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Ghost story. I. Wedge states in the oscillator formalism

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Abstract: This paper is primarily devoted to the ghost wedge states in string field theory formulated with the oscillator formalism. Our aim is to prove, using such formalism, that the wedge states can be expressed as $|n\rangle = \exp\left[\frac{2-n}{2}\left(\mathcal{L}_0 + \mathcal{L}_0^\dagger\right)\right]|0\rangle$, separately in the matter and ghost sector. This relation is crucial for instance in the proof of Schnabl's solution. We start from the exponentials in the rhs and wish to prove that they take precisely the form of wedge states. As a guideline we first re-demonstrate this relation for the matter part. Then we turn to the ghosts. On the way we face the problem of 'diagonalizing' infinite rectangular matrices. We manage to give a meaning to such an operation and to prove that the eigenvalues we obtain satisfy the recursion relations of the wedge states.

Keywords: String Field Theory, Ghost Wedge States.

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1. Introduction

The analytic solution of string field theory [1] has been found by means of the CFT language. All the most recent developments in this field are also formulated by means of the same powerful formalism [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12], while the 'old' oscillator formalism [13, 14, 15, 16] has remained in the shadow. This is in sharp contrast with the developments in vacuum string field theory of a few years ago, when the two formalisms played a parallel role in the search for solutions (but in that case the ghost sector of the theory was almost irrelevant). One of the reasons for this asymmetry is certainly the extreme simplicity some tools needed in order to prove the string field theory solution, take in the conformal field theory language. The typical example in this sense is provided by the wedge states, an essential tool in the construction of the solution, which take an astonishingly simple form in the conformal language. The second reason is, on the other hand, the almost unapproachably complicated form of the oscillator formalism, especially in the ghost sector.

It is nevertheless clear that it would be highly desirable to derive Schnabl's solution [1] also in the oscillator formalism. On one hand it is disturbing that this formalism, which has served so well in many developments in string theory, seems to have become suddenly unfit to deal with the most recent progress. On the other hand finding a more algebraic way to formulate the analytic solution to string field theory may open new prospects and suggest new solutions. For instance, in vacuum string field theory the oscillator formalism opened the way to finding a "complete" set of orthonormal projectors [18, 20], a sort of canonical way to classify solutions.

The road map to an algebraic proof of Schnabl's solution has been laid down by Okawa in [2]. In order for this to be applicable to the oscillator approach (apart from minor details) two main conditions should be fulfilled. The first is the representation of the wedge states as an exponential of $\mathcal{L}_0 + \mathcal{L}_0^\dagger$

$$|n\rangle = e^{\frac{2-n}{2}\left(\mathcal{L}_0+\mathcal{L}_0^\dagger\right)}|0\rangle \tag{1.1}$$

The second is a meaningful description of the star product of such states as $c_1|0\rangle$ or of the wedge states themselves. This paper will deal with the first problem. We will show below that eq.(1.1) can be given a meaning separately in the matter sector and in the ghost sector, the latter being the crux of the problem. The oscillator formalism proves to be, if not as simple as the conformal formalism, at least as effective.

The first problem we meet in dealing with the ghost sector is the normal ordering. We are familiar with what we call below the *conventional* normal ordering, which is very handy in the analysis of the perturbative string spectrum. This option is discussed in detail in section 8, but, even if it is correct, it does not appear to be a convenient choice. We use instead the *natural* normal ordering, which is required by the SL(2,R) invariant vacuum. This appears to be the right option. The second problem is the use of asymmetric bases, that is we are obliged to use matrices which act on two different bases on the left and on the right. This complication can be dealt with first of all because the relevant matrices commute and then because we can find the basis that diagonalizes them. This allows us to reduce the relation (1.1) to a relation of eigenvalues and show that the relevant recursion relations are satisfied.

The paper is organized as follows. In section 2 we re-derive the equivalence (1.1) for the matter sector. The problem has already been solved in the oscillator formalism, but we redo it here as a guideline for the ghost sector, and also because the derivation is partially new and simpler. Section 3 contains preliminary materials concerning the ghost sector. Section 4 concerns the integration of what we call the Kostelecky–Potting equations, [21]. In section 5 we diagonalize the infinite matrices we need in our problem. In section 6 we prove the recursion relation implied by eq.(1.1) for the ghost sector.

In section 7, for completeness, we repeat the same oscillator analysis for the twisted ghost sector, although twisted ghosts do not seem to be relevant to the proof of Schnabl's analytic solution. In section 8 we discuss the already mentioned conventional normal ordering option and illustrate its difficulties. Section 9 contains some conclusions. Several appendices contain auxiliary materials and samples of calculations that are needed in the course of the paper.

2. Matter wedge states

The exponential in the RHS of (1.1)factorizes in matter and ghost part, therefore it must be possible to deal with the two sectors separately. Let us start with the matter sector. Our problem is therefore to prove the representation of matter wedge states, i.e. to prove that

$$|n\rangle = e^{-\frac{n-2}{2}(\mathcal{L}_0+\mathcal{L}_0^\dagger)}|0\rangle = \mathcal{N}_n\, e^{-\frac{1}{2}a^\dagger S_n a^\dagger}|0\rangle \tag{2.1}$$

where S_n and $\mathcal{N}_n$ are the appropriate matrix and normalization factor for the n–th matter wedge state in the discrete oscillator basis [31]. This problem has already been solved recently by Fuchs and Kroyter, [30, 8, 29]. We repeat here in detail the derivation because it is a useful guide to the more complicated case of the ghost wedge states, but also because our derivation, although inspired by Fuchs and Kroyter's one, is rather different and does not require any particular regularization except the one provided by special functions such as Gamma, Dilogarithm and Hypergeometric functions.

Let us start from the vacuum that appears in (2.1). It is defined in the usual way as $a_n|0\rangle = 0$ for $n \geq 0$. The matter Virasoro generators ($n > 0$ and $\alpha_n = \sqrt{n}a_n$) are

$$L_n^{(X)} = \alpha_0\sqrt{n}\, a_n + \sum_{k=1}^{\infty} \sqrt{k(k+n)}\, a_k^\dagger a_{n+k} + \frac{1}{2}\sum_{k=1}^{n-1} \sqrt{k(n-k)}\, a_k\, a_{n-k} \tag{2.2}$$

$$L_0^{(X)} = \frac{1}{2}\alpha_0^2 + \sum_{k=1}^{\infty} k\, a_k^\dagger\, a_k \tag{2.3}$$

$$L_{-n}^{(X)} = \alpha_0\sqrt{n}\, a_n^\dagger + \sum_{k=1}^{\infty} \sqrt{k(k+n)}\, a_{n+k}^\dagger a_k + \frac{1}{2}\sum_{k=1}^{n-1} \sqrt{k(n-k)}\, a_{n-k}^\dagger\, a_k^\dagger \tag{2.4}$$

Since the zero mode does not come into play in the problem we are dealing with and, on the other hand, it commutes with everything, we are at liberty to ignore it altogether.

Now from the definition of $\mathcal{L}_0 + \mathcal{L}_0^\dagger$ we have

$$\begin{aligned}\mathcal{L}_0 + \mathcal{L}_0^\dagger &= 2L_0^{(X)} + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{4n^2-1}(L_{2n}^{(X)} + L_{-2n}^{(X)}) \\ &= a^\dagger A a^\dagger + a\, B\, a + a^\dagger C\, a\end{aligned} \tag{2.5}$$

where the matrices A, B, C are:

$$A_{pq} = B_{pq} = \frac{1}{2}\sum_{n=1}^{\infty} \ell_n \sqrt{pq}\delta_{p+q,n} \tag{2.6}$$

$$C_{pq} = \sum_{n\geq 1} \ell_n \sqrt{pq}\,(\delta_{p+n,q} + \delta_{q+n,p}) + p\ell_0\delta_{p,q} \tag{2.7}$$

where

$$\ell_{2n+1} = 0, \qquad \ell_{2n} = \frac{2(-1)^{n+1}}{4n^2-1}, \qquad n \geq 0, \qquad \ell_n = 0, \quad n < 0 \tag{2.8}$$

These matrices are symmetric, moreover they vanish for odd $p+q$. Therefore they commute with the twist matrix $\hat{C}$:

$$\hat{C}\, A = A\, \hat{C}, \qquad \hat{C}\, C = C\, \hat{C}$$

2.1 The KP equations

The way to prove (2.1) has been shown years ago by Kostelecky and Potting, [21]. The idea is to factorize the exponential of (2.5) as follows

$$e^{t\left(a^\dagger A\, a^\dagger + a^\dagger C\, a + a\, B\, a\right)} = e^{\eta}\, e^{a^\dagger \alpha a^\dagger}\, e^{a^\dagger \gamma\, a}\, e^{a\,\beta\, a} \tag{2.9}$$

where we have introduced an arbitrary parameter t. Therefore α, β, γ and η are to be understood as functions of t. Now one differentiate both sides, commutes to the left and equates. The result is

$$A = \dot{\alpha} - \frac{d\,e^{\gamma}}{dt} e^{-\gamma}(\alpha + \alpha^T) + \frac{1}{2}(\alpha + \alpha^T) e^{-\gamma^T}(\dot{\beta} + \dot{\beta}^T) e^{-\gamma}(\alpha + \alpha^T) \tag{2.10}$$

$$B = e^{-\gamma^T} \dot{\beta} e^{-\gamma} \tag{2.11}$$

$$C = \frac{d\,e^{\gamma}}{dt} e^{-\gamma} - (\alpha + \alpha^T) e^{-\gamma^T}(\dot{\beta} + \dot{\beta}^T) e^{-\gamma} \tag{2.12}$$

$$0 = \dot{\eta} - \mathrm{Tr}(\alpha e^{-\gamma^T}(\dot{\beta} + \dot{\beta}^T) e^{-\gamma}) \tag{2.13}$$

We will refer to these as the KP equations. In our case we have $A = B = B^T$. So from (2.11) we get $\beta = \beta^T$. Moreover we are interested only in the symmetric part of α, so we symmetrize (2.10). Calling the symmetric part with the same symbol α: $\frac{1}{2}(\alpha + \alpha^T) \to \alpha$, finally one gets

$$\dot{\alpha} = A + \{C, \alpha\} + 4\alpha B \alpha \tag{2.14}$$

$$\dot{\beta} = e^{\gamma^T}\, B\, e^{\gamma} \tag{2.15}$$

$$\frac{d\,e^{\gamma}}{dt} e^{-\gamma} = C + 4\alpha B \tag{2.16}$$

$$\dot{\eta} = 2\,\mathrm{Tr}(\alpha B) \tag{2.17}$$

We are actually interested only in α and η, with the initial condition $\alpha(0) = 0$. If $CA = AC$, it is easy to integrate (2.14) and obtain

$$\alpha(t) = A \frac{\sinh\left(\sqrt{C^2 - 4A^2}\, t\right)}{\sqrt{C^2 - 4A^2} \cosh\left(\sqrt{C^2 - 4A^2}\, t\right) - C \sinh\left(\sqrt{C^2 - 4A^2}\, t\right)} \tag{2.18}$$

where we have assumed that $C^2 - 4A^2$ is a positive operator. It so happens that in our case A and C commute and $C^2 - 4A^2$ is indeed positive, so that the solution to the KP equations is precisely (2.18).

2.2 A and C commute

Let us show that A and C commute. We have

$$A_{pq} = B_{pq} = \frac{1}{2}\sqrt{pq}\,\ell_{p+q} \tag{2.19}$$

$$C_{pq} = \sqrt{pq}\,\ell_{|p-q|} \tag{2.20}$$

These matrices are symmetric and vanish for odd $p+q$.

The commutator between the two is

$$(AC - CA)_{pq} = \frac{\sqrt{pq}}{2} \sum_{l=1}^{\infty} l\,(\ell_{p+l}\,\ell_{|l-q|} - \ell_{|p-l|}\,\ell_{q+l}) \tag{2.21}$$

One can show that numerically this commutator vanishes for all p and q, but this can be shown also analytically.

To start with suppose that p, q are both even. It follows that the summation extend over all even $l \to 2l$. Therefore, in this case, after some algebra,

$$(AC - CA)_{pq} = 4\sqrt{pq}(-1)^{\frac{p+q}{2}} \Big(f(p,q) - f(q,p)\Big) \tag{2.22}$$

where

$$\begin{aligned} f(p,q) &= \sum_{l=1}^{\infty} \frac{l}{((p+2l)^2-1)((q-2l)^2-1)} \\ &= -\frac{1}{2((p+q)^2-4)} \left(1 - \frac{1}{2}\frac{p-q}{p+q}\left(\psi(\frac{1}{2}+\frac{p}{2}) - \psi(\frac{1}{2}-\frac{q}{2})\right)\right) \end{aligned} \tag{2.23}$$

ψ is the dilogarithm function $\psi(z) = \frac{d}{dz}\log\Gamma(z)$. One of its remarkable properties is that

$$\psi(\frac{1}{2}+z) - \psi(\frac{1}{2}-z) = \pi\,\mathrm{Tan}(\pi z)$$

Using this one can easily prove (since p, q are both even) that $f(p,q) = f(q,p)$. Therefore (2.22) vanishes identically.

Similarly, when p, q are both odd, it follows that the summation extends over odd $l \to 2l+1$. Therefore, after some elementary algebra

$$(AC - CA)_{pq} = -2\sqrt{pq}(-1)^{\frac{p+q}{2}} \Big(g(p,q) - g(q,p)\Big) \tag{2.24}$$

where

$$\begin{aligned} g(p,q) &= \sum_{l=0}^{\infty} (2l+1) \frac{1}{(p+2l)(p+2l+2)(q-2l)(q-2l-2)} \\ &= \frac{2+p-q-2pq}{2pq((p+q)^2-4)} + \frac{p-q}{2(p+q)} \cdot \frac{\psi(\frac{p}{2}) - \psi(-\frac{q}{2})}{(p+q)^2-4} \end{aligned} \tag{2.25}$$

It is not immediately evident, but using

$$\psi(z) - \psi(-z) = -\pi\,\mathrm{Cot}(\pi z) - \frac{1}{z}$$

one can prove that indeed

$$g(p,q) = g(q,p)$$

The two functions (2.23,2.25) become singular when $p = q = 1$, but from the initial definition (2.22) it is evident that the commutator in this special case is 0. In conclusion, again, (2.24) vanishes. Therefore A and C commute.

2.3 Diagonalization of K_1

To know more about A and C we can diagonalize them, which can be done by finding a basis of common eigenvectors. To this purpose it is crucial to notice that both A and C commute with the matrix F representing the operator $K_1 = L_1 + L_1^\dagger$. For K_1 can be written as

$$K_1 = a^\dagger F\, a \tag{2.26}$$

where

$$F_{pq} = \sqrt{pq}\,(\delta_{p+1,q} + \delta_{q+1,p}) \tag{2.27}$$

We have

$$[\mathcal{L}_0 + \mathcal{L}_0^\dagger, K_1] = -2a^\dagger FA\, a^\dagger + 2aFA\, a + a^\dagger [C,F]\, a \tag{2.28}$$

where we have utilized $B = A = A^T$ and $F = F^T$. It is elementary to prove with the CFT language that the left hand side of (2.28) vanishes.

Now AF is not zero, but we should symmetrize it, because it appears among two a's or two $a^\dagger$'s. We get in fact

$$(AF + FA)_{pq} = \frac{1}{2}\sqrt{pq}\,\left((p+q+2)\ell_{p+q+1} + (p+q-2)\ell_{p+q-1}\right) = 0 \tag{2.29}$$

The result is easily obtained by inserting the expression for ℓ_n's.

Likewise, for $p > q$ we get

$$[C,F]_{pq} = \sqrt{pq}\,((q-p-2)\ell_{p-q+1} + (q-p+2)\ell_{p-q-1}) = 0$$

and a similar result for $q > p$.

The last equation means in particular that C and F must be simultaneously diagonalizable. On the other hand, from eq.(2.29) we get that A and F anticommute. But $\{\hat{C}, F\} = 0$. Therefore $[\tilde{A}, F] = 0$, where $\tilde{A} = \hat{C}\, A$. So also $\tilde{A}$ must be simultaneously diagonalizable with F.

As it turns out, this problem has already been solved by Rastelli, Sen and Zwiebach, [22], who determined the non degenerate basis of eigenvectors of K_1. For the reader's convenience we summarize the relevant results. K_1 represents the action of the differential operator

$$\mathcal{K}_1 \equiv (1+z^2)\frac{d}{dz} \tag{2.30}$$

in the z plane. We can use this to find its eigenvectors and its eigenvalues. Introduce the sequence $v = \{v_n\}$ and define

$$v \cdot a^\dagger = \sum_{n=1}^{\infty} v_n\, a_n^\dagger$$

Then, forgetting the zero mode a_0, we get

$$[K_1, v\cdot a^\dagger] = (F\, v)\cdot a^\dagger \tag{2.31}$$

A related operator is

$$(\tilde{F})_{nm} = \sqrt{\frac{m}{n}}\, F_{nm} = (n-1)\delta_{n,m+1} + (n+1)\delta_{n+1,m} \tag{2.32}$$

Now define

$$f_w(z) = \sum_{n=1}^{\infty} w_n\, z^n, \qquad v_n = \sqrt{n}\, w_n \tag{2.33}$$

Then

$$\mathcal{K}_1\, f_w(z) = f_{\tilde{F}\,w}(z) + w_1 \tag{2.34}$$

Now it is easy to integrate

$$(1+z^2)\frac{df(z)}{dz} = \kappa\, f(z) + w_1$$

with suitable boundary conditions, we get

$$f_\kappa(z) = -\frac{1}{\kappa}\left(1 - e^{\kappa\ \arctan(z)}\right) \tag{2.35}$$

Equating (2.33) to (2.35) we can extract the eigenfunctions

$$w_n(\kappa) = \frac{1}{2\pi i}\oint dz\, \frac{f_\kappa(z)}{z^{n+1}} \tag{2.36}$$

The eigenfunctions found in this way, when suitably normalized, form a complete orthonormal system.

2.4 Diagonalization of $\Delta, \tilde{A}$ and C

In the solution (2.18) one of the building blocks is the term $\Delta = C^2 - 4A^2 = C^2 - 4\tilde{A}^2$, the discriminant in the integration of the KP differential equation. It is possible to cast it in a very simple form. Indeed one finds (see Appendix A)

$$\begin{aligned}(C^2 - 4\,A^2)_{pq} &= 4\sqrt{pq}\,(-1)^{p+q}\,\frac{p+q}{2(p-q)((p-q)^2-4)}\\ &\quad\cdot\left(\psi\left(\frac{1}{2}-\frac{p}{2}\right) - \psi\left(\frac{1}{2}+\frac{p}{2}\right) + \psi\left(\frac{1}{2}+\frac{q}{2}\right) - \psi\left(\frac{1}{2}-\frac{q}{2}\right)\right)\\ &= 4\sqrt{pq}\,(-1)^{p+q}\,\frac{p+q}{2(p-q)((p-q)^2-4)}\,(\pi\tan(\pi q) - \pi\tan(\pi p))\\ &= \frac{1}{2}\pi^2\left(p^2\,\delta_{p,q} + \frac{1}{4}\sqrt{pq}\,(p+q)(\delta_{p,q+2} + \delta_{p+2,q})\right)\end{aligned} \tag{2.37}$$

Therefore Δ is a Jacobi matrix (it has only three nonvanishing diagonal lines) and it is clearly positive. We know that, in the $v_n(\kappa)$ basis, this matrix is diagonal, and in the form (2.37) it is rather easy to find the corresponding eigenvalues.

