CERN-PRE 84-034

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22/84/E.P.

ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

ABOUT QUANTIZATION OF GRAVITY

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1984 TRIESTE

22/84/EP.

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Questa breve nota è dedicata al collega e maestro Bruno Ferretti per il Suo 70-esimo compleanno. La accompagnano la riconoscenza per quanto ha fatto per la fisica teorica in Europa e in Italia e l'espressione della più profonda stima ed amicizia.

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(+) This work has been partially supported by the Ministero della Pubblica Istruzione of Italy under a grant 1984.

1. Over the past years the interest on gravity as a quantum field theory has been refocused by the many promising developments, like grand-unification schemes and supergravity, toward a unified description of fundamental interactions [1]. The ultimate unification has to include gravity. This will depend on our capability to set up a consistent formulation of gravity at the quantum level and to clarify, in particular, the role of the Newton constant. According to the standard point of view the dimensionality of G_N ($G_N \simeq (10^{-33}\,\mathrm{cm})^2$) is considered to be the clear signal of the non renormalizable nature of quantum gravity and the large value of the Planck mass, $M_{PL}=G_N^{-1/2} \sim 10^{19}$ GeV, represents the most familiar example of the hierarchy problems in Particle Physics.

The underlying hope is that the eventual success will arise from the unique symmetry properties of the theory, which is invariant, in its simplest version, under the local coordinate reparametrizations $x \to x'=f(x)$. This suggest a deep similarity in meaning and methods with non abelian gauge theories.

It has been recognized that the most appropriate approach to perform the quantization of physical system of this sort is represented by the functional formalism [2] and we shall therefore choose this framework to present some general remarks about the problem.

The starting point is represented by the quantity (an Euclidean formulation is used).

$$Z = \int e^{-A}\, d\Omega \tag{1}$$

where two elements appear, the classical action $A = \int \mathcal{L}\, d^4x$ and the path integral measure $d\Omega$. It is clear that the quantum description is actually a hard problem and the generating functional (1) only represents a skeleton to support this effort. Indeed, different forms of the measure correspond to different quantization prescriptions, all leading to the same classical limit.

For problems like gravity where a completely satisfactory quantization procedure is not yet available, the selection of the volume can therefore represent an essential step toward the correct quantum formulation of the theory.

In this note we want to present a simple argument, based on dimensional considerations, for the determination of the functional volume in the gravitational case.

2. In order to illustrate our way of proceeding we start with a few remarks about the action A.

Following Einstein, the action of pure gravity is determined by the requirement of general invariance and simplicity and is customarily written in the form [3]

$$A = \frac{1}{G_N} \int d^4x \sqrt{g}\, (R + \Lambda) \tag{2}$$

G_N and Λ are the Newton and cosmological (dimensional) constants whose presence can be considered somehow surprising. Indeed A is dilatational invariant (*) and we expect that no dimensional parameters should appear in this case.

Formally the matter can be settled and G_N made to disappear by redefining the field $g_{\mu\nu}$, i.e.

$$g_{\mu\nu} \to \frac{1}{G_N} g_{\mu\nu}\,, \quad \Lambda \to \frac{\Lambda}{G_N} \equiv \lambda \tag{3}$$

This redefinition has actually a deeper meaning and amounts to ascribing to the new field the dimensionality +2 (in units of mass) which results from its transformation properties under dilatations [4] (one can

(*) being dilatation the particular coordinate transformation

$$x'^{\mu} = x^{\mu}/\rho$$

call this the group theoretical dimensionality).

Consider in fact the rule of transformation

$$\delta g_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -\varepsilon^{\lambda}\partial_{\lambda} g_{\mu\nu} - g_{\mu\lambda}\partial_{\nu}\varepsilon^{\lambda} - g_{\nu\lambda}\partial_{\mu}\varepsilon^{\lambda}; \qquad \delta x^{\mu} = \varepsilon^{\mu}(x). \tag{4}$$

For dilatations one has

$$\varepsilon^{\mu}(x) = -\varepsilon x^{\mu} \tag{5}$$

from which it follows that

$$\delta g_{\mu\nu} = \varepsilon(x\cdot\partial + 2)\, g_{\mu\nu} \tag{6}$$

In the language of field theory this relation identifies a field of dimension $d=2$(*).

It is easy to get convinced that as a consequence of eq. (6) a general invariant (and dilatational invariant) action A does not contain

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(*) Eq. (6) is a particular case of the result one obtains for a tensor $t^{\nu_1\cdots\nu_N}_{\mu_1\cdots\mu_M}$ i.e.

$$\delta t^{\nu_1\cdots\nu_N}_{\mu_1\cdots\mu_M} = \varepsilon(x\cdot\partial + d)\, t^{\nu_1\cdots\nu_N}_{\mu_1\cdots\mu_M}. \tag{6'}$$

with $d=M-N$, the difference between the number of covariant and of controvariant components. It is thus clear that with the transformation (6') we have identified the elementary dimensionality with the group theoretical one "d", arising from the tensorial character of the field.

any dimensional constant and its simplest form is (dim g=+4,dim R=0)

$$A = \int d^4x \sqrt{g}\, (R+\lambda). \tag{7}$$

The mechanism by which the connection with the standard formulation is regained and the Newton constant reappears in the theory will be considered later.

