqslant |t|^{-d/2}\|\varphi\|_1$ for an entire class of magnetic Schrödinger operators $h = -(\nabla - \mathrm{i}\mathbf{A})^2 + V$ in $d \geqslant 2$ dimensions with critically scaling homogeneous potentials $\mathbf{A}$'s and $V$'s, under an additional abstract condition that involves the eigenfunctions of $h$. Such a condition, so far, has been explicitly worked out only in the above-mentioned special cases in $d = 2, 3$ dimensions.

The critical scalings for $\mathbf{A}$ and $V$ represent a threshold for the validity of global-in-time Strichartz estimates. If $\mathbf{A}$ or $V$ decrease as $|x| \to \infty$ *less* than the threshold behaviour, then counterexamples to Strichartz are known: this is the case of Goldberg, Vega, and Visciglia [12] who produced, in the purely electric case, certain Schrödinger operators with homogeneous potentials $V(x) = |x|^{-\sigma}\phi(x/|x|)$, $\sigma \in [0,2)$, in $d \geqslant 2$ dimensions, which belong to $C^3(\mathbb{R}^d \setminus \{0\})$ and do not satisfy (A.1), (A.2) globally in time except for the trivial case $(q,r) = (\infty,2)$; in an analogous fashion, Fanelli and Garcia [11] produced, in the purely magnetic case, certain Schrödinger operators with homogeneous potentials $\mathbf{A}(x) = |x|^{-\sigma}\phi(x/|x|)$, $\sigma \in (0,1)$, in $d \geqslant 3$ dimensions, that belong to $C^\infty(\mathbb{R}^d \setminus \{0\})$ and do not satisfy (A.1), (A.2) globally in time except for the trivial case $(q,r) = (\infty,2)$.

*Local-in-time electromagnetic Strichartz estimates*

The finite-interval version of (A.1)–(A.2) is in fact all what is needed when Strichartz estimates are used to prove local uniqueness of the solution to the Cauchy problem of non-linear Schrödinger equations, which is in turn a key step for the local well-posedness (see, for a comprehensive review, [5], chapter 4, or [22], chapter 3). For finite $I$ the factors that prevent global dispersion of $\mathrm{e}^{-\mathrm{i}th}$ play no obstruction against (A.1)–(A.2). Local-in-time Strichartz estimates can thus be proved also in different schemes than the global-in-time ones, with less restrictions on the potentials. At the price of requiring high regularity on $\mathbf{A}$, or also on $V$, (A.1)–(A.2) are known locally in time up to a moderate amount of growth of $\mathbf{A}$ and $V$ at spatial infinity (respectively, linear and quadratic). Smoothness is a technical requirement needed for constructing the propagator $\mathrm{e}^{-\mathrm{i}th}$ directly in the form of integral operators, owing to underlining micro-local analysis and semi-classical parametrix techniques involved in the proof.

The standard reference for local-in-time magnetic Strichartz estimates is the seminal work of Yajima [23], where (A.1)–(A.2) is proved locally in time under the assumptions

$$\begin{array}{rcll} \mathbf{A} & \in & C^\infty(\mathbb{R}^d,\mathbb{R}^d) & \\ |\partial_x^\beta \mathbf{A}(x)| & \leqslant & C_\beta & \forall\,\beta\in\mathbb{N}_0^d\,,\,|\beta|\geqslant 1 \\ |\partial_x^\beta B_{jk}(x)| & \leqslant & C_\beta\langle x\rangle^{-(1+\delta)} & \forall\,\beta\in\mathbb{N}_0^d\,,\,|\beta|\geqslant 1 \\ V & \in & L^\infty(\mathbb{R}^d)+L^p(\mathbb{R}^d) & p>\frac{d}{2} \end{array} \tag{A.5}$$

for some $\delta > 0$, $C_\beta \geqslant 0$, where $B = (B_{ij})$, $B_{jk} = \partial_j A_k - \partial_k A_j$. At spatial infinity (A.5) allows for an at most linear growth of $\mathbf{A}$, which still includes the physically relevant case of a constant magnetic field, and puts no decay constraint on $V$ (but for its boundedness). Locally, $\mathbf{A}$ must be bounded (smooth) and the allowed local $L^p$-singularity of $V$ corresponds to the same critical case $|V(x)| \sim |x|^{-2}$ as in (A.3) and (A.4). Recently, Mizutani [15] adapted the previous methods for smooth $\mathbf{A}$'s and $V$'s with, respectively, an at most linear and an at most quadratic growth at spatial infinity and proved (A.1)–(A.2) locally in time under the assumptions

$$\begin{array}{rcll} \mathbf{A} & \in & C^\infty(\mathbb{R}^d,\mathbb{R}^d) & \\ |\partial_x^\beta \mathbf{A}(x)| & \leqslant & C_\beta\langle x\rangle^{1-|\beta|} & \forall\,\beta\in\mathbb{N}_0^d \\ |\partial_x^\beta B_{jk}(x)| & \leqslant & C_\beta\langle x\rangle^{1-|\beta|-\delta} & \forall\,\beta\in\mathbb{N}_0^d\,,\,|\beta|\geqslant 1 \\ V & \in & C^\infty(\mathbb{R}^d) & \\ |\partial_x^\beta V(x)| & \leqslant & C_\beta\langle x\rangle^{2-|\beta|} & \forall\,\beta\in\mathbb{N}_0^d \end{array} \tag{A.6}$$

for some $\delta > 0$, $C_\beta \geqslant 0$.

In view of (A.3)–(A.6) above, it has to be remarked that such conditions for the validity of Strichartz estimates are considerably more restrictive than those needed for the mere self-adjointness of $-(\nabla - \mathrm{i}\mathbf{A})^2 + V$, such as conditions (C.1) or (C.2) in section 2.

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