Indeed let us use the representation (Fuchs and Kroyter)

$$v_n(\kappa) = \mathcal{N}(\kappa)\frac{i^{n-1}\sqrt{n}}{2\pi}\int_{-\infty}^{\infty} du\, \frac{e^{-i\kappa u}\tanh^{n-1}(u)}{\cosh^2(u)} \tag{2.38}$$

for the basis of eigenvectors of K_1. Then

$$\begin{aligned}
\sum_{n=1}^{\infty}(C^2-4A^2)_{pn}\, v_n(\kappa) &= \frac{\pi^2}{2}\sum_{n=1}^{\infty}\left(p^2\delta_{p,n}+\frac{1}{4}\sqrt{pn}(p+n)(\delta_{p,n+2}+\delta_{p+2,n})\right)\cdot\\
&\quad\cdot\frac{\mathcal{N}(\kappa)}{2\pi}\int_{-\infty}^{\infty}du\,\frac{e^{-i\kappa u}}{\cosh^2(u)}\, i^{n-1}\sqrt{n}\tanh^{n-1}(u)\\
&= \frac{\pi^2}{2}\left(\frac{i^{p-1}\sqrt{p}\mathcal{N}(\kappa)}{2\pi}\int_{-\infty}^{\infty}du\,\frac{e^{-i\kappa u}}{\cosh^2(u)}\cdot\right.\\
&\quad\left.\cdot\left(p^2\tanh^{p-1}(u)-\frac{1}{2}(p-1)(p-2)\tanh^{p-3}(u)-\frac{1}{2}(p+1)(p+2)\tanh^{p+1}(u)\right)\right)\\
&= -\frac{\pi^2}{4}\left(\frac{i^{p-1}\sqrt{p}\mathcal{N}(\kappa)}{2\pi}\int_{-\infty}^{\infty}du\, e^{-i\kappa u}\frac{\partial^2}{\partial u^2}\left(\frac{\tanh^{p-1}(u)}{\cosh^2(u)}\right)\right)\\
&= \frac{\pi^2}{4}\kappa^2\, v_p(\kappa)
\end{aligned} \tag{2.39}$$

after integration by parts.

What remains for us to do is to derive the eigenvalues of $\tilde{A}$ and C. As for the diagonalization of $\tilde{A}$, it turns out that it has already been done by Rastelli, Sen and Zwiebach, [22]. In fact their operator B_{nm} defined in (5.18) coincides exactly with

$$\tilde{A}_{pq} = \frac{1}{2}(-1)^p\sqrt{pq}\,\ell_{p+q} = \frac{\sqrt{pq}\,(-1)^{\frac{q-p}{2}+1}}{(p+q)^2-1} \tag{2.40}$$

with even $p+q$ and zero otherwise. Therefore, using their result, the eigenvalue of $\tilde{A}$ is

$$\tilde{A}(\kappa) = -\frac{\kappa\pi}{4\sinh(\frac{\kappa\pi}{2})} \tag{2.41}$$

Now, using (2.41), and $\sqrt{\Delta} = \frac{|k|\pi}{2}$ one can obtain

$$C(\kappa) = \frac{\kappa\pi}{2}\frac{\cosh(\frac{\kappa\pi}{2})}{\sinh(\frac{\kappa\pi}{2})} \tag{2.42}$$

where the + sign has been chosen in taking the square root. This is due to the fact that also C is a positive matrix (its main diagonal is given by the sequence $2, 4, 6, ...$, while the off–diagonal terms have alternating signs and decreasing absolute value). Anyhow, the eigenvalue of C could be calculated directly with the same method used for $\tilde{A}(\kappa)$. But since this type of calculations will be used repeatedly in the ghost sector, we will dispense with doing it in this case.

2.5 The wedge states

We have now all at hand to prove the equivalence (2.1). In order to make the comparison with the known formulas for wedge states we multiply everything in (2.18) by $\hat{C}$, the twist matrix, and evaluate $\tilde{\alpha} = \hat{C}\alpha$ at $t = t_n \equiv -\frac{n-2}{2}$. We expect to find (from now on, in this section, we actually denote with the symbols of matrices their eigenvalues)

$$T_n \equiv \hat{C} S_n = -2\,\tilde{\alpha}\left(-\frac{n-2}{2}\right) \tag{2.43}$$

i.e.

$$T_n = 2\tilde{A}\frac{\sinh\left(\sqrt{\Delta}\,\frac{n-2}{2}\right)}{\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\,\frac{n-2}{2}\right) + C\sinh\left(\sqrt{\Delta}\,\frac{n-2}{2}\right)} \tag{2.44}$$

In particular, we should find

$$T_3 = X, \qquad \lim_{n\to\infty} T_n = \frac{2\tilde{A}}{C+\sqrt{\Delta}} \equiv T \tag{2.45}$$

where $S = \hat{C}T$ is the sliver matrix. Replacing the continuous basis and using

$$\tilde{A} = -\frac{\kappa\pi}{4\sinh\left(\frac{\kappa\pi}{2}\right)}, \qquad C = \frac{\kappa\pi}{2}\coth\left(\frac{\kappa\pi}{2}\right) \tag{2.46}$$

this is precisely what one gets.

More generally one must prove the recursion relation

$$T_{n+1} = X\frac{1-T_n}{1-T_n X}, \tag{2.47}$$

or its solution

$$T_n = \frac{T+(-T)^{n-1}}{1-(-T)^n} \tag{2.48}$$

To avoid cumbersome calculations let us proceed as follows. We use the condensed notations

$$\sinh\left(\sqrt{\Delta}\,\frac{n-2}{2}\right) = \mathrm{sh}_n, \qquad \cosh\left(\sqrt{\Delta}\,\frac{n-2}{2}\right) = \mathrm{ch}_n$$

and the trigonometric formulas

$$\mathrm{sh}_{n+1} = \mathrm{sh}_n\mathrm{ch}_3 + \mathrm{ch}_n\mathrm{sh}_3$$
$$\mathrm{ch}_{n+1} = \mathrm{ch}_n\mathrm{ch}_3 + \mathrm{sh}_n\mathrm{sh}_3$$

Then we have in particular

$$T_1 = -2\tilde{A}\frac{\mathrm{sh}_3}{\sqrt{\Delta}\,\mathrm{ch}_3 - C\,\mathrm{sh}_3} \tag{2.49}$$
$$T_2 = 0$$
$$T_3 = 2\tilde{A}\frac{\mathrm{sh}_3}{\sqrt{\Delta}\,\mathrm{ch}_3 + C\,\mathrm{sh}_3} \tag{2.50}$$

Replacing the trigonometric formulas inside T_{n+1} we get

$$T_{n+1} = 2\tilde{A}\frac{\mathrm{sh}_n\mathrm{ch}_3 + \mathrm{ch}_n\mathrm{sh}_3}{\sqrt{\Delta}(\mathrm{ch}_n\mathrm{ch}_3 + \mathrm{sh}_n\mathrm{sh}_3) + C(\mathrm{sh}_n\mathrm{ch}_3 + \mathrm{ch}_n\mathrm{sh}_3)} \tag{2.51}$$

We want to compare this with the RHS of eq.(2.47)

$$T_3\frac{1-T_n}{1-T_nT_3} = 2\tilde{A}\frac{\frac{C-2\tilde{A}}{\sqrt{\Delta}}\mathrm{sh}_n\mathrm{sh}_3 + \mathrm{ch}_n\mathrm{sh}_3}{\sqrt{\Delta}(\mathrm{ch}_n\mathrm{ch}_3 + \mathrm{sh}_n\mathrm{sh}_3) + C(\mathrm{sh}_n\mathrm{ch}_3 + \mathrm{ch}_n\mathrm{sh}_3)} \tag{2.52}$$

Therefore (2.51) and (2.52) coincide if

$$\frac{C-2\tilde{A}}{\sqrt{\Delta}}\mathrm{sh}_3 = \mathrm{ch}_3 \tag{2.53}$$

Notice that this is exactly the condition for which $T_1 = 1$, as it should be. Replacing the eigenvalues of the previous subsection it is elementary to prove (2.53).

As for the normalization constants $\mathcal{N}_n$ they must also satisfy a recursive relation

$$\mathcal{N}_n\,\mathcal{K}\det(1-T_nX)^{-\frac{1}{2}} = \mathcal{N}_{n+1} \tag{2.54}$$

where $\mathcal{K}$ is some constant to be determined. Set

$$\eta_n = 2\int_0^{t_n} dt\,\mathrm{tr}(\alpha B) = 2\int_0^{t_n} dt\,\mathrm{tr}(\tilde{\alpha}\tilde{A}) \tag{2.55}$$

and identify

$$\mathcal{N}_n = e^{\eta_n}$$

Now, inserting the results of the previous subsection one can verify that (2.54) is satisfied, and determine $\mathcal{K}$. The result is expressed in terms of untraced quantities $\hat{\eta}_n$ such that $\eta_n = \mathrm{tr}(\hat{\eta}_n)$. Setting $y = \frac{\pi\kappa}{2}$,

$$\hat{\eta}_n = -\frac{n-2}{4}\,y\coth(y) + \frac{1}{2}\log\left(\frac{\sinh(y)}{\sinh\left(y\frac{n}{2}\right)}\right) \tag{2.56}$$

and $\mathcal{K} = e^{\eta_3}$. Taking the traces corresponds to integrating over κ from $-\infty$ to ∞ and introducing an infinite factor. These objects need to be regularized. But this is a well–known problem when normalizing squeezed states.

3. The ghost sector. Propaedeutics

We would now like to repeat the above analysis also for the ghost sector, and prove that[1]

$$|n\rangle = e^{-\frac{n-2}{2}\left(\mathcal{L}_0^{(g)}+\mathcal{L}_0^{(g)\dagger}\right)}|0\rangle = \mathcal{N}_n\,e^{c^\dagger S_n b^\dagger}|0\rangle \tag{3.1}$$

[1]The RHS of (3.1) is of course meant to represent the ghost wedge states. It should be stressed that defining these states and their star product is a rather non–trivial problem in the oscillator formalism. We postpone the treatment of this issue to a forthcoming paper and, for the time being, we understand that all works as in the matter sector.

(to avoid a proliferation of symbols we use the same ones as in the matter case, but it should be clear that from now on we deal only with ghosts; so S_n is the wedge matrix for the ghost sector, etc.).

A crucial question for the ghost sector is what normal ordering we use. In string theory the conventional normal ordering is the one implied by the vacuum $c_1|0\rangle$, where $|0\rangle$ is the SL(2,R) invariant vacuum, identified by the regularity of the ghost fields $c(z)$ and $b(z)$ at $z=0$. We call *conventional* this normal ordering, while we call *natural* the normal ordering implied by the choice of the $|0\rangle$ vacuum. More explicitly, in the natural n.o. $c_1, c_0, c_{-1}, c_{-2}, \ldots$ are creation operators and the complementary ones annihilation operators, while in the conventional n.o. $c_0, c_{-1}, c_{-2}, \ldots$ are creation operators and the complementary one annihilation operators (with the conjugate prescriptions for the b_n's).

We will see in the forthcoming sections that the natural normal ordering fits in a natural way in the task of solving the KP equations for the ghost sector, while there may be some problems with the conventional normal ordering.

This section is devoted to collect some preparatory material for the ghost sector.

3.1 Virasoro generators with the conventional n.o.

The Virasoro generators for the ghost sector (for some basic information concerning ghosts in string field theory see [13, 15, 16, 17, 19, 20]), normal ordered in the conventional way, are

$$L_n^{(g)} = \sum_{k\geq 1}(2n+k)\, b_k^\dagger c_{k+n} - \sum_{k\geq 0}(n-k)\, c_k^\dagger b_{n+k} - \sum_{k=1}^{n}(n+k)\, c_k b_{n-k} \tag{3.2}$$

$$L_0^{(g)} = \sum_{k\geq 1} k\, c_k^\dagger b_k + \sum_{k\geq 1} k\, b_k^\dagger c_k - 1 \tag{3.3}$$

$$L_{-n}^{(g)} = \sum_{k\geq 1}(k-n)\, b_{n+k}^\dagger c_k + \sum_{k\geq 0}(2n+k)\, c_{k+n}^\dagger b_k + \sum_{k=0}^{n-1}(n+k)\, c_k^\dagger b_{n-k}^\dagger \tag{3.4}$$

Therefore we can write

$$\begin{aligned}\mathcal{L}_0^{(g)} + \mathcal{L}_0^{(g)\dagger} &= 2L_0^{(g)} + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{4n^2-1}(L_{2n}^{(g)} + L_{-2n}^{(g)})\\ &= c^\dagger A\, b^\dagger + c^\dagger C\, b + b^\dagger D c - c\, B\, b - 2\end{aligned} \tag{3.5}$$

where

$$A_{pq} = \sum_{n=1}^{\infty} \ell_n (n+p) \delta_{p+q,n} = (2p+q)\, \ell_{p+q}, \qquad p \geq 0, \quad q \geq 1 \tag{3.6}$$

$$B_{pq} = \sum_{n=1}^{\infty} \ell_n (n+p) \delta_{p+q,n} = (2p+q)\, \ell_{p+q}, \qquad p \geq 1, \quad q \geq 0 \tag{3.7}$$

$$\begin{aligned} C_{pq} &= \sum_{n=1}^{\infty} \ell_n [(p-n)\delta_{p+n,q} + (2n+q)\delta_{q+n,p}] + 2\, p\, \delta_{p,q} \\ &= (2p-q)\, \ell_{|q-p|}, \qquad p \geq 0, \quad q \geq 0 \end{aligned} \tag{3.8}$$

$$\begin{aligned} D_{pq} &= \sum_{n=1}^{\infty} \ell_n [(q-n)\delta_{q+n,p} + (2n+p)\delta_{p+n,q}] + 2\, p\, \delta_{p,q} \\ &= (2q-p)\, \ell_{|q-p|}, \qquad p \geq 1, \quad q \geq 1 \end{aligned} \tag{3.9}$$

In view of these formulas it is natural to introduce two types of indices: the unbarred ones, p, q, running from 0 up to $+\infty$ and the barred ones $\bar{p}, \bar{q}$, running from 1 up to $+\infty$. The above matrices will carry either type of indices: $C = \{C_{pq}\}$ is a square 'long–legged' matrix, $D = \{D_{\bar{p}\bar{q}}\}$ is square 'short–legged'. The matrix $A = \{A_{p\bar{q}}\}$ ($B = \{B_{\bar{p},q}\}$) is instead left–long–legged and right–short–legged (right–long–legged and left–short–legged). We will refer to this kind of matrices as *lame* matrices[2].

We notice that in the overlapping span of the indices we have $A = B$ and $C = D^T$. Actually we will learn in due course that A and B (C and D^T) represent the same operator applied to two different bases.

3.2 The Virasoro generators with the natural n.o.

The Virasoro generators for the ghost sector in the natural normal ordering are ($n > 0$)

$$L_n^{(g)} = \sum_{k \geq 2} (2n+k)\, b_k^{\dagger} c_{k+n} - \sum_{k \geq -1} (n-k)\, c_k^{\dagger} b_{n+k} - \sum_{k=2}^{n+1} (n+k)\, c_k b_{n-k} \tag{3.10}$$

$$L_0^{(g)} = \sum_{k \geq -1} k\, c_k^{\dagger} b_k + \sum_{k \geq 2} k\, b_k^{\dagger} c_k \tag{3.11}$$

$$L_{-n}^{(g)} = \sum_{k \geq 2} (k-n)\, b_{n+k}^{\dagger} c_k + \sum_{k \geq -1} (2n+k)\, c_{k+n}^{\dagger} b_k + \sum_{k=-1}^{n-2} (n+k)\, c_k^{\dagger} b_{n-k}^{\dagger} \tag{3.12}$$

Therefore we can write

$$\begin{aligned} \mathcal{L}_0^{(g)} + \mathcal{L}_0^{(g)\dagger} &= 2 L_0^{(g)} + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{4n^2-1} (L_{2n}^{(g)} + L_{-2n}^{(g)}) \\ &= c^{\dagger} A\, b^{\dagger} + c^{\dagger} C\, b + b^{\dagger} D c - c\, B\, b \end{aligned} \tag{3.13}$$

[2] If the matrices in question were finite the appropriate term would be rectangular matrices, but semi–infinite matrices can always be seen as square, provided we relabel the indices. It is clear that in this context what matters is the bases these matrices are applied to. For this reason we use the *ad hoc* term 'lame'.

where

$$A_{p\bar{q}} = (2p+\bar{q})\ell_{|p+\bar{q}|}, \tag{3.14}$$
$$B_{\bar{p}q} = (2\bar{p}+q)\ell_{|\bar{p}+q|}, \tag{3.15}$$
$$C_{pq} = (2p-q)\,\ell_{|q-p|} \tag{3.16}$$
$$D_{\bar{p}\bar{q}} = (2\bar{q}-\bar{p})\,\ell_{|\bar{q}-\bar{p}|} \tag{3.17}$$

where again we have two types of indices, but, at variance with the previous section, now the unbarred ones run from -1 up to $+\infty$, while the barred ones run from 2 to $+\infty$. Again we have different type of matrices. C is a long–legged square matrix, D is a short–legged square matrix. A and B are 'lame' matrices. We also remark that again for overlapping indices we have $A = B$ and $C = D^T$. At times we will refer to A and B as A–type matrices and to C and D^T as C–type matrices.

3.3 KP equations for the ghosts

Our purpose is to prove a relation similar to (2.9)

$$e^{t(c^\dagger A\, b^\dagger + c^\dagger C\, b + b^\dagger D c - c\, B\, b)} = e^{\eta}\, e^{c^\dagger \alpha\, b^\dagger}\, e^{c^\dagger \gamma\, b}\, e^{b^\dagger \delta\, c}\, e^{c\,\beta\, b} \tag{3.18}$$

where $\alpha, \beta, \gamma, \delta, \eta$ depend on t and they are matrices of the same type as A, B, C, D (that is, α is right–long–legged and left–short–legged matrix, etc.). Proceeding as in KP, we differentiate both sides, commute to the left and obtain the equations

$$A = \dot{\alpha} - C\,\alpha - \alpha\, D^T - \alpha\, B\alpha \tag{3.19}$$
$$C = -\alpha\, B + \frac{de^{\gamma}}{dt} e^{-\gamma} \tag{3.20}$$
$$D = -\alpha^T B^T + \frac{de^{\delta}}{dt} e^{-\delta} \tag{3.21}$$
$$B = e^{-\delta^T}\, \dot{\beta}\, e^{-\gamma} \tag{3.22}$$
$$0 = \dot{\eta} + \mathrm{Tr}\,(B\,\alpha) \tag{3.23}$$

In this paper we are interested only in eq.(3.19) and (3.23). Notice the ordering of factors in the trace in (3.23).