3. Let us now discuss the implications of enforcing the same constraint of dilatational invariance on the path integral measure $d\Omega$. It will be shown that this requires that the integration variables (in the functional sense) are fields of dimensionality 2.

In order to do it we shall use a "poor man's" argument which is based on the continuum generalization of a simple result valid in the discrete case.

Let us consider for a system with N degrees of freedom the volume element in the configuration space and its variation. For $q_m \to q'_m = q_m + \Delta q_m$ one has

$$d\Omega(q) \equiv \Pi_m\, dq_m \Rightarrow d\Omega(q') \equiv \Pi_m\, dq'_m = \Pi_m\, dq_m \det \frac{\partial q'_k}{\partial q_l}$$

$$\simeq d\Omega(q) \det\left(\delta_{kl} + \frac{\partial}{\partial q_l}\Delta q_k\right) \simeq d\Omega(q)\left[1 + \Sigma_m \frac{\partial}{\partial q_m}\Delta q_m\right]$$

namely

$$\Delta\, d\Omega(q) \simeq d\Omega(q)\, \mathrm{tr}\left(\frac{\partial}{\partial q_l}\Delta q_k\right) \tag{8}$$

Assuming that the continuum limit, $q_m \to q(x)$, $\prod_n dq_n \to \prod_x dq(x)$ does not hide any subtlety we finally obtain

$$\Delta\, d\Omega(q) = d\Omega(q)\, \mathrm{Tr}\left(\frac{\delta}{\delta q(x)}\, \Delta q(y)\right) =$$

$$= d\Omega(q) \int dx\, dy\, \delta(x-y) \frac{\delta}{\delta q(x)}\, \Delta q(y) \qquad (8')$$

The particular case of dilatations produces again an interesting condition: taking

$$\Delta q(x) = \varepsilon(x\cdot\partial + d)\, q(x)$$

we find

$$\Delta\, d\Omega(q) = d\Omega(q)\,\varepsilon \int dx\, dy\, \delta(x-y)\, (x\partial + d)\, \delta(x-y)$$

$$= d\Omega(q)\, \varepsilon\, (d-2)\, \delta(0) \int d^4x . \qquad (9)$$

This shows (if the presence of the ill-defined quantity - the number of degrees of freedom- is tolerated on the r.h.s.) that a dilatation invariant functional volume requires as integration variables fields with d=2.

In order to have an immediate appreciation of this condition take the case of a scalar field $\phi(x)$ for which d=1. The appropriate integration variable is then $q(x) = \phi^2(x)$.

If on the other hand we prefer to use the canonical field $\phi(x)$ (which is suggested by the simple expression of the Lagrangian) as fundamental quantity, this leads to the appearance of an additional factor(*):

$$d\Omega(q) = \prod_x d\phi^2(x) = \prod_x \phi(x)\, d\phi(x) = \mathrm{Det}\, \phi(x) \prod_x d\phi(x) \qquad (10)$$

Det $\phi(x)$ has to be understood as the determinant in the functional sense.

Let us now move to the case of gravitation. As possible field variables one can choose $g_{\mu\nu}$, $g^{\mu\nu}$ or the vierbeine V_μ^a, V_a^μ (remember $g_{\mu\nu}(x) = \sum_a V_\mu^a(x)\, V_\nu^a(x)$).
We indicate by q(x) any of these fields and by "d" the corresponding dimensionality (d=2, -2,1,-1 in the order, for the fields listed before). It is then clear, on invariance grounds, that in order to build the appropriate integration variable of dimension d=2 can simply resort to the quantity $g(x) = \det g_{\mu\nu}(x)$ (the determinant is on the tensor indices) so that the functional measure will be of the form

$$d\Omega = \prod_x d\left[q(x) (g(x))^k \right] \qquad (11)$$

where (since $\dim g = 8$) k is given by the expression

$$k = \frac{2-d}{8} \qquad (12)$$

This formula, obtained in Ref. 5, coincides with the result recently derived by K. Fujikawa for the gravitational part of the measure (6). His work follows quite a different path, based on the requirement of the absence

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(*) These results can be given a much more respectable basis in the frame-work of a new formulation of the functional approach we have recently proposed.(5) The main idea there is to achieve a formal equivalence of the Euclidean Field Theory with Classical Statistical Mechanics (in 4+1 dimensions). This requires the introduction of an extra time variable and setting up a complete canonical scheme. As a payback, the invariance properties of the measure can be adequately controlled and the determinantal factor of Eq.(10) is indeed seen to be there.

of anomalies for the B.R.S. supersymmetry associated with general coordinate transformation.

