

THE COMPLETE ONE-LOOP SPIN CHAIN FOR $N = 2$ SUPER YANG-MILLS

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Abstract We show that the complete planar one-loop mixing matrix of the $\mathcal{N} = 2$ Super Yang-Mills theory can be obtained from a reduction of that of the $\mathcal{N} = 4$ theory. For composite operators of scalar fields, this yields an anisotropic XXZ spin chain, whose spectrum of excitations displays a mass gap.

1. INTRODUCTION AND DISCUSSION

The study of the AdS/CFT correspondence in some particular subsectors characterised by large quantum numbers, started with [1], provided a new and very intuitive way to look at the duality between gauge and string theory. The basic idea is that in presence of large quantum numbers one can give a semiclassical description of the string states. This semiclassical description is reproduced on the gauge theory side by the coherent states of a quantum mechanical model, which can be extracted from the study of a particular subsector of the dual gauge theory.

More precisely, the original idea of Berenstein et al. [1] was to regard some gauge-invariant operators of $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory as a discretized version of the physical type IIB string on the plane-wave background. The BMN operators are single trace operators containing a large number of one of the complex scalar fields of $\mathcal{N} = 4$ SYM, with the insertion of few *impurities* given by the other fields of the $\mathcal{N} = 4$ supermultiplet, each of them corresponding to a different excita-

tion of the string. The mass of the string corresponds to the anomalous dimension of the gauge theory operator (see [2] for a review).

It was soon realised [3] that the study of type IIB string on the plane-wave background corresponds to a semiclassical expansion around a particular solution of the $AdS_5 \times S^5$ sigma model, describing a massless point-like string spinning along the equator of S^5 . Then also more general solutions describing extended strings spinning in AdS_5 and/or in S^5 were analysed¹. The corresponding gauge theory operators bring in this case a large number of impurities. This makes very difficult to compute their anomalous dimensions, which are given by the eigenvalues of a large mixing matrix. A very interesting observation was made in [5], where the one-loop mixing matrix for the operators containing scalar impurities was identified with the Hamiltonian of an integrable spin-chain. This allows one to compute the anomalous dimensions of the “long” gauge theory operators by using the Bethe ansatz. The integrability property has then been extended to the full $\mathcal{N} = 4$ dilatation operator and studied also to higher orders in the ’t Hooft coupling (see [6] for a review).

Besides giving evident computational advantages, the relationship with integrable systems brings also new insights on the gauge/string duality. In fact the infrared dynamics of the spin-chain, for coherent state excitations with wavelengths much longer than the distance between two nearest-neighbors sites, is described by a sigma model which can be mapped to that of a string spinning in S^5 [7]. In this way, not only the mass of the string is identified with the anomalous dimensions, but also its shape can be identified with the mean value of the spin on the coherent state built with the gauge theory operators. Moreover, these states probe regions of the string spectrum far away from the states protected by supersymmetry. This suggest a new possible path to extract informations about the string dual of gauge theories, which can be studied also in cases where some (or all) the supersymmetries are broken and the conformal invariance is lost.

Here we discuss the one-loop renormalisation of composite operators in $\mathcal{N} = 2$ SYM theory, and we observe that the *complete* mixing matrix can be simply obtained from a reduction of that of the maximally supersymmetric $\mathcal{N} = 4$ SYM theory. Then, by focusing on the subsector of composite operators of scalar fields, we recover the identification of their mixing matrix with the Hamiltonian of an *anisotropic* XXZ spin chain [13].

¹For an up-to-date review, see the Tseytlin’s lectures at this school [4].

Anisotropic (XXZ) spin-chains have been found recently also in the study of some subsectors of $\mathcal{N} = 1$ SYM and in pure Yang–Mills in the light–cone gauge [8]. Also very recently it appeared an analysis of the two–loops dilatation operator in QCD and $\mathcal{N} = 1$ SYM [10]. It would be interesting to see whether the direct relationship between the $\mathcal{N} = 2$ and the $\mathcal{N} = 4$ mixing matrix found here still persist at higher orders. In fact the ground state of the $\mathcal{N} = 2$ XXZ spin chain is protected to all orders of perturbation theory [11], and one can expect that the integrability property could be maintained at higher orders at least in some subsector.

Finally, we observe that the direct relationship between the dilatation operator of the $\mathcal{N} = 2$ and the $\mathcal{N} = 4$ theory that we presented here could be a useful tool to explore a possible dual string theory description of composite operators of the $\mathcal{N} = 2$ SYM.