3.4 Some very useful identities

Before we proceed we need to derive a few basic identities that will turn up in all the subsequent calculations. In the following we use the symbol $\mathcal{A}$ and $\mathcal{C}$ to represent square matrices that are related to A and C but do not coincide exactly with them (they correspond to A and C in the range $1, 2, \ldots, \infty$), but they give rise to sort of universal relations, which can be used in the rest of the paper with just few modifications.

Let first p, q be both odd, then one can prove the following

$$
\begin{aligned}
(\mathcal{A}\mathcal{C})_{pq} &= \sum_{k=1}^{\infty}(2p+k)(2k-q)\,\ell_{p+k}\,\ell_{|k-q|} \\
&= 4(-1)^{\frac{p+q}{2}}\sum_{l=0}^{\infty}\frac{(2p+2l+1)(4l-q+2)}{((p+2l+1)^2-1)((2l-q+1)^2-1)} \\
&= 2(-1)^{\frac{p+q}{2}}\Big[-\Big(\frac{(q-2)(2p+q-1)}{q(p+q)(p+q-2)}+\frac{(p-1)(2p+q+2)}{p(p+q)(p+q-2)}\Big) \\
&\quad + \frac{4p^2+3pq+q^2-4}{(p+q)((p+q)^2-4)}\Big(\psi(\frac{p}{2})-\psi(-\frac{q}{2})\Big)\Big]
\end{aligned}
\tag{3.24}
$$

and

$$
\begin{aligned}
(\mathcal{C}\mathcal{A})_{pq} &= \sum_{k=1}^{\infty}(2p-k)(2k+q)\,\ell_{q+k}\,\ell_{|k-p|} \\
&= 4(-1)^{\frac{p+q}{2}}\sum_{l=0}^{\infty}\frac{(2p-2l-1)(4l+q+2)}{((p-2l-1)^2-1)((2l+q+1)^2-1)} \\
&= 2(-1)^{\frac{p+q}{2}}\Big[-\Big(\frac{(q+2)(2p+q+1)}{q(p+q)(p+q-2)}+\frac{(p+1)(2p+q-2)}{p(p+q)(p+q-2)}\Big) \\
&\quad + \frac{4p^2+3pq+q^2-4}{(p+q)((p+q)^2-4)}\Big(\psi(-\frac{p}{2})-\psi(\frac{q}{2})\Big)\Big]
\end{aligned}
\tag{3.25}
$$

Using $\psi(\frac{p}{2})-\psi(-\frac{p}{2})=-\frac{2}{p}$ we get exactly

$$
(\mathcal{A}\mathcal{C})_{pq} = (\mathcal{C}\mathcal{A})_{pq} \tag{3.26}
$$

This is valid only for $p, q > 1$, since for $p = q = 1$ there is a singularity in the previous formulas. This case must be calculated separately. One gets

$$
[\mathcal{A},\mathcal{C}]_{11} = \sum_{l=0}^{\infty}\frac{16\,l(l+1)}{(2l+1)(2l+3)(4l^2-1)} = \frac{\pi^2}{8} \tag{3.27}
$$

Let now p, q be both even. We have

$$
\begin{aligned}
(\mathcal{A}\mathcal{C})_{pq} &= \sum_{l=1}^{\infty}(2p+2l)(4l-q)\,\ell_{p+2l}\,\ell_{|2l-q|} \\
&= 8(-1)^{\frac{p+q}{2}}\sum_{l=1}^{\infty}\frac{(p+l)(4l-q)}{((p+2l)^2-1)((2l-q)^2-1)} \\
&= 2(-1)^{\frac{p+q}{2}}\Big[-\Big(\frac{(q-2)(2p+q-1)}{(q-1)(p+q)(p+q-2)}+\frac{(p-1)(2p+q+2)}{(p+1)(p+q)(p+q-2)}\Big) \\
&\quad + \frac{(2p-q)^2-4}{(p+q)((p+q)^2-4)}\Big(\psi(\frac{1}{2}+\frac{p}{2})-\psi(\frac{1}{2}-\frac{q}{2})\Big)\Big]
\end{aligned}
\tag{3.28}
$$

and

$$
\begin{aligned}
(\mathcal{C}\mathcal{A})_{pq} &= \sum_{l=1}^{\infty}(2p-2l)(4l+q)\,\ell_{q+2l}\,\ell_{|2l-p|} \\
&= 8(-1)^{\frac{p+q}{2}}\sum_{l=1}^{\infty}\frac{(p-l)(4l+q)}{((p-2l)^2-1)((2l+q)^2-1)} \\
&= 2(-1)^{\frac{p+q}{2}}\Bigg[-\left(\frac{(q+2)(2p+q-2)}{(q+1)(p+q)(p+q-2)}+\frac{(p+1)(2p+q-2)}{(q-1)(p+q)(p+q-2)}\right) \\
&\quad+\frac{(2p-q)^2-4}{(p+q)((p+q)^2-4)}\left(\psi(\frac{1}{2}-\frac{p}{2})-\psi(\frac{1}{2}+\frac{q}{2})\right)\Bigg]
\end{aligned}
\tag{3.29}
$$

Using $\psi(\frac{1}{2}-\frac{p}{2})=\psi(\frac{1}{2}+\frac{p}{2})$ for p even, we get

$$
(\mathcal{A}\mathcal{C}-\mathcal{C}\mathcal{A})_{pq}=8(-1)^{\frac{p+q}{2}}\,\frac{pq}{(p^2-1)(q^2-1)}=2\,u_p\,u_q \tag{3.30}
$$

where $u_p=p\,\ell_p$.

Another important relations is the one concerning $\mathcal{C}^2-\mathcal{A}^2$. Proceeding as in Appendix A one gets, for odd p,q,

$$
(\mathcal{C}^2-\mathcal{A}^2)_{pq}=\frac{\pi^2}{2}\left((p^2-2)\,\delta_{p,q}+\frac{1}{2}p(p-1)\,\delta_{q,p+2}+\frac{1}{2}p(p+1)\,\delta_{p,q+2}\right) \tag{3.31}
$$

while for even p,q

$$
\begin{aligned}
(\mathcal{C}^2-\mathcal{A}^2)_{pq} &= 8\frac{pq\,(-1)^{\frac{p+q}{2}}}{(p^2-1)(q^2-1)} \\
&\quad+\frac{\pi^2}{2}\left((p^2-2)\delta_{p,q}+\frac{1}{2}p(p-1)\,\delta_{q,p+2}+\frac{1}{2}p(p+1)\,\delta_{p,q+2}\right)
\end{aligned}
\tag{3.32}
$$

All these equations will be repeatedly applied in the sequel.

4. Integrating the KP equations in the ghost sector. Natural n.o.

In this section our purpose is to integrate the KP equations in the ghost sector. We concentrate on the case of the natural n.o., which provides a clean framework to solve the problem and postpone till the end of the paper the discussion of the problems with conventional n.o. In order to be able to tackle the integration of the KP equation we need some remarkable identities satisfied by the A,B,C,D matrices, analogous to the ones in the matter sector.

4.1 Commuting matrices

First let us show that $AD^T=CA$. Suppose p and $\bar{q}$ are even. Then

$$
\begin{aligned}
(AD^T)_{p\bar{q}} &= \sum_{\bar{k}=2}^{\infty}A_{p\bar{k}}\,D^T_{\bar{k}\bar{q}}=\sum_{l=2}^{\infty}(2p+2l)(4l-\bar{q})\ell_{|\bar{q}+2l|}\,\ell_{|2l-p|} \\
&= 8(-1)^{\frac{p+\bar{q}}{2}}\sum_{l=1}^{\infty}\frac{(p+l)(4l-\bar{q})}{((p+2l)^2-1)((2l-\bar{q})^2-1)}
\end{aligned}
\tag{4.1}
$$

having set $\kappa = 2l$. We notice that the RHS of this equation coincide with (3.28). Similarly

$$(CA)_{p\bar{q}} = \sum_{\bar{k}=-1}^{\infty} C_{pk}\, A_{k\bar{q}} = \sum_{l=0}^{\infty}(2p-2l)(4l+\bar{q})\ell_{|p+2l|}\,\ell_{|2l-\bar{q}|}$$

$$= 8(-1)^{\frac{p+\bar{q}}{2}}\left[\sum_{l=1}^{\infty}\frac{(p-l)(4l+\bar{q})}{((p-2l)^2-1)((2l+\bar{q})^2-1)} + \frac{p\bar{q}}{(p^2-1)(\bar{q}^2-1)}\right] \tag{4.2}$$

The first term in the square bracket on the RHS of this equation is nothing but eq.(3.29). Putting everything together and using eq.(3.30), we find

$$(AD^T - CA)_{p\bar{q}} = 8(-1)^{\frac{p+\bar{q}}{2}}\left[\frac{p\bar{q}}{(p^2-1)(\bar{q}^2-1)} - \frac{p\bar{q}}{(p^2-1)(\bar{q}^2-1)}\right] = 0 \tag{4.3}$$

This derivation becomes singular in the case $p = 0, \bar{q} = 2$. This case has to be handled separately. One easily gets

$$(AD^T)_{02} = -\frac{1}{4}, \qquad (CA)_{02} = -\frac{1}{4} \tag{4.4}$$

Similarly for p and $\bar{q}$ odd we have

$$(AD^T)_{p\bar{q}} = \sum_{\bar{k}=3}^{\infty} A_{p\bar{k}}\, D^T_{\bar{k}\bar{q}} = 4(-1)^{\frac{p+\bar{q}}{2}}\left[\sum_{l=0}^{\infty}\frac{(2p+2l+1)(4l-\bar{q}+2)}{((p+2l+1)^2-1)((2l-\bar{q}+1)^2-1)}\right.$$
$$\left. - \frac{(2p+1)(2-\bar{q})}{((p+1)^2-1)((q-1)^2-1)}\right] \tag{4.5}$$

and

$$(CA)_{p\bar{q}} = \sum_{\bar{k}=-1}^{\infty} C_{pk}\, A_{k\bar{q}} = 4(-1)^{\frac{p+\bar{q}}{2}}\left[\sum_{l=0}^{\infty}\frac{(2p-2l-1)(4l+\bar{q}+2)}{((p-2l-1)^2-1)((2l+\bar{q}+1)^2-1)}\right.$$
$$\left. + \frac{(2p+1)(\bar{q}-2)}{((p+1)^2-1)((q-1)^2-1)}\right] \tag{4.6}$$

having set $k = 2l+1$. The first term in square brackets in (4.5) coincides with eq.(3.24), while the corresponding term in the RHS of (4.6) coincides with (3.25). Therefore they are equal. It follows that the RHS of (4.5) equals the RHS of (4.6). The previous analysis does not hold for $p = -1$ and $q = 3$, but one can prove directly that

$$(AD^T)_{-1,3} = -\frac{1}{4} = (CA)_{-1,3} \tag{4.7}$$

Therefore we can conclude that

$$AD^T = CA \tag{4.8}$$

which is the appropriate 'commutation relation' for these kind of matrices. In the same way one can prove that

$$BC = D^T B \tag{4.9}$$

It follows at once from these results that, for instance,

$$CAB = AD^T B = ABC \tag{4.10}$$

4.2 Integration of the KP equations

We are now ready to integrate the KP equations. We concentrate on

$$\dot{\alpha} = A + C\alpha + \alpha D^T + \alpha B\alpha \tag{4.11}$$

with the initial condition $\alpha(0) = 0$. The solution to

$$\dot{\eta} = -\mathrm{Tr}\,(B\,\alpha) \tag{4.12}$$

is obvious once we know $\alpha(t)$. Let us assume that

$$\alpha\, D^T = C\,\alpha \tag{4.13}$$

and multiply (4.11) from the right by B. Defining $\theta = \alpha B$, eq.(4.11) becomes

$$\dot{\theta} = AB + 2C\theta + \theta^2 \tag{4.14}$$

since $C\theta = \theta C$ on the basis of (4.13). Since $CAB = ABC$, eq.(4.14) can be now easily integrated like a numerical equation, provided also $\theta AB = AB\theta$. The result is

$$\theta(t) = AB\frac{\sinh\left(\sqrt{C^2 - AB}\,t\right)}{\sqrt{C^2 - AB}\cosh\left(\sqrt{C^2 - AB}\,t\right) - C\sinh\left(\sqrt{C^2 - AB}\,t\right)} \tag{4.15}$$

It is easy to see that the commutativity hypotheses made above are all satisfied. Notice that $\theta(t)$ is a square long–legged matrix. From θ we can extract α by multiplying from the right by the inverse of B (concerning the invertibility of B, see the end of the next section). Let us denote the resulting solution by α_1:

$$\alpha_1(t) = \frac{\sinh\left(\sqrt{C^2 - AB}\,t\right)}{\sqrt{C^2 - AB}\cosh\left(\sqrt{C^2 - AB}\,t\right) - C\sinh\left(\sqrt{C^2 - AB}\,t\right)}\,A \tag{4.16}$$

This solution is left–long–legged and right–short–legged.

If we multiply (4.11) from the left by B, we can get another solution, say $\lambda(t) = B\,\alpha(t)$. Proceeding in the same way as above we find

$$\lambda(t) = BA\frac{\sinh\left(\sqrt{(D^T)^2 - BA}\,t\right)}{\sqrt{(D^T)^2 - BA}\cosh\left(\sqrt{(D^T)^2 - BA}\,t\right) - D^T\sinh\left(\sqrt{(D^T)^2 - BA}\,t\right)} \tag{4.17}$$

This solution is represented by a square short-legged matrix. So we can extract another form of the solution to the KP equation

$$\alpha_2(t) = A\frac{\sinh\left(\sqrt{(D^T)^2 - BA}\,t\right)}{\sqrt{(D^T)^2 - BA}\cosh\left(\sqrt{(D^T)^2 - BA}\,t\right) - D^T\sinh\left(\sqrt{(D^T)^2 - BA}\,t\right)} \tag{4.18}$$

However commuting A to the right from (4.18) we get (4.16) and viceversa. Therefore the two solutions are one and the same. This unique solution will be referred to as $\alpha(t)$.

4.3 Other useful formulas

A crucial role is clearly played in the above formulas by $(D^T)^2 - BA$ and $C^2 - AB$. They turn out to be matrices of a very simple form. Indeed using (3.31,3.32) we can prove that, for odd p, q

$$\begin{aligned}(C^2 - AB)_{pq} &= \sum_{k=-1} C_{pk}C_{kp} - \sum_{k=2} A_{pk}B_{kp} = (\mathcal{C}^2 - \mathcal{A}^2)_{pq} + C_{p0}C_{0q} \\ &= \frac{\pi^2}{2}\left((p^2-2)\,\delta_{p,q} + \frac{1}{2}p(p-1)\,\delta_{q,p+2} + \frac{1}{2}p(p+1)\,\delta_{p,q+2}\right)\end{aligned} \tag{4.19}$$

and for even p, q

$$\begin{aligned}(C^2 - AB)_{pq} &= \sum_{k=-1} C_{pk}C_{kp} - \sum_{k=2} A_{pk}B_{kp} = (\mathcal{C}^2 - \mathcal{A}^2)_{pq} + C_{p,-1}C_{-1,q} + A_{p,1}A_{1,q} \\ &= \frac{\pi^2}{2}\left((p^2-2)\delta_{p,q} + \frac{1}{2}p(p-1)\,\delta_{q,p+2} + \frac{1}{2}p(p+1)\,\delta_{p,q+2}\right)\end{aligned} \tag{4.20}$$

i.e., in general,

$$(C^2 - AB)_{pq} = \frac{\pi^2}{2}\left((p^2-2)\delta_{p,q} + \frac{1}{2}p(p-1)\,\delta_{q,p+2} + \frac{1}{2}p(p+1)\,\delta_{p,q+2}\right) \tag{4.21}$$

These equalities hold for square long–legged matrices. As above one can prove in a similar way

$$((D^T)^2 - BA)_{\bar{p}\bar{q}} = \frac{\pi^2}{2}\left((\bar{p}^2-2)\delta_{\bar{p},\bar{q}} + \frac{1}{2}\bar{p}(\bar{p}-1)\,\delta_{\bar{q},\bar{p}+2} + \frac{1}{2}\bar{p}(\bar{p}+1)\,\delta_{\bar{p},\bar{q}+2}\right) \tag{4.22}$$

for square short–legged matrices.

5. Diagonalization. Natural n.o.

We have seen in the previous section that the A–type and C–type matrices commute (although not in the form square matrices do, but in a more elaborate form fit for rectangular–type matrices). This has allowed us to integrate the KP equations. Now we would like to proceed further in the direction of proving the equivalence (3.1). It proves to be very difficult if not impossible to do so in the discrete matrix language used so far. As in the matter case it is convenient to pass to a continuous basis. The A and C–type matrices are however not easy to diagonalize, it is convenient to choose a simpler operator that commutes with them and has a nondegenerate spectrum. This is $K_1 = L_1 + L_{-1}$, as suggested by Rastelli, Sen and Zwiebach, [22].

In the ghost sector with the natural normal ordering we get

$$K_1 = \sum_{p,q\geq -1} c_p^\dagger\, G_{pq}\, b_q + \sum_{p,q\geq 2} b_{\bar{p}}^\dagger\, H_{\bar{p}\bar{q}}\, c_{\bar{q}} - 3c_2\, b_{-1} \tag{5.1}$$

where

$$\begin{aligned}G_{pq} &= (p-1)\delta_{p+1,q} + (p+1)\delta_{p-1,q}, \\ H_{\bar{p}\bar{q}} &= (\bar{p}+2)\delta_{\bar{p}+1,\bar{q}} + (\bar{p}-2)\delta_{\bar{p}-1,\bar{q}}\end{aligned} \tag{5.2}$$

Therefore G is a square long–legged matrix and H a square short–legged one. In the common overlap we have $G = H^T$.