Similarly if a matter part is present, for instance a scalar or a spinor field for which d=0 (see eq.(6') , the variable $g^{1/4}\phi(x)$ has to be used.

If for some reason it is preferable to select as elementary fields the canonical Lagrangian variables, then the relevant path integral measures turn out to be

$$
\begin{aligned}
d\Omega(g_{\mu\nu}) &= \Pi_x \Pi_{\mu\nu\geq}\, dg_{\mu\nu}(x) \\
d\Omega(g^{\mu\nu}) &= (\mathrm{Det}\,g)^5\, \Pi_x \Pi_{\mu\geq\nu}\, dg^{\mu\nu}(x) \\
d\Omega(V_\mu^a) &= (\mathrm{Det}\,g)^2\, \Pi_x \Pi_{\mu,a}\, dV_\mu^a(x) \\
d\Omega(V_a^\mu) &= (\mathrm{Det}\,g)^6\, \Pi_x \Pi_{\mu,a}\, dV_a^\mu(x)
\end{aligned}
\tag{13}
$$

characterized in most cases by the presence of functional determinants.

These factors can be alternatively viewed as additional terms in the action according to the relation

$$\mathrm{Det}\,(g(x))^m\, e^{-\int \mathcal{L}\, d^4x} = e^{-\int d^4x \{\mathcal{L} - m\delta(0) \log g(x)\}} \tag{14}$$

4. One immediately notices that owing to the presence of the $\log g(x)$ quantity an expansion around the configuration $V_\mu^a = 0$ is, in general, forbidden. This is a gratifying feature: in fact in the case of gravity the non polynomial character of the Lagrangian requires a special field theoretical treatment, which consists in expanding the quantum field around a non vanishing background solution. Our formulation can therefore be considered as providing a theoretical motivation for the special treatment one applies in quantum gravity.

This fact leads to a natural way for clarifying the role of the dimensional Newton constant in the framework of a fully conformal invariant quantum field theory. The following argument is offered.

The simpleste form of a classical background solution is the constant one $g_{\mu\nu}(x)|_{class.} = \mathrm{const.}\,\delta_{\mu\nu}$. Keeping into account the dimensionality of the field and the quantum nature of the problem we shall write

$$
\begin{aligned}
\langle 0| g_{\mu\nu}(x) |0\rangle &= G_N^{-1}\, \delta_{\mu\nu}\,, \\
\langle 0| V_\mu^a(x) |0\rangle &= G_N^{-1/2}\, \delta_{\mu a}
\end{aligned}
\tag{15}
$$

The Newton constant is introduced through the vacuum expectation value of the gravitational fields: the vacuum is consequently no longer dilatational (and general) invariant while Poincarè symmetry is preserved. Spontaneous symmetry breaking provides, as in the case of models à la Goldstone or à la Higgs, the mechanism to slip into the theory a dimensional quantity.

We can say then that the strength of the Newton force is not characterized by what dimensionalconstant one writes in front of the Lagrangian but rather by the separation between background and effective field e.g.

$$V_\mu^a(x) = G_N^{-1/2}\, \delta_{\mu a} + \varphi_\mu^a(x)\,. \tag{16}$$

The dimensional parameter G_N characterizes the low energy behaviour of the amplitude, counting the number of gravitons emitted and absorbed.

A point which has to be further clarified remains the connection between the Newton constant and the underlying Lagrangian. To this aim let us turn to the effective Lagrangian of eq. 14 which in terms of the vierbein field can be written as ($V \equiv \det V_\mu^a = g^{1/2}$)

$$\mathcal{L} = \frac{1}{2} \partial_\mu V_\nu^a \partial_{\mu'} V_{\nu'}^{a'} C^{\mu\nu a}_{\mu'\nu' a'} + \lambda V - 4\,\delta(0) \log V(x). \tag{17}$$

In order to give a meaning to the quantity $\delta^{(4)}(0)$ a phenomenological cut-off k_M can be introduced i.e.

$$\delta^{(4)}(0) \approx k_M^4 \tag{18}$$

The relation between the two dimensional quantities G_N and K_M now present , can be easily established if one uses the constant solution (15) as an input for the equations of motion which follow from the Lagrangian (17). This finally leads to the simple formula[7]

$$\lambda M_{PL}^4 = 4 k_M^4 . \tag{19}$$

The conclusion emerging from this result is that the Planck mass does not coincide with the cut-off of the theory. The large size of M_{PL} (on the hadron scale) appears to be related to the extreme smallness of the (non vanishing!) cosmological constant.

This argument is highly qualitative, of course, and contrasts with the current wisdom. The point of view we illustrated in this note suggests nevertheless the possibility of an alternative look at quantum gravity and may represent a promising avenue for further research.

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