2. OPERATOR MIXING IN $\mathcal{N} = 2$ SYM

We start by writing the Lagrangian of $\mathcal{N} = 2$ Super Yang–Mills in Weyl notations

$$L_E = \frac{2}{g^2} \text{Tr} \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + (D_\mu \phi)^\dagger D_\mu \phi + \psi \sigma^\mu D^\mu \bar{\psi} + \lambda \sigma^\mu D^\mu \bar{\lambda} - i\sqrt{2} (\psi[\bar{\phi}, \lambda] + \bar{\psi}[\phi, \bar{\lambda}]) + \frac{1}{2} [\bar{\phi}, \phi]^2 \right), \quad (1.1)$$

in terms of the euclidean σ -matrices $\sigma^\mu = (\mathbf{1}, i\tau^i)$, τ^i being the Pauli matrices. The field ϕ is the complex scalar field of the $\mathcal{N} = 2$ Super Yang–Mills, the two Weyl spinors λ and ψ are the fermionic superpartners and the covariant derivative reads $D_\mu \phi = \partial_\mu \phi - i[A_\mu, \phi]$.

We are interested in studying the planar, one–loop renormalization of composite operators of the elementary fields appearing in the Lagrangian (1.1). We observe that the mixing matrix in this approximation can be directly obtained from a reduction of that of $\mathcal{N} = 4$ SYM, which can be found in [6]. In fact, the $\mathcal{N} = 4$ theory can be seen as an $\mathcal{N} = 2$ SYM coupled with an hypermultiplet in the adjoint representation. The Feynman diagrams contributing to the mixing of operators containing only fields of the $\mathcal{N} = 2$ vector multiplet are the same in the two theories, except for the self–energy. However, we will show that the wave function renormalisation for the fields in the Lagrangian (1.1) is exactly the same as in the $\mathcal{N} = 4$ theory. Thus we can read the mixing matrix of the $\mathcal{N} = 2$ SYM simply by restricting the indices of that of the $\mathcal{N} = 4$ SYM to the $\mathcal{N} = 2$ vector multiplet. Similar arguments based on the inspection of the Feynman diagrams were used to connect the one–loop light–cone mixing matrices of Yang–Mills theories with $0 \leq \mathcal{N} \leq 4$ supersymmetry

[12, 8] and more recently to obtain the complete one-loop mixing of QCD composite operators in a covariant formalism [9].

Let us focus for the moment on the mixing of operators containing only the two real scalar fields of the $\mathcal{N} = 2$ vector multiplet, related to the complex field as $\phi = 1/\sqrt{2}(\varphi_1 + i\varphi_2)$ ²

$$\mathcal{O} = \text{Tr}(\varphi_{i_1} \dots \varphi_{i_l} \varphi_{i_{l+1}} \dots \varphi_{i_L}) . \quad (1.2)$$

We then study the correlator

$$\langle \varphi^{i_L} \dots \varphi^{i_{l+1}} \varphi^{i_l} \dots \varphi^{i_1} \mathcal{O} \rangle = \mathcal{Z}_\varphi^{L/2} \mathcal{Z}_\mathcal{O} \langle \varphi_r^{i_L} \dots \varphi_r^{i_{l+1}} \varphi_r^{i_l} \dots \varphi_r^{i_1} \mathcal{O}_r \rangle , \quad (1.3)$$

where \mathcal{Z}_φ is the usual wave-function renormalization needed to make finite the two-point function $\langle \bar{\varphi}_r(x) \varphi_r(y) \rangle$ and $\mathcal{Z}_\mathcal{O}$ is the renormalization factor for the composite operator. The operators (1.2) mix among themselves at the quantum level, and $\mathcal{Z}_\mathcal{O}$ is a matrix carrying the indices of the real fields. Actually the operators (1.2) mix at one-loop also with operators containing derivatives of the scalar fields, but the mixing matrix is triangular [13]. Then, for the computation of the anomalous dimensions we can neglect this mixing and study only the correlators (1.3). By using the large N approximation, we focus on the nearest-neighbors interaction

$$\langle \dots \varphi_{i_{l+1}}(x) \varphi_{i_l}(y) \dots \text{Tr}(\dots \varphi_{j_l} \varphi_{j_{l+1}} \dots)(z) \rangle \quad (1.4)$$

The corresponding one-loop diagrams are displayed in Fig.(1.1).

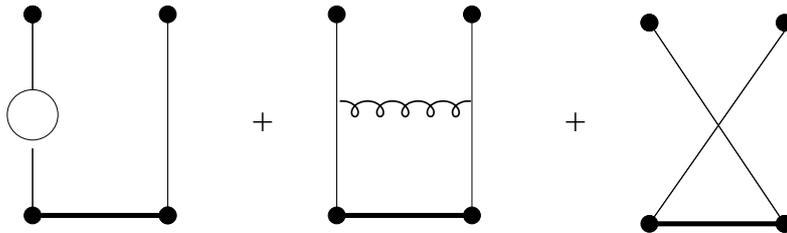


Figure 1.1 Feynman diagrams contributing at one-loop. The thick horizontal line joins the fields belonging to the composite operator.