Since we want to transfer the relevant information contained in K_1, in its oscillator form, to infinite matrices and replace the effect of oscillator anticommutation relations with matrix operation, it is clear that the last term in the RHS of (5.1) is an awful nuisance. Fortunately the effect of this term can be incorporated in the matrices G and H provided we suitably enlarge them. More precisely we can write

$$K_1 = \sum_{\dot{p}q} c^\dagger_{\dot{p}}\, G_{\dot{p}q}\, b_q + \sum_{\tilde{p},\bar{q}} b^\dagger_{\tilde{p}}\, H_{\tilde{p}\bar{q}}\, c_{\bar{q}} \tag{5.3}$$

where G, H are now lame matrices

$$\begin{aligned} G_{\dot{p}q} &= (\dot{p}-1)\delta_{\dot{p}+1,q} + (\dot{p}+1)\delta_{\dot{p}-1,q}, \\ H_{\tilde{p}\bar{q}} &= (\tilde{p}+2)\delta_{\tilde{p}+1,\bar{q}} + (\tilde{p}-2)\delta_{\tilde{p}-1,\bar{q}} \end{aligned} \tag{5.4}$$

where $\dot{p}$ runs from -2 to ∞ and $\tilde{q}$ from 1 to ∞. We will refer to these indices as the *stretched* ones and to the corresponding matrix enlargement as *stretching.* Since this implies two additional terms corresponding to $G_{-2,-1}$ and $H_{1,2}$, it would seem that we are double–counting, but this is not the case if we use the following prescription: in each of this new terms in K_1 we have one daggered and one undaggered operator, for instance b_{-1} appears in the $H_{1,2}$ term as the daggered operator $b^\dagger_1$ and has nontrivial anticommutaion relation with the undaggered operator c_1 (not with $c^\dagger_{-1}$). That is the anticommutation rules are nontrivial only between daggered and conjugate undaggered operators. This allows us to transfer all the content of the oscillator anticommutation relations to the product of matrices. In order to see that this is the correct prescription one can check that these rules reproduce the correct commutation relations of K_1 with the conformal $b(z)$ and $c(z)$ fields (see below).

In general in conformal field theory, for a primary field of weight h, we have

$$[\mathcal{L}_0 + \mathcal{L}_0^\dagger, \phi(z)] = (\mathrm{arctg}(z) + \mathrm{arctg}(1/z))\left((1+z^2)\,\partial\phi(z) + 2\,z\,h\,\phi(z)\right) \tag{5.5}$$

With these formulas one can compute the commutator between K_1 and $\mathcal{L}_0 + \mathcal{L}_0^\dagger$. Using the property $\frac{\partial}{\partial z}(\mathrm{arctg}(z) + \mathrm{arctg}(1/z)) = 0$, it is elementary to see that

$$[[\mathcal{L}_0 + \mathcal{L}_0^\dagger, K_1]\,,\, \phi(z)] = 0$$

for any primary field. This must be true in particular for $c(z)$ and $b(z)$.

Therefore, returning to the oscillator representation of $\mathcal{L}_0 + \mathcal{L}_0^\dagger$ in terms of matrices and of K_1 in terms of G and H (5.1), we expect to find

$$0 = [K_1, \mathcal{L}_0 + \mathcal{L}_0^\dagger] = \quad c^\dagger(GA + AH^T)b^\dagger - c(BG + H^TB)b + c^\dagger[G,C]b + b^\dagger[H,D]c$$

We indeed find

$$(GA + AH^T)_{p\bar{q}} = (2p+\bar{q})(p+\bar{q}+2)\ell_{p+\bar{q}+1} + (2p+\bar{q})(p+\bar{q}-2)\ell_{p+\bar{q}-1} = 0$$

We remark that in this equation the internal summations are the ordinary ones, over a 'long' index n in GA and a 'short' index $\bar{n}$ in AH^T. Similarly

$$[G,C]_{pq} = (2p-q)(p-q-2)\ell_{|p-q-1|} + (2p-q)(p-q+2)\ell_{|p-q+1|} = 0$$

The internal dummy index is an ordinary 'long' one in GC, while it is a stretched one $\dot{n}$ in CG. This is due to the previous stretching of G and consequent stretching of C. This carries into the game a new term (which is precisely the one that is produced by the third term in the RHS of (5.1)).

In a similar way we can prove $(BG + H^T B)_{\bar{p}\bar{q}} = 0$ and $[H,D]_{\bar{p}\bar{q}} = 0$. This is true only because of the extension of the internal summations due to the stretching of H, G and corresponding stretching of B, D, following the above recipe.

We notice that since the twist matrix $\hat{C}$ anticommutes with G and H we get the 'commutation' rules (fit for lame matrices)

$$(G\tilde{A} - \tilde{A}H^T)_{p\bar{q}} = 0 = (\tilde{B}G - H^T\tilde{B})_{\bar{p}\bar{q}} \tag{5.6}$$

The stretching does not affect the matrix A, while B is both left and right–stretched, C and D are right–stretched. We would like to notice that matrix stretching introduced above carries a stretching also in the commutators and consequently new rows or columns. For instance we have $(G\tilde{A} - \tilde{A}H^T)_{-2,\bar{q}} \neq 0$ or $(GA + AH^T)_{p1} \neq 0$, and similar relations for the other commutators. This nonzero terms however do not interfere with our diagonalization because they are outside the range of the bases to which we will apply these commutation relations.

It is important to remark that this technical complication is not intrinsic to the problem we want to solve, i.e. diagonalizing the solution of the KP equations, but rather to the technique we use to solve it. In fact our true aim is to prove eq.(5.20) below and the like. Since we cannot do it directly, because of the complexity of the problem, we resort to the operator K_1, which can be easily diagonalized and has a nondegenerate spectrum, and use the fact that operators that commute with it must have the same eigenvectors. It is in taking this detour that we meet the complication of having to use stretched matrices. Fortunately, as we will see, this implies only some awkwardnesses in the formulas, but disappears in the final results.

5.1 The $\mathcal{H}^{(2)}$ space

We call $\mathcal{H}^{(2)}$ the space spanned by the eigenvectors of K_1 belonging to the weight 2 basis, [24, 25]. These are obtained as follows. Let us start from the action of K_1 on the weight 2 field $b(z)$

$$\mathcal{K}_1 b(z) \equiv [K_1, b(z)] = (1+z^2)\partial b(z) + 4z\, b(z) = -\sum_n \left((n+1)\, b_{n-1} + (n-1) b_{n+1}\right) z^{-n-2}$$

On the other hand, on the operator side,

$$
\begin{aligned}
[K_1, b(z)] &= \left[c^\dagger G\, b + b^\dagger H\, c\,,\ \sum_{n\geq -1} b_n\, z^{-n-2} + \sum_{n>1} b_n^\dagger z^{n-2} \right] \\
&= -\sum_n \left((n-1)\, b_{n+1} + (n+1) b_{n-1}\right)\, z^{-n-2}
\end{aligned} \tag{5.7}
$$

We stress that in order for this and the corresponding equation for the c field to be true we have to use stretched H, G matrices.

Therefore $\mathcal{K}_1$ and K_1 with the adjoint action, generate the same conformal transformation on the z–plane. Now let us integrate the differential equation

$$
\mathcal{K}_1 f_\kappa^{(2)}(z) = (1+z^2)\partial f_\kappa^{(2)}(z) + 4z\, f_\kappa(z) = \kappa f_\kappa^{(2)}(z)
$$

The result is the generating function

$$
f_\kappa^{(2)}(z) = \left(\frac{1}{1+z^2}\right)^2 e^{\kappa \arctan(z)} = 1 + \kappa z + \left(\frac{\kappa^2}{2} - 2\right) z^2 + \dots \tag{5.8}
$$

The unnormalized basis (weight 2 basis) is given by

$$
f_\kappa^{(2)}(z) = \sum_{n=2} V_n^{(2)}(\kappa)\, z^{n-2} \tag{5.9}
$$

i.e.

$$
V_n^{(2)}(\kappa) = \frac{1}{2\pi i} \oint dz\, \frac{e^{\kappa \arctan(z)}}{(1+z^2)^2} \frac{1}{z^{n-1}} \tag{5.10}
$$

Notice the different labeling with respect to [24, 25]. Following [24, 25], (see also Appendix B), we normalize the eigenfunctions as follows

$$
\tilde{V}_n^{(2)}(\kappa) = \sqrt{A_2(\kappa)} V_n^{(2)}(\kappa) \tag{5.11}
$$

where

$$
A_2(\kappa) = \frac{\kappa(\kappa^2+4)}{\sinh\left(\frac{\pi\kappa}{2}\right)}
$$

5.2 The $\mathcal{H}^{(-1)}$ space

We call $\mathcal{H}^{(-1)}$ the space spanned by the eigenvectors of K_1 belonging to the weight -1 basis. It is obtained in the same way as above by replacing the $b(z)$ with the $c(z)$ field. In short, we start with

$$
\mathcal{K}_1 f_\kappa^{(-1)}(z) = (1+z^2)\partial f_\kappa^{(-1)}(z) - 2z\, f_\kappa(z) = \kappa f_\kappa^{(-1)}(z)
$$

which is dictated by the transformation of the c field, and integrate the differential equation. The result is the generating function

$$
f_\kappa^{(-1)}(z) = (1+z^2)\, e^{\kappa \arctan(z)} = 1 + \kappa z + \left(\frac{\kappa^2}{2} + 1\right) z^2 + \dots \tag{5.12}
$$

The unnormalized basis is given by

$$f_\kappa^{(-1)}(z) = \sum_{n=-1} V_n^{(-1)}(\kappa)\, z^{n+1} \tag{5.13}$$

i.e.

$$V_n^{(-1)}(\kappa) = \frac{1}{2\pi i} \oint dz\, (1+z^2)\, e^{\kappa \arctan(z)} \frac{1}{z^{n+2}} \tag{5.14}$$

The normalized one is

$$\tilde{V}_n^{(-1)}(\kappa) = \sqrt{A_{-1}(\kappa)} V_n^{(-1)}(\kappa), \qquad \sqrt{A_{-1}(\kappa)} = \frac{1}{2}\mathcal{P}\frac{1}{\kappa}\frac{\sqrt{A_2(\kappa)}}{\kappa^2+4} \tag{5.15}$$

where $\mathcal{P}$ denotes the principal value.

Before proceeding let us briefly comment about the two bases $V_n^{(2)}(\kappa)$ and $V_n^{(-1)}(\kappa)$ (a more complete account is presented in Appendix B, based on the results of [24, 25, 27]). We can promote $V_n^{(2)}(\kappa)$ to a complete orthonormal basis in $\mathcal{H}^{(2)}$ by multiplying them by suitable normalization constants $N_n^{(2)}$. The so obtained new states, $\hat{V}_n^{(2)}$, are however not anymore eigenstates of H^T (see below). We can do the same with $V_n^{(-1)}(\kappa)$ and get a complete orthonormal basis $\hat{V}_n^{(-1)}(\kappa)$, but the latter are not eigenfunctions of G. This may seem to be a drawback, but in fact what we need is another property: the weight 2 and -1 bases, $\tilde{V}_n^{(2)}(\kappa)$ and $\tilde{V}_n^{(-1)}(\kappa)$ are biorthogonal, see Appendix B.

5.3 Diagonalization of K_1

It is now easy to verify that, as expected, the matrices representing K_1 are diagonal in the above bases. Let us start with the weight 2 basis.

$$\begin{aligned}
\sum_{n=2}^{\infty} H^T_{pn}\, V_n^{(2)}(\kappa) &= \frac{1}{2\pi i}\oint dz\, \frac{e^{\kappa\arctan(z)}}{(1+z^2)^2}\,\frac{1}{z^{n-1}}\,((p-1)\,\delta_{p+1,n} + (p+1)\,\delta_{p-1,n})\\
&= \frac{1}{2\pi i}\oint dz\, \frac{e^{\kappa\arctan(z)}}{(1+z^2)^2}\left(\frac{p-1}{z^p} + \frac{p+1}{z^{p-2}}\right)\\
&= -\frac{1}{2\pi i}\oint dz\, e^{\kappa\arctan(z)}\frac{d}{dz}\left(\frac{1}{1+z^2}\frac{1}{z^{p-1}}\right)\\
&= \kappa\, V_p^{(2)}(\kappa)
\end{aligned} \tag{5.16}$$

as expected. In the weight -1 basis we have

$$\begin{aligned}
\sum_{n=-1}^{\infty} V_n^{(-1)}(\kappa)\, G_{nq} &= \frac{1}{2\pi i}\oint dz (1+z^2)\, e^{\kappa\arctan(z)}\\
&\quad \cdot \sum_{n=-1}^{\infty} \frac{1}{z^{n+2}}\,((n-1)\delta_{n+1,q} + (n+1)\delta_{n-1,q})\\
&= -\frac{1}{2\pi i}\oint e^{\kappa\arctan(z)}\frac{d}{dz}\left((1+z^2)^2\frac{1}{z^{q+2}}\right)\\
&= \kappa\, V_q^{(-1)}(\kappa)
\end{aligned} \tag{5.17}$$

In other words $V_n^{(2)}(\kappa)$ right–diagonalizes H^T, while $V_n^{(-1)}(\kappa)$ left–diagonalizes G. We remark that the stretching of G and H does not affect the above equations.

5.4 Diagonalization of $C^2 - AB$

The weight 2 basis right–diagonalizes $(D^T)^2 - BA$. Using (4.22) we get

$$\begin{aligned}
&\sum_{n=2}^{\infty}((D^T)^2 - BA)_{pn}\, V_n^{(2)}(\kappa) = \\
&= \frac{\pi^2}{2}\frac{1}{2\pi i}\oint dz\, \frac{e^{\kappa \arctan(z)}}{(1+z^2)^2}\left(\frac{p^2-2}{z^{p-1}} + \frac{1}{2}\frac{p^2-p}{z^{p+1}} + \frac{1}{2}\frac{p^2+p}{z^{p-3}}\right) \\
&= \frac{\pi^2}{2}\frac{1}{2\pi i}\oint dz\, \frac{e^{\kappa \arctan(z)}}{(1+z^2)^2}\cdot\frac{1}{2}(1+z^2)^2\frac{d}{dz}\left((1+z^2)\frac{d}{dz}\left(\frac{1}{1+z^2}\frac{1}{z^{p-1}}\right)\right) \\
&= \frac{\pi^2\kappa^2}{4}\frac{1}{2\pi i}\oint dz\, \frac{e^{\kappa \arctan(z)}}{(1+z^2)^2}\frac{1}{z^{p-1}} = \frac{\pi^2\kappa^2}{4}V_p^{(2)}(\kappa) \qquad (5.18)
\end{aligned}$$

Similarly the weight -1 basis left–diagonalizes $C^2 - AB$. Using (4.21) one gets

$$\begin{aligned}
\sum_{n=-1}^{\infty} V_n^{(-1)}(\kappa)\,(C^2 - AB)_{nq} &= \frac{\pi^2}{4}\frac{1}{2\pi i}\oint dz(1+z^2)\, e^{\kappa \arctan(z)} \\
&\quad\cdot\left(2\frac{q^2-2}{z^{q+2}} + \frac{(q-2)(q-3)}{z^q} + \frac{(q+2)(q+3)}{z^{q+4}}\right) \\
&= \frac{\pi^2}{4}\frac{1}{2\pi i}\oint dz(1+z^2)\, e^{\kappa \arctan(z)} \\
&\quad\cdot\frac{1}{1+z^2}\frac{d}{dz}\left((1+z^2)\frac{d}{dz}\left((1+z^2)^2\frac{1}{z^{q+2}}\right)\right) \\
&= \frac{\pi^2}{4}\kappa^2\, V_p^{(-1)}(\kappa) \qquad (5.19)
\end{aligned}$$

5.5 Eigenvalue of $\tilde{A}$ in the weight 2 basis

It is of course incorrect to speak about the eigenvalue of a lame matrix like $\tilde{A}$, because $\tilde{A}$ transform short–legged vectors into long–legged ones. Therefore the eigenvalue equation, strictly speaking, cannot be satisfied. What we really mean here is the following. We wish to prove that

$$\sum_{\bar{q}=2}^{\infty} A_{\bar{p}\bar{q}}\, V_{\bar{q}}^{(2)}(\kappa) = \mathfrak{a}(\kappa) V_{\bar{p}}^{(2)}(\kappa), \qquad (5.20)$$

$$\sum_{\bar{q}=2}^{\infty} A_{a\bar{q}}\, V_{\bar{q}}^{(2)}(\kappa) = 0, \qquad a = -1, 0, 1 \qquad (5.21)$$

This is precisely what we need in order to demonstrate the result in the next section. Eq.(5.21) is proven in Appendix D1. Next let us call $\mathcal{A}$ the square submatrix $\mathcal{A}_{\bar{p}\bar{q}} = A_{\bar{p}\bar{q}} = B_{\bar{p}\bar{q}}$. Thanks to (5.21) it is easy to realized that, when applied to the basis $V_{\bar{p}}^{(2)}(\kappa)$ the relation (5.6), $G\tilde{A} - \tilde{A}H^T = 0$, reduces to $[\mathcal{A}, H^T] = 0$. This and (5.16) are enough to conclude that $\mathcal{A}$ is diagonal in the weight 2 basis. This is precisely eq.(5.20). From this we also see that the stretching introduced at the beginning of this section is completely harmless.

Now let us compute the eigenvalue $\mathfrak{a}(\kappa)$, following the method of [22]. We must have

$$\sum_{n=2}^{\infty} \tilde{\mathcal{A}}_{2,n}\, V_n^{(2)}(\kappa) = \mathfrak{a}(\kappa)\, V_2^{(2)}(\kappa) = \mathfrak{a}(\kappa) \tag{5.22}$$

because $V_2^{(2)}(\kappa) = 1$. Now, for $n = 2l$ even, we have

$$\tilde{\mathcal{A}}_{2,n} \equiv \tilde{A}_{2,n} = (-1)^{\frac{n}{2}} \left(\frac{3}{n+1} - \frac{1}{n+3} \right) \tag{5.23}$$

The strategy consists in: (*step 1*) writing a differential equation for $\mathfrak{a}(\kappa)$ and integrating it; then, (*step 2*), expressing the integral in terms of hypergeometric functions. Finally, (*step 3*), using the properties of the latter to simplify the result. We will work out this case in some detail as a sample of several computation of the same kind.