Concerning the first, it turns out that in $\mathcal{N} = 2$ Super Yang-Mills all the self-energy diagrams cancel. This means that in the convention we

²This mixing has been studied in [13]. Here we use the same conventions: the generators of the gauge group are normalized as $\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$ and the relations between the bare and renormalized quantities are $g = Z_g g_r$ and $\varphi = Z_\varphi^{1/2} \varphi_r$.

choose for the lagrangian (1.1) the only renormalization for the fields is that associated to the gauge coupling, *i.e.* $\mathcal{Z}_\varphi^{1/2} \equiv \mathcal{Z}_g$. On the other hand from the knowledge of the β -function

$$\beta(g) \equiv \mu \frac{\partial}{\partial \mu} g_r = -g\mu \frac{\partial}{\partial \mu} \log \mathcal{Z}_g = -\frac{g^3}{16\pi^2} 2N \quad (1.5)$$

one can derive the expression for \mathcal{Z}_g

$$\mathcal{Z}_g = 1 - \frac{g^2 N}{16\pi^2} \frac{\mu^{-2\epsilon}}{\epsilon} . \quad (1.6)$$

Indeed (1.5) follows from (1.6) after taking the $\epsilon \rightarrow 0$ limit. Then we get

$$\mathcal{Z}_\varphi = 1 - \frac{g^2 N}{4\pi^2} \frac{\mu^{-2\epsilon}}{2\epsilon} \quad (1.7)$$

giving $\gamma_\varphi = 1/2\mu\partial/\partial\mu\mathcal{Z}_\varphi = g^2 N/8\pi^2$ exactly as for the $\mathcal{N} = 4$ theory in the Feynman gauge [14]. The $\mathcal{Z}_\varphi^{L/2}$ factor in (1.3) takes care of half of the contribution associated to the wave function renormalisation. We have thus to multiply the composite operator by a factor

$$\mathcal{Z}^{(g)\dots j_l j_{l+1} \dots}_{\dots i_l i_{l+1} \dots} = \mathbf{1} - \frac{g^2 N}{8\pi^2} \frac{\mu^{-2\epsilon}}{2\epsilon} \delta_{i_l}^{j_l} \delta_{i_{l+1}}^{j_{l+1}} . \quad (1.8)$$

for each nearest-neighbor.

Coming now to the other two diagrams of Fig.(1.1), the one-loop correction associated to the gluon exchange is³

$$\mathcal{Z}^{(gluon)\dots j_l j_{l+1} \dots}_{\dots i_l i_{l+1} \dots} = \mathbf{1} + \frac{g^2 N}{16\pi^2} \frac{\mu^{-2\epsilon}}{2\epsilon} \delta_{i_l}^{j_l} \delta_{i_{l+1}}^{j_{l+1}} , \quad (1.9)$$

and that associated to the four-scalar interaction

$$\mathcal{Z}^{(four\ sc.)\dots j_l j_{l+1} \dots}_{\dots i_l i_{l+1} \dots} = \mathbf{1} + \frac{g^2 N}{16\pi^2} \frac{\mu^{-2\epsilon}}{2\epsilon} \left(2\delta_{i_l}^{j_{l+1}} \delta_{i_{l+1}}^{j_l} - \delta_{i_l}^{j_l} \delta_{i_{l+1}}^{j_{l+1}} - \delta_{i_l i_{l+1}} \delta^{j_l j_{l+1}} \right) . \quad (1.10)$$

As anticipated, the contributions coming from the gluon exchange and from the four-scalar interaction are the same as in the $\mathcal{N} = 4$ case [5] except that now the indices $i_l, i_{l+1}, j_l, j_{l+1}$ run only over two values and not six because $\mathcal{N} = 2$ Super Yang-Mills has only two real scalars. Adding the three contributions in (1.8), (1.9) and (1.10) we get

$$\mathcal{Z}^{\dots j_l j_{l+1} \dots}_{\dots i_l i_{l+1} \dots} = \mathbf{1} - \frac{g^2 N}{16\pi^2} \frac{\mu^{-2\epsilon}}{2\epsilon} \left(\delta_{i_l i_{l+1}} \delta^{j_l j_{l+1}} + 2\delta_{i_l}^{j_l} \delta_{i_{l+1}}^{j_{l+1}} - 2\delta_{i_l}^{j_{l+1}} \delta_{i_{l+1}}^{j_l} \right) . \quad (1.11)$$

³For details on the computations, we refer to [13].