Step 1

Let us write

$$F(z) = \sum_{l=1}^{\infty} \frac{(-1)^l}{2l+1} V_{2l}^{(2)}(\kappa) z^{2l+1} \tag{5.24}$$

$$G(z) = \sum_{l=1}^{\infty} \frac{(-1)^l}{2l+3} V_{2l}^{(2)}(\kappa) z^{2l+3} \tag{5.25}$$

Then

$$\frac{dF}{dz} = \sum_{l=1}^{\infty} (-1)^l V_{2l}^{(2)}(\kappa)\, z^{2l} = -\frac{z^2}{2} \left(f_\kappa^{(2)}(iz) + f_\kappa^{(2)}(-iz) \right) \tag{5.26}$$

$$\frac{dG}{dz} = \sum_{l=1}^{\infty} (-1)^l V_{2l}^{(2)}(\kappa)\, z^{2l+2} = -\frac{z^4}{2} \left(f_\kappa^{(2)}(iz) + f_\kappa^{(2)}(-iz) \right) \tag{5.27}$$

where $f_\kappa^{(2)}(z)$ was defined above. Let us set

$$H(z) = 3F(z) - G(z) \tag{5.28}$$

We get

$$\frac{dH}{dz} = -\frac{3z^2 - z^4}{2} \left(f_\kappa^{(2)}(iz) + f_\kappa^{(2)}(-iz) \right)$$

Therefore

$$\mathfrak{a}(\kappa) \equiv H(1) = -\int_0^1 dz\, h(z) \cosh\left(\kappa \tan^{-1}(iz) \right) \tag{5.29}$$

where

$$h(z) = \frac{3z^2 - z^4}{(1-z^2)^2} \tag{5.30}$$

We will need

$$\tan^{-1}(iz) = i\tanh^{-1}(z) = \frac{i}{2}\log\left(\frac{1+z}{1-z}\right)$$

As it is evident from this equation in the integrand of (5.29) there is a branch point at 1, therefore this expression of the integral is formal and we have to specify its meaning.

Step 2

We do it in the following way. We rewrite

$$\begin{aligned} H(1) &= \frac{1}{2}\int_0^1 dz\left(2z^2\,(1+z)^{\zeta-2}(1-z)^{-\zeta-2} + z^2\,(1+z)^{\zeta-1}(1-z)^{-\zeta-1}\right) + \\ &+(\kappa\to-\kappa) = \Theta(\kappa)+\Theta(-\kappa) \end{aligned} \tag{5.31}$$

where $\zeta = \frac{i\kappa}{2}$.

Using the basic integral representation of the hypergeometric function (C.1)

$$\Theta(\kappa) = -\frac{1}{\zeta(1-\zeta)(1+\zeta)}\left[2\,F\,(2-\zeta,3;2-\zeta;-1) - \frac{1+\zeta}{2-\zeta}\,F\,(1-\zeta,3;3-\zeta;-1)\right] \tag{5.32}$$

Here $F(a,b;c;z) \equiv {}_2F_1(a,b;c;z)$. For other properties of the hypergeometric functions, see Appendix C.

Step 3

The strategy now consists in bringing the hypergeometric functions to the form (C.2), (C.4) or (C.3). To this end one uses (C.5) or (C.6). One then uses, for instance, the properties of the $\mathcal{G}$ special function

$$\mathcal{G}(1+z)+\mathcal{G}(z) = \frac{2}{z}, \qquad \mathcal{G}(1-z)+\mathcal{G}(z) = \frac{2\pi}{\sin(\pi z)}$$

Using (C.3) one gets $F(2-\zeta,3;2-\zeta;-1) = \frac{1}{8}$. The other expression is more complicated. Using three times (C.6) on $F\,(1-\zeta,3;3-\zeta;-1)$ one gets a sum of terms of the form (C.3) or (C.4). Finally

$$\Theta(\kappa) = -\frac{1}{4(\zeta-\zeta^3)} + \frac{1}{2} + \frac{1}{4\zeta} + \frac{\zeta}{2}\,\mathcal{G}(1-\zeta) \tag{5.33}$$

Then

$$\begin{aligned} \mathfrak{a}(\kappa) \equiv H(1) &= \Theta(\kappa)+\Theta(-\kappa) = 1 + \frac{\zeta}{2}\left(\mathcal{G}(1-\zeta)-\mathcal{G}(1+\zeta)\right) \\ &= 1+\frac{\zeta}{2}\left(\mathcal{G}(1-\zeta)+\mathcal{G}(\zeta)-\frac{2}{\zeta}\right) = \frac{\pi\kappa}{2}\frac{1}{\sinh\left(\frac{\pi\kappa}{2}\right)} \end{aligned} \tag{5.34}$$

We have done an independent check of this result by considering the equation

$$\sum_{n=2}^{\infty}\tilde{\mathcal{A}}_{3,n}\,V_n^{(2)}(\kappa) = \mathfrak{a}(\kappa)\,V_3^{(2)}(\kappa) = \kappa\,\mathfrak{a}(\kappa) \tag{5.35}$$

Analytically this case is very complicated, but numerically one can easily show that it gives the same result (5.34).

5.6 The eigenvalues of $\tilde{B}, D^T$ in the weight 2 basis, and $\tilde{A}$ in the weight -1 basis

In a very similar way one can prove that $\tilde{A}$, when applied from the right to the weight -1 basis has the same eigenvalue (5.34). On the other hand the eigenvalue of D^T in the weight 2 basis is given by $\mathfrak{c}(\kappa)$

$$\mathfrak{c}(\kappa) = \frac{\pi\kappa}{2}\coth\left(\frac{\pi\kappa}{2}\right) \tag{5.36}$$

All these results are derived in Appendix D. As for $\tilde{B}$, due to eq.(5.21), it is easy to show that the product $\tilde{B}\tilde{A}$ boils down to $\mathcal{A}^2$ when acting on the weight 2 basis. Therefore the eigenvalue of $\tilde{B}$ (in the sense explained at the beginning of the previous section) is $\mathfrak{a}(\kappa)$. It follows that, if we consider for instance the solution (4.17) applied to the weight 2 basis, it makes sense to multiply it from the left by $\tilde{B}^{-1}$ and end up with (4.18).

6. Ghost wedge states

We are now in the position to draw the conclusions of our long analysis on the ghost sector. Let us consider the solution (4.18) to the KP equations. We apply it to the weight 2 basis from the left. We remark that due to the form of (4.18), since A is not stretched[3] the stretching of B and D disappears, and we can safely use (5.18): $(D^T)^2 - BA$ is diagonal with eigenvalue $\frac{\pi^2\kappa^2}{4}$. Moreover D^T is diagonal in this basis with eigenvalue $\mathfrak{c}(\kappa)$, and $\tilde{A}$ too (in the sense explained above) with eigenvalue

$$\mathfrak{a}(\kappa) = \frac{\pi\kappa}{2\sinh\left(\frac{\pi\kappa}{2}\right)}$$

Notice the different sign and factor of 2, with respect to the matter sector. So $\alpha_2(t) = \alpha(t)$ is diagonal too in the weight 2 basis with eigenvalue given by (4.18) where all the matrices have been replaced by the corresponding eigenvalues. Let us call $\alpha(\kappa, t)$ the resulting eigenvalue.

Let us return to eq.(3.1). In this equation we have to compute in particular $c^\dagger\alpha b^\dagger$. Using the formalism of Appendix B, we can write this term as follows

$$\begin{aligned} c^\dagger\alpha(t)b^\dagger &= \sum_{n=-1,m=2} c_n^\dagger \alpha_{nm}(t) b_m^\dagger = \sum_{n=-1,m=2}^{\infty} \int d\kappa\, d\kappa'\, \tilde{c}(\kappa)\tilde{V}_n^{(-1)}(\kappa)\tilde{\alpha}_{nm}(t)\tilde{V}_m^{(2)}(\kappa')\, b(\kappa') \\ &= \sum_{n=2}^{\infty} \int d\kappa\, d\kappa'\, \tilde{c}(\kappa)\tilde{V}_n^{(-1)}(\kappa)\tilde{\alpha}(\kappa,t)\tilde{V}_n^{(2)}(\kappa')b(\kappa') = \int d\kappa\, \tilde{c}(\kappa)\tilde{A}(\kappa)\, b(\kappa) \end{aligned} \tag{6.1}$$

(see Appendix B for the definitions of $\tilde{c}(\kappa)$ and $b(\kappa)$, and the biorthogonality relation used in (6.1)). The same can be done for any power of $\alpha(t), \tilde{A}$ and D^T and $\Delta = (D^T)^2 - BA$. Therefore, from now on we will simply deal with the corresponding eigenvalues.

[3] In solving the KP equations we must use stretched matrices, but it is easy to realize that the solution is unstretched.

Now let us apply this to the construction of the ghost wedge states (as in the matter case, in the rest of this section we use the matrix symbols to actually represent the corresponding eigenvalues). We should have

$$T_n \equiv \hat{C} S_n = \tilde{\alpha}\left(-\frac{n-2}{2}\right) \tag{6.2}$$

where, as usual, a tilde represents a twisted matrix $\tilde{\alpha} = \hat{C}\alpha$. In detail

$$T_n = -\tilde{A}\frac{\sinh\left(\sqrt{\Delta}\,\frac{n-2}{2}\right)}{\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\,\frac{n-2}{2}\right) + D^T\sinh\left(\sqrt{\Delta}\,\frac{n-2}{2}\right)} \tag{6.3}$$

where $\sqrt{\Delta} = \frac{\pi|\kappa|}{2}$, while

$$\tilde{A} = \frac{\kappa\pi}{2\sinh\left(\frac{\kappa\pi}{2}\right)}, \qquad D^T = \frac{\kappa\pi}{2}\coth\left(\frac{\kappa\pi}{2}\right) \tag{6.4}$$

Comparing with the expression of the wedge state in the matter sector, we see that the expression of T_n in terms of κ are exactly the same, because the relative minus sign and factor of 2 in the definition of T_n are compensated by the change of sign and the factor of 2 in the expression of the eigenvalue of $\tilde{A}$. Therefore we do not need to repeat the demonstration of section 2.5. We know it is true that[4]

$$T_{n+1} = X\frac{1 - T_n}{1 - T_n X}, \tag{6.5}$$

with $T_1 = 1$, $T_2 = 0$,

$$T_3 = X, \qquad T = -\frac{\tilde{A}}{D^T + \sqrt{\Delta}} = e^{-\frac{\pi|\kappa|}{2}} \tag{6.6}$$

where $X = \hat{C}\tilde{V}^{11}$ and $\tilde{V}^{11}$ is (the eigenvalue of) the matrix of ghost Neumann coefficients in the naturally normal ordered three–string vertex.

As in the matter sector the normalization constants $\mathcal{N}_n$ must also satisfy a recursion relation

$$\mathcal{N}_n\,\mathcal{K}\det\left(1 - T_n X\right) = \mathcal{N}_{n+1} \tag{6.7}$$

where $\mathcal{K}$ is some constant to be determined. We have

$$\eta_n = -\int_0^{t_n} dt\,\mathrm{tr}(B\alpha) \tag{6.8}$$

Using the simple relations of these quantities with their matter counterpart and identifying

$$\mathcal{N}_n = e^{\eta_n}$$

[4]It must be stressed once again that (6.5) makes sense if the symbols denote eigenvalues, not matrices. The true matrix correspondent of (6.5) is more complicated and requires an adequate preparation (see footnote at the beginning of section 3).

we can verify that (6.7) is satisfied, and determine $\mathcal{K}$. We remark that in order to define the trace in (6.8) the completeness relation (B.15) is needed.

In this section we have solved the problem by applying $\alpha(t)$ to the weight 2 basis from the left. We believe that the same result can be obtained by applying it to the weight -1 basis from the right, even though the calculations turn out to be more involved. For this reason we have started the relevant analysis (see, for instance, App. D3), but we have not completed it.

7. Twisted ghost sector

In this section, for completeness we deal with the twisted ghost sector, [23]. This sector, due to its features very close to the those of the matter sector, allows for a simplified treatment. The twist changes the geometrical nature of the ghost fields so that b gets weight 1 and c weight 0.

The unordered Virasoro generators for the twisted ghosts are

$$L_n = -\sum_k k b_{n+k} c_{-k}$$

Once they are normal ordered

$$L_n^{(g)} = \sum_{k\geq 1}(n+k)\, b_k^\dagger c_{k+n} + \sum_{k\geq 0} k\, c_k^\dagger b_{n+k} - \sum_{k=1}^{n} k\ c_k b_{n-k} \tag{7.1}$$

$$L_0^{(g)} = \sum_{k\geq 1} k\, c_k^\dagger b_k + \sum_{k\geq 1} k\, b_k^\dagger c_k \tag{7.2}$$

$$L_{-n}^{(g)} = \sum_{k\geq 1} k\, b_{n+k}^\dagger c_k + \sum_{k\geq 0}(n+k)\, c_{k+n}^\dagger b_k + \sum_{k=0}^{n-1} k\, c_k^\dagger b_{n-k}^\dagger \tag{7.3}$$

Therefore, for the ghost part, we can write

$$\begin{aligned}\mathcal{L}_0 + \mathcal{L}_0^\dagger &= 2L_0^{(g)} + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{4n^2-1}(L_{2n}^{(g)} + L_{-2n}^{(g)}) \\ &= c^\dagger\, \mathbf{A}\, b^\dagger + c^\dagger \mathbf{C}\, b + b^\dagger \mathbf{D} c - c\, \mathbf{B}\, b\end{aligned} \tag{7.4}$$

where we have introduced boldface matrices for a reason that will be clear in a moment,

$$\mathbf{A}_{pq} = \sum_{n=0}^{\infty} \ell_n p \delta_{p+q,n} = p\,\ell_{p+q}, \qquad p \geq 0, q \geq 1 \tag{7.5}$$

$$\mathbf{B}_{pq} = \sum_{n=0}^{\infty} \ell_n p \delta_{p+q,n} = -p\,\ell_{p+q}, \qquad p \geq 1, q \geq 0 \tag{7.6}$$

$$\begin{aligned}\mathbf{C}_{pq} &= \sum_{n=0}^{\infty} \ell_n [p\delta_{p+n,q} + (n+q)\delta_{q+n,p}] + 2\,p\,\delta_{p,q} \\ &= p(\ell_{q-p} + \ell_{p-q}) = p\ell_{|p-q|}, \qquad p \geq 0, q \geq 0\end{aligned} \tag{7.7}$$

$$\begin{aligned}\mathbf{D}_{pq} &= \sum_{n=0}^{\infty} \ell_n [q\delta_{q+n,p} + q\delta_{p+n,q}] + 2\,p\,\delta_{p,q} \\ &= q(\ell_{q-p} + \ell_{p-q}) = q\ell_{|p-q|}, \qquad p \geq 1, q \geq 1\end{aligned} \tag{7.8}$$

The matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ have the following structure.

$$\mathbf{A} = \begin{pmatrix} 0 & | & 0 \\ --- & | & --- \\ 0 & | & A \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 0 & | & 0 \\ --- & | & --- \\ \vec{u} & | & C \end{pmatrix} \tag{7.9}$$

$$\mathbf{B} = \begin{pmatrix} 0 & | & 0 \\ --- & | & --- \\ \vec{u} & | & A \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 0 & | & 0 \\ --- & | & --- \\ 0 & | & C^T \end{pmatrix} \tag{7.10}$$

where $\vec{u} = \{u_p\}$ and $u_p = p\,\ell_p$. The matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are referred to as the large matrices, while A and C will be called small matrices.

In analogy with what was done in section 4, we can immediately check that $\mathbf{A}\mathbf{D}^T - \mathbf{C}\mathbf{A} = 0 \Longleftrightarrow [A, C] = 0$, while

$$\mathbf{B}\,\mathbf{C} - \mathbf{D}^T\,\mathbf{B} = \begin{pmatrix} 0 & | & 0 \\ --------- & | & -------------- \\ A\,\vec{u} & | & AC - CA \end{pmatrix} \tag{7.11}$$

On the other hand there is a direct relation with the corresponding matter matrices

$$A^{(m)} = \frac{1}{2} E^{-1} A\, E, \qquad C^{(m)} = E^{-1} C\, E, \tag{7.12}$$

where $E_{nm} = \sqrt{n}\delta_{nm}$ and the label $^{(m)}$ of course denote the matter counterpart. It follows in particular that

$$[A, C] = 0 \tag{7.13}$$

Using this relation with the matter part we also get

$$\begin{aligned}(C^2 - A^2)_{pq} &= \left(E(C^{(m)2} - 4A^{(m)2})E^{-1}\right)_{pq} \\ &= \frac{\pi^2}{2}\left(p^2\delta_{p,q} + \frac{1}{4}p(p+q)(\delta_{p,q+2} + \delta_{q,p+2})\right)\end{aligned} \tag{7.14}$$

This relation does not hold if we include the zero mode (we do not have the analogs of the relations in section 4.3). However, in the untwisted ghost sector we do not have to repeat the derivation. Here life is easier.

7.1 The KP equation

Now we would like to integrate the KP equation

$$\dot{\hat{\alpha}} = \mathbf{A} + \mathbf{C}\hat{\alpha} + \hat{\alpha}\mathbf{D}^T + \hat{\alpha}\,\mathbf{B}\hat{\alpha} \tag{7.15}$$

where $\hat{\alpha}$ represent the large matrix solution, while the small one will be denoted by α. Looking at the relation (3.18) we can guess the form to be expected for $\hat{\alpha}$. It can only be

$$\hat{\alpha} = \begin{pmatrix} 0 & | & 0 \\ --- & | & --- \\ \vec{f} & | & \alpha \end{pmatrix}, \tag{7.16}$$

which breaks down to the equations

$$\dot{\vec{f}} = (C + \alpha A)\,\vec{f} \tag{7.17}$$

$$\dot{\alpha} = A + \{\alpha, C\} + \alpha\,A\,\alpha \tag{7.18}$$

It is consistent to set $\vec{f} = 0$ (this is actually the solution one gets by solving (7.15) recursively starting from $\mathbf{A}$). Therefore only the second equation remains and can be solved precisely as in the matter case since the small matrices A and C commute:

$$\alpha(t) = A\frac{\sinh\left(\sqrt{C^2 - A^2}\,t\right)}{\sqrt{C^2 - A^2}\cosh\left(\sqrt{C^2 - A^2}\,t\right) - C\sinh\left(\sqrt{C^2 - A^2}\,t\right)} \tag{7.19}$$

7.2 Representation of K_1

Let us consider the $K_1 = L_1 + L_{-1}$ operator in the twisted ghost case. We get

$$K_1 = \sum_{p,q\geq 0} c_p^\dagger\,G_{pq}\,b_q + \sum_{p,q\geq 1} b_p^\dagger\,H_{pq}\,c_q + c_1\,b_0 \tag{7.20}$$

where

$$G_{pq} = p(\delta_{p+1,q} + \delta_{p-1,q}),$$
$$H_{pq} = q(\delta_{p+1,q} + \delta_{p-1,q}) = G_{qp} \tag{7.21}$$

So G is a long–legged square matrix, while H is a short–legged square one. However this distinction is pointless here because the nonvanishing 0–th column present in G is immaterial in what follows. Also in this, like in the untwisted case, we have the problem of the additional term in the RHS of eq.(7.20) and we can remedy in the same way, by absorbing it in the first two terms at the price of introducing stretched matrices. G becomes a left–stretched matrix with the first index running from -1 to ∞, and H a left–stretched matrix with the first index starting from 0 instead of 1, and retaining nontrivial anticommutation rules only between daggered and undaggered operators.

All the conditions for K_1 to commute with $\mathcal{L}_0 + \mathcal{L}_0^\dagger$ are satisfied. In fact it is easy to prove that

$$[G, \mathbf{C}]_{pq} = p(p-q+2)\,\ell_{|p-q+1|} + p(p-q-2)\,\ell_{|p-q-1|} = 0, \qquad p, q \geq 0$$

provided in $\mathbf{C}G$ the summation index runs from -1 to ∞. Similarly one can prove that $G\mathbf{A} + \mathbf{A}H^T = 0$ and $[H, \mathbf{D}] = 0$, which do not require any stretching. Finally $\mathbf{B}G + H^T\mathbf{B} = 0$, provided in $\mathbf{B}G$ the summation index is stretched. Once again we will realize that stretching does not have practical consequences except to help us in the process of diagonalization.

7.3 The weight 1 basis

For weight 1 ghost field b we have

$$\mathcal{K}_1\, b(z) = [K_1, b(z)] = (1+z^2)\partial b(z) + 2z b(z)$$

Integrating the equation

$$\mathcal{K}_1\, f_\kappa^{(1)}(z) = \kappa\, f_\kappa^{(1)}(z)$$

we get

$$f_\kappa^{(1)}(z) \equiv \sum_{n=1}^{\infty} V_n^{(1)}(\kappa) z^{n-1} = \frac{1}{1+z^2} e^{\kappa \arctan(z)} = 1 + \dots \tag{7.22}$$

So, in particular, $V_1^{(1)} = 1$, and

$$V_n^{(1)}(\kappa) = \frac{1}{2\pi i} \oint dz\, \frac{1}{1+z^2} \frac{e^{\kappa \arctan(z)}}{z^n} \tag{7.23}$$

As a verification, for instance, we have

$$\begin{aligned}
\sum_{n=1} H_{pn}^T\, V_n^{(1)}(\kappa) &= \frac{1}{2\pi i} \oint dz\, \frac{1}{1+z^2}\, e^{\kappa \arctan(z)} p \sum_{n=1}^{\infty} (\delta_{p+1,n} + \delta_{p-1,n})\, \frac{1}{z^n} \\
&= -\frac{1}{2\pi i} \oint dz\, e^{\kappa \arctan(z)} \frac{d}{dz} \left(\frac{1}{z^p} \right) \\
&= \kappa\, V_p^{(1)}(\kappa)
\end{aligned} \tag{7.24}$$

and, similarly,

$$\sum_{n=1} V_n^{(1)}(\kappa) H_{nq} = \kappa\, V_q^{(1)}(\kappa)$$

In the same way we can compute

$$\sum_{n=1}^{\infty}(C^2-A^2)_{pn}\,V_n^{(1)}(\kappa) = \frac{1}{2\pi i}\oint dz\,\frac{1}{1+z^2}\,e^{\kappa\arctan(z)}$$

$$\cdot\frac{\pi^2}{2}\sum_{n=1}^{\infty}\left(p^2\delta_{p,n}+\frac{1}{4}p(p+n)(\delta_{p,n+2}+\delta_{n,p+2})\frac{1}{z^n}\right)$$

$$= \frac{\pi^2}{4}\frac{1}{2\pi i}\oint dz\,\frac{e^{\kappa\arctan(z)}}{1+z^2}\,(1+z^2)\frac{d}{dz}\left((1+z^2)\frac{d}{dz}\frac{1}{z^p}\right)$$

$$= \frac{\pi^2\kappa^2}{4}\frac{1}{2\pi i}\oint dz\,\frac{e^{\kappa\arctan(z)}}{1+z^2}\,\frac{1}{z^p} = \frac{\pi^2\kappa^2}{4}\,V_p^{(1)}(\kappa) \tag{7.25}$$

7.4 The eigenvalue of $\tilde{A}$ and C

We proceed as in section 5.5, although here life is easier. In fact, due to the structure of $\mathbf{A}$, see (7.9), the analogue of eq.(5.21) is trivial. Therefore we have simply to prove

$$\sum_{n=1}^{\infty} A_{p,n}\,V_n^{(1)}(\kappa) = \mathfrak{a}(\kappa)\,V_p^{(1)}(\kappa) \tag{7.26}$$

and find $\mathfrak{a}(\kappa)$. On the weight 1 basis $G\tilde{\mathbf{A}}-\tilde{\mathbf{A}}H^T=0$ reduces to $H^T\tilde{A}-\tilde{A}H^T=0$. Thanks to this and (7.24) we conclude that (7.26) is true. So we set out to compute $\mathfrak{a}(\kappa)$.

Since $V_1^{(1)}(\kappa)=1$ we have

$$\sum_{n=1}^{\infty}\tilde{A}_{1,n}V_n^{(1)}(\kappa) = \mathfrak{a}(\kappa)\,V_1^{(1)}(\kappa) = \mathfrak{a}(\kappa) \tag{7.27}$$

On the other hand

$$\tilde{A}_{1n} = (-1)^{\frac{n+1}{2}}\left(\frac{1}{n}-\frac{1}{n+2}\right)$$

Therefore

$$\mathfrak{a}(\kappa) = -\sum_{l=0}^{\infty}(-1)^l\left(\frac{1}{2l+1}-\frac{1}{2l+3}\right)\,V_{2l+1}^{(1)}(\kappa) = G(1)-F(1) \tag{7.28}$$

where

$$F(z) = \sum_{l=0}^{\infty}(-1)^l\frac{1}{2l+1}V_{2l+1}^{(1)}(\kappa)z^{2l+1} \tag{7.29}$$

$$G(z) = \sum_{l=0}^{\infty}(-1)^l\frac{1}{2l+3}V_{2l+1}^{(1)}(\kappa)z^{2l+3} \tag{7.30}$$

So

$$\frac{dF}{dz} = \sum_{l=0}^{\infty}(-1)^lV_{2l+1}^{(1)}z^{2l} = \frac{1}{2}\left(f_\kappa^{(1)}(iz)+f_\kappa^{(1)}(-iz)\right) \tag{7.31}$$

$$\frac{dG}{dz} = \sum_{l=0}^{\infty}(-1)^lV_{2l+1}^{(1)}z^{2l+2} = \frac{z^2}{2}\left(f_\kappa^{(1)}(iz)+f_\kappa^{(1)}(-iz)\right) \tag{7.32}$$

Let us define

$$H(z) = G(z) - F(z) \tag{7.33}$$

We have $H(0) = 0$. So that $\mathfrak{a}(\kappa) = H(1)$.

Now

$$\begin{aligned}\frac{dH}{dz} &= -\cosh\left(\kappa \arctan(iz)\right)\\ &= -\frac{1}{2}\Big((1+z)^{\zeta}(1-z)^{-\zeta} + (\zeta \to -\zeta)\Big)\end{aligned} \tag{7.34}$$

where $\zeta = i\kappa/2$. Therefore

$$H(1) = -\frac{1}{2}\int_0^1 dz\, \Big((1+z)^{\zeta}(1-z)^{-\zeta} + (\zeta \to -\zeta)\Big) \tag{7.35}$$

$$= -\frac{1}{2(1-\zeta)}\left(F(-\zeta, 1; 2-\zeta; -1) + \zeta \to -\zeta\right) \tag{7.36}$$

Now we use (C.5,C.6) and obtain

$$H(1) = -\frac{1}{2(1-\zeta)}\left(-\zeta\, F(1-\zeta, 1; 2-\zeta; -1) - (1-\zeta)\, F(-\zeta, 1; 1-\zeta; -1) + (\zeta \to -\zeta)\right)$$

Next use (C.4)

$$H(1) = \frac{\zeta}{4}\left[\mathcal{G}(-\zeta) - \mathcal{G}(1-\zeta) - \mathcal{G}(\zeta) + \mathcal{G}(1+\zeta)\right] = \frac{\zeta}{4}\left(-\frac{4\pi}{\sin(\pi\zeta)}\right) \tag{7.37}$$

i.e.

$$\mathfrak{a}(\kappa) = \frac{\pi\kappa}{2\sinh\left(\frac{\pi\kappa}{2}\right)} \tag{7.38}$$

From the above results for $C^2 - A^2$ and A we can determine the eigenvalue of C up to a sign

$$\mathfrak{c}(\kappa) = \pm\frac{\pi\kappa}{2}\coth\left(\frac{\pi\kappa}{2}\right) \tag{7.39}$$

That the correct sign is actually + is shown in Appendix E.

7.5 Wedge states for twisted ghosts

We have shown above that the solution to the KP equation in the twisted case reduces to the small matrix $\alpha(t)$. Thanks to the relation (7.12) with the matter we can now conclude that

$$\alpha(t) = E\, \alpha^{(m)}(t)\, E^{-1} \tag{7.40}$$

Therefore we can dispense with the full discussion of the wedge states, because we can use the results of the matter sector. We must simply bear in mind the sign difference between the ghost and matter sector. So, for instance,

$$X^{(tw)} = -EX^{(m)}E^{-1}, \qquad T^{(tw)} = -ET^{(m)}E^{-1} \tag{7.41}$$

where the LHS matrices refer to the twisted ghosts. Moreover the scalar function η is the same for matter and twisted ghosts. We can conclude that the equivalence (3.18) holds for twisted ghost sector as well.

In this derivation we do not need to explicitly use the weight 0 basis, but of course, such a basis is needed in order for the formalism to work. A short discussion is given in Appendix E.

8. The conventional n.o.

In this last part of the paper we discuss the conventional normal ordering introduced in section 3.1 and its problems. Eventually it will be clear that the treatment with this normal ordering is very awkward, to say the least. However we cannot completely exclude that the formalism may work with this normal ordering either.

8.1 Commuting matrices

Let us return to the matrices of subsection 3.1. Using the formulas of the subsection 3.4 we can easily show some properties that may help us integrating the KP equations.

Suppose p and $\bar{q}$ are even. Then

$$\begin{aligned}(AD^T)_{p\bar{q}} &= \sum_{\bar{k}=2}^{\infty} A_{p\bar{k}}\, D^T_{\bar{k}\bar{q}} = \sum_{l=1}^{\infty}(2p+2l)(4l-\bar{q})\ell_{|\bar{q}-2l|}\,\ell_{2l+p} \\ &= 8(-1)^{\frac{p+\bar{q}}{2}} \sum_{l=1}^{\infty} \frac{(p+l)(4l-\bar{q})}{((p+2l)^2-1)(((2l-\bar{q})^2-1)}\end{aligned} \tag{8.1}$$

having set $\bar{k}=2l$. We notice that the RHS of this equation coincide with (3.28). Similarly

$$\begin{aligned}(CA)_{p\bar{q}} &= \sum_{k=0}^{\infty} C_{pk}\, A_{k\bar{q}} = \sum_{l=0}^{\infty}(2p-2l)(4l+\bar{q})\ell_{p+2l}\,\ell_{|2l-\bar{q}|} \\ &= 8(-1)^{\frac{p+\bar{q}}{2}}\left[\sum_{l=1}^{\infty} \frac{(p-l)(4l+\bar{q})}{((p-2l)^2-1)(((2l+\bar{q})^2-1)} + \frac{p\bar{q}}{(p^2-1)(\bar{q}^2-1)}\right]\end{aligned} \tag{8.2}$$

The first term in the square bracket on the RHS of this equation is nothing but eq.(3.29). Putting everything together we find

$$(AD^T - CA)_{p\bar{q}} = 8(-1)^{\frac{p+\bar{q}}{2}}\left[\frac{p\bar{q}}{(p^2-1)(\bar{q}^2-1)} - \frac{p\bar{q}}{(p^2-1)(\bar{q}^2-1)}\right] = 0 \tag{8.3}$$

This derivation becomes singular in the case $p=0, \bar{q}=2$. This case has to be dealt with separately. One easily gets

$$(AD^T)_{02} = -\frac{1}{4}, \qquad (CA)_{02} = -\frac{1}{4} \tag{8.4}$$

Similarly for p and $\bar{q}$ odd we have

$$(AD^T)_{p\bar{q}} = \sum_{\bar{k}=1}^{\infty} A_{p\bar{k}}\, D^T_{\bar{k}\bar{q}} = 4(-1)^{\frac{p+\bar{q}}{2}} \sum_{l=0}^{\infty} \frac{(2p+2l+1)(4l-\bar{q}+2)}{((p+2l+1)^2-1)(((2l-\bar{q}+1)^2-1)} \tag{8.5}$$

and

$$(CA)_{p\bar{q}} = \sum_{\bar{k}=0}^{\infty} C_{pk}\, A_{k\bar{q}} = 4(-1)^{\frac{p+\bar{q}}{2}} \sum_{l=0}^{\infty} \frac{(2p-2l-1)(4l+\bar{q}+2)}{((p-2l-1)^2-1)(((2l+\bar{q}+1)^2-1)} \tag{8.6}$$

having set $k = 2l+1$. The first term in square brackets in (8.5) coincides with eq.(3.24), while the corresponding term in the RHS of (8.6) coincides with (3.25). Therefore they are equal. It follows that the RHS of (8.5) equals the RHS of (8.6). The previous analysis does not hold for $p = 1$ and $q = 1$. By a direct calculation we find

$$(AD^T - CA)_{1,1} = [\mathcal{A}, \mathcal{C}]_{1,1} = \frac{\pi^2}{8} \tag{8.7}$$

on the basis of (3.27). This is the first obstacle brought in by the conventional n.o.. One could adopt the attitude that this is a very marginal non–commutativity and adopt a prescription in order to eliminate it. One possibility in this sense is to state that $(AD^T - CA)_{p,q}$ should be calculated for generic complex p, q and then analytically continued to integral p, q. The result is 0 and the above discrepancy is eliminated. Let us assume this attitude and continue the analysis.

Therefore we can conclude that

$$AD^T = CA \tag{8.8}$$

which is the appropriate 'commutation relation' for these kind of matrices.

In the same way one can prove that

$$BC = D^T B \tag{8.9}$$

It follows again from these results that, for instance,

$$CAB = AD^T B = ABC \tag{8.10}$$

Another important identity is the one concerning $C^2 - AB$. For odd p, q we get

$$\begin{aligned}(C^2 - AB)_{pq} &= \sum_{k=1} C_{pk}C_{kp} - \sum_{k=1} A_{pk}B_{kp} = (\mathcal{C}^2 - \mathcal{A}^2)_{pq} \\ &= \frac{\pi^2}{2}\left((p^2-2)\,\delta_{p,q} + \frac{1}{2}p(p-1)\,\delta_{q,p+2} + \frac{1}{2}p(p+1)\,\delta_{p,q+2}\right)\end{aligned} \tag{8.11}$$

and for even p, q

$$\begin{aligned}(C^2 - AB)_{pq} &= \sum_{k=0} C_{pk}C_{kp} - \sum_{k=1} A_{pk}B_{kp} = (\mathcal{C}^2 - \mathcal{A}^2)_{pq} + C_{p0}C_{0q} \\ &= (\mathcal{C}^2 - \mathcal{A}^2)_{pq} - 8(-1)^{\frac{p+q}{2}} \frac{pq}{(p^2-1)(q^2-1)} \\ &= \frac{\pi^2}{2}\left((p^2-2)\delta_{p,q} + \frac{1}{2}p(p-1)\,\delta_{q,p+2} + \frac{1}{2}p(p+1)\,\delta_{p,q+2}\right)\end{aligned} \tag{8.12}$$

These two are identities valid for long–legged square matrices. We can prove in a similar way

$$((D)^T - BA)_{\bar{p}\bar{q}} = \frac{\pi^2}{2}\left((\bar{p}^2-2)\delta_{\bar{p},\bar{q}} + \frac{1}{2}\bar{p}(\bar{p}-1)\,\delta_{\bar{q},\bar{p}+2} + \frac{1}{2}\bar{p}(\bar{p}+1)\,\delta_{\bar{p},\bar{q}+2}\right) \tag{8.13}$$

for square short–legged matrices.

8.2 Integration of the KP equations

The relevant KP equations

$$\dot{\alpha} = A + C\alpha + \alpha D^T + \alpha B\alpha \tag{8.14}$$

$$\tag{8.15}$$

and

$$\dot{\eta} = -\mathrm{Tr}\,(\alpha\, B) \tag{8.16}$$

can be integrated as in section 4.2. Proceeding in the same way it is easy to find the two solutions

$$\alpha_1(t) = \frac{\sinh\left(\sqrt{C^2 - AB}\, t\right)}{\sqrt{C^2 - AB}\cosh\left(\sqrt{C^2 - AB}\, t\right) - C\sinh\left(\sqrt{C^2 - AB}\, t\right)} A \tag{8.17}$$

and

$$\alpha_2(t) = A \frac{\sinh\left(\sqrt{(D^T)^2 - BA}\, t\right)}{\sqrt{(D^T)^2 - BA}\cosh\left(\sqrt{(D^T)^2 - BA}\, t\right) - D^T\sinh\left(\sqrt{(D^T)^2 - BA}\, t\right)} \tag{8.18}$$

These two solutions have the same functional form as in subsection 4.2, but, of course, they have different legs. If the commutation rules of the previous subsection hold (for that we need to adopt the analytic continuation argument), we can use the same argument as in subsection 4.2 and conclude that these two solutions coincide.

8.3 Other difficulties

In the ghost sector with the conventional normal ordering we get

$$K_1 = \sum_{p,q\geq 0} c_p^\dagger\, G_{pq}\, b_q + \sum_{p,q\geq 1} b_{\bar{p}}^\dagger\, H_{\bar{p}\bar{q}}\, c_{\bar{q}} - c_1\, b_0 + c_0^\dagger b_1^\dagger \tag{8.19}$$

where the expression G, H are the same as in (5.2) and G is a is a square long–legged matrix and H a square short–legged one, but with a different length with respect to section 5: long legs run from 0 to $+\infty$, short legs from 1 to $+\infty$. This is not without consequences. Absorbing the last two terms on the RHS of (8.19) in the first two, as we have done in the natural n.o. case, requires stretching both legs of G and H, and this complicates a lot the task of finding simple commutation rules with A, B, C, D. Trying then to diagonalize the matrices involved we face similar problems. The matrix A for instance is right–short–legged, but not as short as the weight 2 basis, which starts at $n = 2$. We do not exclude that all these discrepancies might somehow be fixed. But at the moment we do not see how to keep under control the consequences of the new prescriptions that are needed. For this reason we leave the use of the conventional n.o. as an open problem.

9. Conclusions

Our precise aim in this paper was to prove the equivalence (1.1) in the oscillator formalism, in the matter and in the ghost sector separately. And, starting from the RHS of (1.1) we have succeeded in proving the recursion relations characteristic of the diagonalized wedge states matrices that feature in the LHS. This result is of course important for the proof of the Schnabl's solution in the oscillator language, but it is interesting in itself because wedge states are a powerful approximation tool in SFT.