In conclusion, we see that matrix of anomalous dimensions for the operators (1.2) can be directly obtained from that of $\mathcal{N} = 4$ theory [5] by restricting its indices to that of the two real scalar fields of the $\mathcal{N} = 2$ vector multiplet, $i = 1, 2$ ⁴. The same applies to the mixing matrix of operators containing gluons and fermions. In fact, if we consider in $\mathcal{N} = 4$ the matrix elements with indices running only on the fields of the $\mathcal{N} = 2$ vector multiplet, by definition the contribution of the extra particles in the hypermultiplet can only appear in loops, since they cannot appear as external states. In particular at the one-loop level they only appear in the self-energy diagrams. We remark that the wave function renormalisation for the gluon and the fermion fields is the same of that of the scalar field (1.7), as one expects from supersymmetry⁵. The remaining Feynman diagrams are the same in $\mathcal{N} = 4$ and $\mathcal{N} = 2$ theories. Then we conclude that the complete one-loop mixing matrix of composite operators in $\mathcal{N} = 2$ SYM can be directly read from that of the $\mathcal{N} = 4$ theory.

3. SCALAR OPERATORS AND THE XXZ SPIN CHAIN

Let us now come to the relation with the spin-chain, focusing on the sector of operators (1.2). Quite naturally the two scalar fields of the $\mathcal{N} = 2$ SYM can be interpreted as different orientations of a spin and then the whole gauge invariant operator formed just by scalars can be seen as a spin chain. The cyclicity of the trace makes the chain closed and implies that the physical states of the chain corresponding to the gauge theory operators have zero total momentum. Before to write down the spin chain Hamiltonian we observe that the operators containing only products of the complex scalar field ϕ have vanishing anomalous dimensions. In fact these operators, when represented in terms of the real fields φ_i , are symmetric and traceless in the real indices $i = 1, 2$, and this ensures the vanishing of their one-loop anomalous dimensions computed from (1.11). This suggest to take them as the ground state of the spin chain and to identify the two orientations of a spin with the

⁴Another difference is that the 't Hooft coupling that appears in (1.11) is the renormalised running coupling $\lambda_r = g_r^2 N$. However, the substitution $\lambda \rightarrow \lambda_r$ induces only higher order corrections. With this remark in mind, we will write our results in terms of the bare coupling λ to simplify the notation.

⁵In general the supersymmetry can be broken when using the dimensional regularisation scheme, but in this case one can check with an explicit computation that the self-energies are all the same (in particular they are all vanishing for $\mathcal{N} = 2$).

following 2-vectors

$$\bar{\phi} \rightarrow |+\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi \rightarrow |-\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (1.12)$$

In this basis, the matrix of anomalous dimensions $\gamma_{\mathcal{O}} \equiv \mathcal{Z}_{\mathcal{O}}^{-1} \mu \frac{\partial}{\partial \mu} \mathcal{Z}_{\mathcal{O}}$, with $\mathcal{Z}_{\mathcal{O}}$ given by (1.11) reads [13]

$$\gamma_{\mathcal{O}} = \frac{\lambda}{16\pi^2} H_{\text{XXZ}}, \quad (1.13)$$

where

$$H_{\text{XXZ}} = -\frac{1}{2} \sum_{l=1}^L [(\sigma^x)_l (\sigma^x)_{l+1} + (\sigma^y)_l (\sigma^y)_{l+1} + \Delta ((\sigma^z)_l (\sigma^z)_{l+1} - \mathbf{1}_l \mathbf{1}_{l+1})] \quad (1.14)$$

is the Hamiltonian of an XXZ spin chain. For the $\mathcal{N} = 2$ theory the value of the anisotropy parameter is $\Delta = 3$. As anticipated, the ground state of the spin chain corresponds to the protected operator $\mathcal{O}_{vac} \equiv \text{Tr}(\phi^L)$. The excited states are associated to spin flips along the chain, which in the field theory language correspond to the insertion of ‘‘impurities’’ $\bar{\phi}$ in the operator \mathcal{O}_{vac} . In [13] we studied the spectrum associated to these excitations. The energy associated to one impurity turns out to be

$$E_n = \frac{\lambda}{8\pi^2} \left[(\Delta - 1) + \frac{2\pi^2}{L^2} n^2 \right] \quad (1.15)$$

We thus see that the presence of a non-trivial anisotropy parameter $\Delta > 1$ implies the presence of a *mass gap* of the order of the 't Hooft coupling λ in the spectrum.

Acknowledgments

It is a pleasure to thank the organizers of the Cargèse 2004 ASI. Part of the A.T. work was supported by the European Union under RTN contract HPRN-CT-2000-00131.

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