Some aspects of our proof should be underlined. Although at first the task seems to be unwieldy in the ghost sector, because the latter is characterized by asymmetric bases, it is nevertheless possible to do it because the relevant infinite matrices can nevertheless be diagonalized. We have learned from our research that such problems can be dealt with in a way not devoid of elegance. We have also learned that the most suitable choice for normal ordering in the ghost sector is the natural one. This of course means that in future developments (in particular in the continuation of the present program) we have to reformulate the star product, the wedge states and so on in the natural n.o.[5], while so far they have been dealt with essentially with the conventional n.o.. On the other hand the natural normal ordering seems to be the 'natural' one in the Schnabl's proof, otherwise we cannot see how one can deal with such states as $c_1|0\rangle$.

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Appendix

A. Some remarkable identities

In this section we present an explicit sample of calculations needed in this paper.

[5] We have already mentioned that strictly speaking, eq.(1.1) is not really demonstrated before this is done.

We are going to repeatedly use the identity

$$\frac{l}{(p+2l+a)(q+2l+b)} = \frac{1}{2(p-q+a-b)}\left(\frac{p+a}{p+2l+a} - \frac{q+b}{q+2l+b}\right) \tag{A.1}$$

Now assume p, q are both even.

$$\begin{aligned}
A_{pq}^2 &= 2\sqrt{pq}\sum_{l=1}^{\infty} l\frac{(-1)^{p+q}}{(p+2l)^2-1)((q+2l)^2-1)} \\
&= \frac{\sqrt{pq}(-1)^{p+q}}{2}\sum_{l=1}^{\infty} l\left(\frac{1}{p+2l-1} - \frac{1}{p+2l+1}\right)\left(\frac{1}{q+2l-1} - \frac{1}{q+2l+1}\right) \\
&= \frac{\sqrt{pq}(-1)^{p+q}}{2}\sum_{l=1}^{\infty}\left[\frac{1}{2(p-q)}\left(\frac{p-1}{p+2l-1} - \frac{q-1}{q+2l-1}\right)\right. \\
&\quad + \frac{1}{2(p-q)}\left(\frac{p+1}{p+2l+1} - \frac{q+1}{q+2l+1}\right) \\
&\quad - \frac{1}{2(p-q+2)}\left(\frac{p+1}{p+2l+1} - \frac{q-1}{q+2l-1}\right) \\
&\quad \left. - \frac{1}{2(p-q-2)}\left(\frac{p-1}{p+2l-1} - \frac{q+1}{q+2l+1}\right)\right]
\end{aligned} \tag{A.2}$$

where use have been made of (A.1). Now we use the definition of the ψ function as an infinite series

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\frac{z}{n(z+n)}$$

which can be written, using $\frac{z}{n(z+n)} = \frac{1}{n} - \frac{1}{z+n}$,

$$\sum_{n=1}^{\infty}\frac{1}{z+n} = -\log\epsilon - \left(\frac{1}{z} + \gamma + \psi(z)\right) \tag{A.3}$$

where ϵ is a regulator for $\epsilon \to 0$: $-\log\epsilon = \sum_{n=1}^{\infty}(1-\epsilon)^n/n$. Of course anything meaningful must not depend on ϵ.

Replacing (A.3) inside (A.2) one finds that all terms cancel out except those containing ψ, so that one gets

$$\begin{aligned}
A_{pq}^2 &= \frac{\sqrt{pq}(-1)^{p+q}}{8}\left[-\left(\frac{p-1}{p-q} - \frac{p-1}{p-q-2}\right)\psi\left(\frac{p-1}{2}\right) + \left(\frac{q-1}{p-q} - \frac{q-1}{p-q+2}\right)\psi\left(\frac{q-1}{2}\right)\right. \\
&\quad \left. -\left(\frac{p+1}{p-q} - \frac{p+1}{p-q+2}\right)\psi\left(\frac{p+1}{2}\right) + \left(\frac{q+1}{p-q} - \frac{q+1}{p-q-2}\right)\psi\left(\frac{q+1}{2}\right)\right] \\
&= \frac{\sqrt{pq}(-1)^{p+q}}{4}\left[\frac{p-1}{p-q-2}\psi\left(\frac{p-1}{2}\right) + \frac{q-1}{p-q+2}\psi\left(\frac{q-1}{2}\right)\right. \\
&\quad \left. - \frac{p+1}{p-q+2}\psi\left(\frac{p+1}{2}\right) - \frac{p+1}{p-q-2}\psi\left(\frac{q+1}{2}\right)\right] \\
&= \frac{\sqrt{pq}(-1)^{p+q}}{2(p-q)}\left[\frac{p+q}{(p-q)^2-4}\psi\left(\frac{p+1}{2}\right) - \frac{p+q}{(p-q)^2-4}\psi\left(\frac{q+1}{2}\right) - \frac{2(p-q)}{(p-q)^2-4}\right]
\end{aligned} \tag{A.4}$$

For C^2 one proceeds in the same way

$$C^2_{pq} = 8\sqrt{pq}\sum_{l=1}^{\infty} l\frac{(-1)^{p+q}}{(p-2l)^2-1)((q-2l)^2-1)}$$

$$= -\frac{\sqrt{pq}(-1)^{p+q}}{2}\left[\left(\frac{p-1}{p-q}-\frac{p-1}{p-q-2}\right)\psi\left(\frac{1+p}{2}\right)-\left(\frac{q-1}{p-q}-\frac{q-1}{p-q+2}\right)\psi\left(\frac{1+q}{2}\right)\right.$$
$$\left.+\left(\frac{p+1}{p-q}-\frac{p+1}{p-q+2}\right)\psi\left(\frac{3+p}{2}\right)-\left(\frac{q+1}{p-q}-\frac{q+1}{p-q-2}\right)\psi\left(\frac{3+q}{2}\right)\right]$$

where use has been made of

$$\psi\left(\frac{1}{2}+z\right)=\psi\left(\frac{1}{2}-z\right)+\pi\tan(\pi\, z) \tag{A.5}$$

This can be reduced to

$$C^2_{pq} = \frac{2\sqrt{pq}(-1)^{p+q}}{(p-q)}\left[\frac{p+q}{(p-q)^2-4}\psi\left(\frac{1-p}{2}\right)-\frac{p+q}{(p-q)^2-4}\psi\left(\frac{1-q}{2}\right)-\frac{2(p-q)}{(p-q)^2-4}\right]$$

Therefore $C^2_{pq} - 4A^2_{pq}$ is the same as in eq.(2.37).

The p, q both odd case can be dealt with in a similar way.

B. The weight 2 and -1 bases

.

A series of bases of weight s were introduced in [24, 25]. They are expressed in terms of generating functionals $f^{(s)}_\kappa(z)$, where s is integer or half–integer. In order to normalize them the quadratic form

$$\langle f|g\rangle = \frac{1}{\pi\,\Gamma(2s-1)}\int_{|z|\leq 1} d^2z\,(1-|z|^2)^{2s-2}\overline{g(z)}\,f(z) \tag{B.1}$$

is used. This leads to the normalized generating functions

$$\tilde{f}^{(s)}_\kappa(z) = \sqrt{A_s(\kappa)}f^{(s)}_\kappa(z) \tag{B.2}$$

where $f^{(s)}_\kappa(z)$ are the generating functions we have used in the text (eigenfunctions of $\mathcal{K}_1$) and

$$A_s(\kappa) = \frac{2^{2s-2}}{\pi}\,\Gamma\left(s+\frac{i\kappa}{2}\right)\Gamma\left(s-\frac{i\kappa}{2}\right), \tag{B.3}$$

which satisfies

$$A_{s+1}(\kappa) = (\kappa^2+4s^2)A_s(\kappa)$$

In [24, 25]

$$\tilde{f}^{(s)}_\kappa(z) \equiv |\kappa, s\rangle(z) \equiv \langle z, s|\kappa, s\rangle = \overline{\langle\kappa, s|z, s\rangle} \tag{B.4}$$

where we have introduced two continuous basis, the κ basis and the z basis, [26]. The normalization (B.3) is such that

$$\langle \kappa, s|\kappa', s\rangle = \delta(\kappa, \kappa') \tag{B.5}$$

Let us also introduce the discrete basis $|n, s\rangle$, such that

$$\langle n, s|z, s\rangle = N_n^{(s)} z^{n-s}, \qquad n \geq s \tag{B.6}$$

The discrete basis satisfies $\langle n, s|m, s\rangle = \delta_{n,m}$, therefore it is the basis of the space of square–summable sequences. Normalizing according to (B.1) one gets

$$N_n^{(s)} = \sqrt{\frac{\Gamma(n+s)}{\Gamma(n-s+1)}} \tag{B.7}$$

The polynomials (B.6) form a complete set of orthonormalized eigenfunctions of L_0. We notice that the normalization constant $N_n^{(s)}$ are singular corresponding to the ghost zero modes (i.e. for $n = 0$, $s = 0$ and $n = -1, 0, 1$ for $s = -1$). In these cases we can use s as a regulator

$$\begin{aligned} &N_0^{(s)} = \frac{1}{\sqrt{s}}, \qquad s \approx 0 \\ &N_1^{(s)} = N_{-1}^{(s)} = \frac{1}{\sqrt{2(s+1)}}, \qquad N_0^{(s)} = \frac{1}{\sqrt{-1-s}}, \qquad s \approx -1 \end{aligned} \tag{B.8}$$

Inserting a discrete basis in (B.4) we get

$$\tilde{f}_\kappa^{(s)}(z) = \langle z, s|\kappa, s\rangle = \sum_{n=s} \langle z, s|n, s\rangle\langle n, s|\kappa, s\rangle = \sum_{n=s} N_n^{(s)} \langle n, s|\kappa, s\rangle z^{n-s} \tag{B.9}$$

Therefore

$$\tilde{V}_n^{(s)}(\kappa) = N_n^{(s)} \langle n, s|\kappa, s\rangle \tag{B.10}$$

Now, after defining

$$\hat{V}_n^{(s)}(\kappa) = \langle n, s|\kappa, s\rangle = \frac{\sqrt{A_s(\kappa)}}{N_n^{(s)}} V_n^{(s)}(\kappa) \tag{B.11}$$

one can prove the orthonormality relation

$$\int_{-\infty}^{\infty} d\kappa \, \hat{V}_n^{(s)}(\kappa)\hat{V}_m^{(s)}(\kappa) = \delta_{n,m} \tag{B.12}$$

And introducing a complete set of discrete states in (B.5) one gets the completeness relation

$$\sum_{n=s} \hat{V}_n^{(s)}(\kappa)\hat{V}_n^{(s)}(\kappa') = \delta(\kappa, \kappa') \tag{B.13}$$

In conclusion, all the spaces $\mathcal{H}^{(s)}$ possess a complete orthonormal basis given by $\hat{V}_n^{(s)}$. However *this is not a basis of eigenfunctions of the matrices G or H*. The eigenfunctions of G or H are given by the $\tilde{V}_n^{(s)}$ or by the corresponding unnormalized ones, $V_n^{(s)}$. What is relevant for the latter is a *biorthogonality* relation.

Let us concentrate on the conjugate cases $s = -1$ and $s = 2$, that is the weight -1 and weight 2 basis of section 3. In an unpublished paper, using the same techniques as in [24], Belov and Lovelace, [27], showed that

$$\int_{-\infty}^{\infty} d\kappa\, \tilde{V}_n^{(-1)}(\kappa)\, \tilde{V}_m^{(2)}(\kappa) = \delta_{n,m}, \qquad n \geq 2 \tag{B.14}$$

that is the two basis are biorthogonal. Moreover we have the completeness relation

$$\sum_{n=2}^{\infty} \tilde{V}_n^{(-1)}(\kappa)\, \tilde{V}_n^{(2)}(\kappa') = \delta(\kappa, \kappa') \tag{B.15}$$

We would like now to give an example of application of these bases which underlies our approach in this paper. Let us consider $c^\dagger A b^\dagger$. It can be written as follows

$$\begin{aligned} c^\dagger A b^\dagger &= \sum_{n=-1,m=2} c_n^\dagger A_{nm} b_m^\dagger = \sum_{n=-1,m=2}^{\infty} \int d\kappa\, d\kappa'\, \tilde{c}(\kappa) \tilde{V}_n^{(-1)}(\kappa) \tilde{A}_{nm} \tilde{V}_m^{(2)}(\kappa')\, b(\kappa') \\ &= \sum_{n=2}^{\infty} \int d\kappa\, d\kappa'\, \tilde{c}(\kappa) \tilde{V}_n^{(-1)}(\kappa) \tilde{A}(\kappa) \tilde{V}_n^{(2)}(\kappa') b(\kappa') = \int d\kappa\, \tilde{c}(\kappa) \tilde{A}(\kappa)\, b(\kappa) \end{aligned} \tag{B.16}$$

where we have introduced

$$(-1)^n\, c_n^\dagger = \int d\kappa\, \tilde{c}(\kappa)\, V_n^{(-1)}(\kappa), \qquad b_n^\dagger = \int d\kappa\, b(\kappa)\, V_n^{(2)}(\kappa) \tag{B.17}$$

and used (B.14).

We can reverse (B.17) again by means of (B.14)

$$\tilde{c}(\kappa) = \sum_{n=2}^{\infty} (-1)^n c_n^\dagger\, V_n^{(2)}(\kappa), \qquad b(\kappa) = \sum_{n=2}^{\infty} b_n^\dagger\, V_n^{(-1)}(\kappa) \tag{B.18}$$

We note that in (B.16) the modes c_a with $a = -1, 0, 1$ are irrelevant. Moreover the eigenfunctions corresponding to these three zero modes are excluded from the completeness relation (B.15). We could qualitatively phrase the reason for that by saying that they do not carry more information about the system than what is already contained in the remaining eigenvectors. They can in fact be expressed in terms of the latter due to the relations

$$\int_{-\infty}^{\infty} d\kappa\, \tilde{V}_{-1}^{(-1)}(\kappa)\, \tilde{f}_\kappa^{(2)}(z) = \frac{z}{(1+z^2)^2} \tag{B.19}$$

$$\int_{-\infty}^{\infty} d\kappa\, \tilde{V}_0^{(-1)}(\kappa)\, \tilde{f}_\kappa^{(2)}(z) = \frac{1}{(1+z^2)} \tag{B.20}$$

$$\int_{-\infty}^{\infty} d\kappa\, \tilde{V}_1^{(-1)}(\kappa)\, \tilde{f}_\kappa^{(2)}(z) = \frac{z}{(1+z^2)} + \frac{z}{(1+z^2)^2} \tag{B.21}$$

For let us write $\tilde{V}_a^{(-1)}(\kappa) = \sum_{n=2}^{\infty} b_{an} \tilde{V}_n^{(-1)}(\kappa)$, for $a = -1, 0, 1$ and plug these expressions in the LHS of (B.19,B.20,B.21). By expanding both sides of the equations in powers of z

and equating coefficients of the same powers we can determine all the b_{an} coefficients (they are the same as the coefficients of the power series expansion in z in rhs of (B.19,B.20,B.21). Therefore the $\tilde{V}_a^{(-1)}(\kappa)$ do not contain additional information with respect to the set of $\tilde{V}_n^{(-1)}(\kappa)$ with $n \geq 2$.

It would seem that, using this result, we can come to the absurd conclusion that $\hat{V}_a^{(-1)}(\kappa)$ can be expanded in terms of $\hat{V}_n^{(-1)}(\kappa)$ with $n \geq 2$. However the true relation is

$$\hat{V}_a^{(-1)}(\kappa) = \sum_{n=2}^{\infty} b_{an} \frac{N_n^{(-1)}}{N_a^{(-1)}} \hat{V}_n^{(-1)}(\kappa) \tag{B.22}$$

But due to the properties (B.8) this equation means nothing but $0 = 0$.

C. Properties of the hypergeometric functions

In this section we collect some properties of the hypergeometic function $F(a, b; c, z) \equiv {}_2F_1(a, b; c; z)$ and other special functions that we need in various derivations of this paper. For their derivation, see for instance [32]. We start with the integral representation of the hypergeometric function

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt\, t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} \tag{C.1}$$

which is valid when $Re(c) > Re(b) > 0$ and $|Arg(1-z)| < \pi$.

Next we need some identities valid for the special case of the argument $z = -1$:

$$F(a, b; 1 + a - b; -1) = 2^{-a} \frac{\Gamma(1 + a - b)\sqrt{\pi}}{\Gamma(1 - b + \frac{1}{2}a)\,\Gamma(\frac{1}{2} + \frac{a}{2})} \tag{C.2}$$

$$F(a, b; b, -1) = 2^{-a} \tag{C.3}$$

and

$$F(a, 1; a + 1; -1) = \frac{a}{2}\left(\psi\left(\frac{1}{2} + \frac{a}{2}\right) - \psi\left(\frac{a}{2}\right)\right) \equiv \frac{a}{2}\,\mathcal{G}(a) \tag{C.4}$$

where ψ is the dilogarithm function and $\mathcal{G}$ is another special function defined by (C.4).

Next we need other raising and lowering parameters relations:

$$F = -\frac{1}{c-a-1}\left(a\,F(a+1) - (c-1)F(c-1)\right) \tag{C.5}$$

and

$$F = -\frac{1}{c-a-b}\left(a(1-z)\,F(a+1) - (c-b)F(b-1)\right) \tag{C.6}$$

In these formulas the arguments are not indicated if they are the obvious ones a, b, c, z.

In order to raise b one can use

$$F(a, b; c; -1) = -\frac{3 + 3b - c - a}{c - b - 1} F(a, b + 1; c; -1) + 2\frac{b+1}{c-b-1} F(a, b + 2; c; -1) \tag{C.7}$$

$$F(a, b; c; -1) = \frac{3c - b - a + 1}{2c} F(a, b; c + 1; -1) + \frac{(c - b + 1)(c - a + 1)}{2c(c+1)} F(a, b; c + 2; -1) \tag{C.8}$$

D. The eigenvalues of $\tilde{A}$ and D^T

D.1 Zero modes of $\mathcal{A}$

Here we would like to prove eq.(5.21). Let us compute

$$\begin{aligned}\sum_{n=2}^{\infty} A_{-1,n} V_n^{(2)}(\kappa) &= \frac{1}{2\pi i} \oint dz \, \frac{e^{\kappa \arctan(z)}}{(1+z^2)^2} \sum_{l=1}^{\infty} \frac{2(-1)^l}{(2l+1)z^{2l}} \\ &= \frac{1}{2\pi i} \oint dz \, \frac{e^{\kappa \arctan(z)}}{(1+z^2)^2} \left(-2 + 2z \, \arctan\left(\frac{1}{z}\right)\right) \end{aligned} \tag{D.1}$$

Similarly

$$\sum_{n=2}^{\infty} A_{0,n} V_n^{(2)}(\kappa) = \frac{1}{2\pi i} \left(z + (1-z^2) \, \arctan\left(\frac{1}{z}\right)\right) \tag{D.2}$$

$$\sum_{n=2}^{\infty} A_{1,n} V_n^{(2)}(\kappa) = \frac{1}{2\pi i} \oint dz \, \frac{e^{\kappa \arctan(z)}}{(1+z^2)^2} \left(-2 + 2z \, \arctan\left(\frac{1}{z}\right)\right) \tag{D.3}$$

Now we use

$$\arctan\left(\frac{1}{z}\right) = \frac{i}{2} \log\left(\frac{iz+1}{iz-1}\right) = -\arctan(z) \pm \frac{\pi}{2} \tag{D.4}$$

It is evident that inserting (D.4) in the RHS's of (D.1,D.2,D.3) we get 0, because there is no pole left at the origin. So eq.(5.21) is justified. The function $\arctan\left(\frac{1}{z}\right)$ has two branch point at $\pm i$, with a cut between them. There are two sheets and the sign reflects the choice of which sheet we choose to do the integration on. Both choices give a vanishing result. The point is that $\arctan\left(\frac{1}{z}\right)$ converge for large z, but of course we can continue it analytically to small z, where we realize that it does not have poles at the origin, so the integrals vanish. In other words summing all the powers $1/z^n$ eliminates these poles at $z = 0$.

Let us also add that we have obtained the same vanishing result (5.21) applying the same technique we used in calculating the eigenvalues in section 5.5.

D.2 The eigenvalue of D^T in the weight 2 basis

Using the same technique as in section5.5 we can calculate the eigenvalue of D^T. Since $V_2^{(2)}(\kappa) = 1$ we have

$$\sum_{n=2}^{\infty} D^T_{2,n} V_n^{(2)}(\kappa) = \mathfrak{c}(\kappa) \, V_2^{(2)}(\kappa) = \mathfrak{c}(\kappa) \tag{D.5}$$

On the other hand

$$D^T_{2,n} = -(-1)^{\frac{n}{2}} \left(\frac{3}{n-1} - \frac{1}{n-3}\right)$$

Therefore

$$\mathfrak{c}(\kappa) = -\sum_{l=1}^{\infty} (-1)^l \left(\frac{3}{2l-1} - \frac{1}{2l-3}\right) V_{2l}^{(2)}(\kappa) = -3F(1) + G(1) \tag{D.6}$$

where

$$F(z) = \sum_{l=1}^{\infty}(-1)^l \frac{1}{2l-1} V_{2l}^{(2)}(\kappa) z^{2l-1} \tag{D.7}$$

$$G(z) = \sum_{l=1}^{\infty}(-1)^l \frac{1}{2l-3} V_{2l}^{(2)}(\kappa) z^{2l-3} \tag{D.8}$$

So

$$\frac{dF}{dz} = \sum_{l=1}^{\infty}(-1)^l V_{2l}^{(2)} z^{2l-2} = -\frac{1}{2}\left(f_\kappa^{(2)}(iz) + f_\kappa^{(2)}(-iz)\right) \tag{D.9}$$

$$\frac{dG}{dz} = \sum_{l=1}^{\infty}(-1)^l V_{2l}^{(2)} z^{2l-4} = -\frac{1}{2z^2}\left(f_\kappa^{(2)}(iz) + f_\kappa^{(2)}(-iz)\right) \tag{D.10}$$

Let us define

$$H(z) = G(z) - 3F(z) \tag{D.11}$$

So that $\mathfrak{c}(\kappa) = H(1)$. Notice that near 0, $H(z) = \frac{1}{z} + \ldots$.

Now

$$\frac{dH}{dz} = \frac{3z^2-1}{z^2(1-z^2)^2}\cosh\left(\kappa \arctan(iz)\right) \tag{D.12}$$

We have to eliminate the singularity at $z = 0$. We use that $V_3^{(2)}(\kappa) = \kappa$. So

$$\sum_{n=2}^{\infty} D_{3,n}^T V_n^{(2)}(\kappa) = \mathfrak{c}(\kappa)\, V_3^{(2)}(\kappa) = \kappa\, \mathfrak{c}(\kappa)$$

On the other hand

$$D_{3,2l+1}^T = -2(-1)^l\left(\frac{2}{2l-1} - \frac{1}{2l-3}\right)$$

Therefore

$$\kappa\mathfrak{c}(\kappa) = -2\sum_{l=1}^{\infty}(-1)^l\left(\frac{2}{2l-1} - \frac{1}{2l-3}\right) V_{2l+1}^{(2)}(\kappa) = 2(G'(1) - 2F'(1)) \tag{D.13}$$

where

$$F'(z) = \sum_{l=1}^{\infty}(-1)^l \frac{1}{2l-1} V_{2l+1}^{(2)}(\kappa) z^{2l-1} \tag{D.14}$$

$$G'(z) = \sum_{l=1}^{\infty}(-1)^l \frac{1}{2l-3} V_{2l+1}^{(2)}(\kappa) z^{2l-3} \tag{D.15}$$

So

$$\frac{dF'}{dz} = \sum_{l=1}^{\infty}(-1)^l V^{(2)}_{2l+1} z^{2l-2} = \frac{i}{2z}\left(f^{(2)}_\kappa(iz) - f^{(2)}_\kappa(-iz)\right) \tag{D.16}$$

$$\frac{dG'}{dz} = \sum_{l=1}^{\infty}(-1)^l V^{(2)}_{2l+1} z^{2l-4} = \frac{i}{2z^3}\left(f^{(2)}_\kappa(iz) - f^{(2)}_\kappa(-iz)\right) \tag{D.17}$$

Let us define

$$H'(z) = G'(z) - 2F'(z) \tag{D.18}$$

We have $\kappa\mathfrak{c}(\kappa) = 2H'(1)$. Near 0 we have $H'(z) = \frac{\kappa}{z} + \dots$. Therefore the combination $\kappa H - H' = \hat{H}$ vanishes at 0.

Now

$$\frac{dH'}{dz} = \frac{i(1-2z^2)}{z^3(1-z^2)^2}\sinh\left(\kappa \arctan(iz)\right)$$

We have

$$\hat{H}(0) = 0, \qquad \hat{H}(1) = \frac{\kappa}{2}\mathfrak{c}(\kappa) \tag{D.19}$$

So,

$$\begin{aligned}\hat{H}(1) = \int_0^1 dz \Bigg\{ & \frac{3z^2-1}{2z^2}\frac{\kappa}{(1-z^2)^2}\left[\left(\frac{1+z}{1-z}\right)^{\frac{i\kappa}{2}} + \left(\frac{1+z}{1-z}\right)^{-\frac{i\kappa}{2}}\right] \\ & - \frac{i}{2\,z^3}\frac{1-2z^2}{(1-z^2)^2}\left[\left(\frac{1+z}{1-z}\right)^{\frac{i\kappa}{2}} - \left(\frac{1+z}{1-z}\right)^{-\frac{i\kappa}{2}}\right]\Bigg\}\end{aligned} \tag{D.20}$$

Setting $\zeta = \frac{i\kappa}{2}$ and using the hypergeometric function $F = {}_2F_1$ we can write formally

$$\begin{aligned}\frac{1}{i}\,\hat{H}(1) = \Gamma(-1-\zeta)\Bigg[& 3\zeta\,\frac{\Gamma(1)}{\Gamma(-\zeta)}\,F(2-\zeta,1;-\zeta;-1) \\ & -\zeta\,\frac{\Gamma(-1)}{\Gamma(-2-\zeta)}\,F(2-\zeta,-1;-2-\zeta;-1) - \frac{\Gamma(-2)}{\Gamma(-3-\zeta)}\,F(2-\zeta,-2;-3-\zeta;-1) \\ & +2\frac{\Gamma(0)}{\Gamma(-1-\zeta)}\,F(2-\zeta,0;-1-\zeta;-1) - (\zeta\to-\zeta)\Bigg]\end{aligned} \tag{D.21}$$

which is evidently ill–defined. The usual integral representation of the hypergeometric function (C.1) is valid when $Re(c) > Re(b) > 0$ and $|Arg(1-z)| < \pi$. The last condition is satisfied, but the former are not. We modify the b parameter by shifting it $b \to b+\beta$ so that (D.21) will become

$$\begin{aligned}\frac{1}{i}\,\hat{H}(1) = \Gamma(-1-\zeta)\Bigg[& 3\zeta\,\frac{\Gamma(1+\beta)}{\Gamma(\beta-\zeta)}\,F(2-\zeta,1+\beta;\beta-\zeta;-1) \\ & -\zeta\,\frac{\Gamma(-1+\beta)}{\Gamma(\beta-2-\zeta)}\,F(2-\zeta,-1+\beta;\beta-2-\zeta;-1) \\ & -\frac{\Gamma(-2+\beta)}{\Gamma(\beta-3-\zeta)}\,F(2-\zeta,-2+\beta;\beta-3-\zeta;-1) \\ & +2\frac{\Gamma(\beta)}{\Gamma(\beta-1-\zeta)}\,F(2-\zeta,\beta;\beta-1-\zeta;-1) - (\zeta\to-\zeta)\Bigg]\end{aligned} \tag{D.22}$$

Since we know the value of $\mathfrak{c}(\kappa)$ up to the sign from the knowledge of $(D^T)^2 - BA$ and $\mathfrak{a}(\kappa)$, we have only to check the sign. Therefore it is enough to evaluate this expression numerically. This can be done for instance with Mathematica. The limit $\beta \to 0$ is a complicated function of ζ, but coincides exactly with $\mathfrak{c}(\kappa)$ we obtain from $((D^T)^2 - AB)(\kappa)$ and $\mathfrak{a}(\kappa)$ for any value of κ, provided we choose the $+$ sign, i.e. we get eq.(5.36). Of course it would be desirable to derive this result analytically.

D.3 The eigenvalue of $\tilde{A}$ in the weight -1 basis

It is important to verify that that the weight -1 basis left–diagonalizes $\tilde{A}$ with the same eigenvalue as in the weight 2 basis. Here by eigenvalue of $\tilde{A}$ we actually mean the eigenvalue of $\tilde{\mathcal{A}}$. We expect that the contribution of the 0 modes is irrelevant as in Appendix D1. In that case we must have

$$\sum_{n=-1}^{\infty} V_n^{(-1)}(\kappa)\tilde{A}_{n,2} = \mathfrak{a}(\kappa)\, V_2^{(-1)}(\kappa) = \frac{1}{6}\kappa(4+\kappa^2)\mathfrak{a}(\kappa)) \tag{D.23}$$

as $V_2^{(-1)}(\kappa) = \frac{1}{6}\kappa(4+\kappa^2)$. For $n = 2l$ we have

$$\tilde{A}_{2l,2} = 4(-1)^l \frac{1}{2l+3} \tag{D.24}$$

Now define

$$F(z) = \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+3} V_{2l}^{(-1)}(\kappa) z^{2l+3} \tag{D.25}$$

so that $\mathfrak{a}(\kappa)\, V_2^{(-1)}(\kappa) = 4F(1)$. On the other hand we have

$$\begin{aligned} \frac{dF}{dz} &= \sum_{l=0}^{\infty} (-1)^l V_{2l}^{(-1)}(\kappa) z^{2l+2} = -\frac{iz}{2}\left(f_\kappa^{(-1)}(iz) - f_\kappa^{(-1)}(-iz)\right) \\ &= -iz(1-z^2)\sinh(k \arctan(iz)) \end{aligned} \tag{D.26}$$

where $f_\kappa^{(-1)}$ is the weight -1 basis generating function. Integrating this equation and noting $F(0) = 0$ we obtain

$$\begin{aligned} F(1) &= -\frac{i}{2}\int_0^1 dzz\left((1+z)^{1+\zeta}(1-z)^{1-\zeta} - (1+z)^{1-\zeta}(1-z)^{1+\zeta}\right) \\ &= -\frac{i}{2}\left(\frac{1}{(3-\zeta)(2-\zeta)}F(-1-\zeta,2;4-\zeta;-1)\right. \\ &\quad \left. -\frac{1}{(3+\zeta)(2+\zeta)}F(-1+\zeta,2;4+\zeta;-1)\right) \end{aligned} \tag{D.27}$$

where $\zeta = \frac{ik}{2}$. Putting everything together we can write

$$\mathfrak{a}(\kappa) = -\frac{12i}{k(4+k^2)}\left(\frac{1}{(3-\zeta)(2-\zeta)}F(-1-\zeta,2;4-\zeta;-1) - (\zeta \to -\zeta)\right) \tag{D.28}$$

We can check numerically that this result is exactly the $\mathfrak{a}(\kappa)$ obtained from the b–basis. This can also be checked analytically by repeatedly applying the equations in Appendix C. We begin by eq.(C.7), which gives

$$\mathfrak{a}(\kappa) = -\frac{12i}{k(4+k^2)}\left(\frac{1}{2(2-\zeta)} + \frac{\zeta}{(3-\zeta)(2-\zeta)}F(-1-\zeta,1;4-\zeta;-1) - (\zeta\to-\zeta)\right) \tag{D.29}$$

where we have used $F(a,0;c;-1) = 1$. Now we can apply eq.(C.8) to obtain

$$\begin{aligned}\mathfrak{a}(\kappa) &= -\frac{12i}{k(4+k^2)}\left[\frac{1}{2(2-\zeta)} + \frac{\zeta}{(2-\zeta)}\left(-\frac{1}{2}F(-1-\zeta,1;2-\zeta;-1)\right.\right.\\ &+ \left.\left.\frac{7-2\zeta}{4(2-\zeta)}F(-1-\zeta,1;3-\zeta;-1)\right) - (\zeta\to-\zeta)\right]\end{aligned} \tag{D.30}$$

Repeating the same thing three times we will finally get

$$\begin{aligned}\mathfrak{a}(\kappa) &= -\frac{12i}{k(4+k^2)}\left[-\frac{\zeta}{(2-\zeta)}\left(\frac{11-16\zeta+2\zeta^2(7-2\zeta)}{6}F(-1-\zeta,1;-1-\zeta;-1)\right.\right.\\ &+ \left.\left.\frac{2\zeta(2-\zeta)(1-\zeta)}{3}F(-1-\zeta,1;-\zeta;-1) + \frac{1}{2(2-\zeta)}\right) - (\zeta\to-\zeta)\right]\end{aligned}$$

Using (C.3) and (C.4) we can write this as

$$\begin{aligned}\mathfrak{a}(\kappa) &= -\frac{12i}{k(4+k^2)}\left[\frac{1}{2(2-\zeta)} - \frac{\zeta}{(2-\zeta)}\left(\frac{11-16\zeta+2\zeta^2(7-2\zeta)}{12}\right.\right.\\ &+ \left.\left.\frac{2\zeta(2-\zeta)(1-\zeta^2)}{6}\left(\psi(-\frac{1+\zeta}{2}) - \psi(-\frac{\zeta}{2})\right)\right) - (\zeta\to-\zeta)\right].\end{aligned} \tag{D.31}$$

Now we can proceed as in section 7.4 and put back $\zeta = \frac{i\kappa}{2}$ to get

$$\mathfrak{a}(\kappa) = \frac{\pi\kappa}{2}\mathrm{csch}(\frac{\pi k}{2}) \tag{D.32}$$

E. The twisted ghost sector

E.1 The weight 0 basis

Just as in the untwisted sector in the twisted one two bases are involved as well. The first is the weight 1 basis we have introduced in section 7. The second basis corresponds to weight 0. For weight 0 ghost field c we have

$$\mathcal{K}_1\, c(z) = [K_1, c(z)] = (1+z^2)\partial c(z)$$

Integrating this equation we get that, if

$$\mathcal{K}_1\, f^{(0)}_\kappa(z) = \kappa\, f^{(0)}_\kappa(z)$$

then

$$f^{(0)}_\kappa(z) = e^{\kappa\arctan(z)} = 1 + \kappa z + \ldots \tag{E.1}$$

We set

$$f_\kappa^{(0)}(z) = \sum_{n=0}^{\infty} V_n^{(0)}(\kappa) z^n \tag{E.2}$$

So, in particular, $V_0^{(0)} = 1$, and

$$V_n^{(0)}(\kappa) = \frac{1}{2\pi i} \oint dz \, \frac{e^{\kappa \arctan(z)}}{z^{n+1}} \tag{E.3}$$

So that, for instance, we can verify that

$$\begin{aligned}
\sum_{n=1} V_n^{(0)}(\kappa)\, G_{nq} &= \frac{1}{2\pi i} \oint dz\, e^{\kappa \arctan(z)} n \sum_{n=0}^{\infty} (\delta_{q,n+1} + \delta_{q+1,n}) \frac{1}{z^{n+1}} \\
&= -\frac{1}{2\pi i} \oint dz\, e^{\kappa \arctan(z)} \frac{d}{dz} \left((1+z^2) \frac{1}{z^{q+1}} \right) \\
&= \kappa\, V_q^{(0)}(\kappa)
\end{aligned} \tag{E.4}$$

and, similarly,

$$\sum_{n=0} H_{pn} V_n^{(0)}(\kappa) = \kappa\, V_p^{(0)}(\kappa)$$

E.2 The eigenvalue of C

The commutation rule $[H, \mathbf{D}]$=0, i.e $[H^T, \mathbf{D}^T] = 0$, when applied to the weight 1 basis, reduces to $[H^T, C] = 0$. Therefore we conclude that C is diagonal in this basis. Since $V_1^{(1)}(\kappa) = 1$ we have

$$\sum_{n=1}^{\infty} C_{1,n} V_n^{(1)}(\kappa) = \mathfrak{c}(\kappa)\, V_1^{(1)}(\kappa) = \mathfrak{c}(\kappa) \tag{E.5}$$

On the other hand

$$C_{1n} = (-1)^{\frac{n+1}{2}} \left(\frac{1}{n-2} - \frac{1}{n} \right)$$

Therefore

$$\mathfrak{c}(\kappa) = -\sum_{l=0}^{\infty} (-1)^l \left(\frac{1}{2l-1} - \frac{1}{2l+1} \right) V_{2l+1}^{(1)}(\kappa) = -F(1) + G(1) \tag{E.6}$$

where

$$F(z) = \sum_{l=0}^{\infty} (-1)^l \frac{1}{2l-1} V_{2l+1}^{(1)}(\kappa) z^{2l-1} \tag{E.7}$$

$$G(z) = \sum_{l=0}^{\infty} (-1)^l \frac{1}{2l+1} V_{2l+1}^{(1)}(\kappa) z^{2l+1} \tag{E.8}$$

So

$$\frac{dF}{dz} = \sum_{l=0}^{\infty}(-1)^l V_{2l+1}^{(1)} z^{2l-2} = \frac{z^{-2}}{2}\left(f_\kappa^{(1)}(iz) + f_\kappa^{(1)}(-iz)\right) \tag{E.9}$$

$$\frac{dG}{dz} = \sum_{l=0}^{\infty}(-1)^l V_{2l+1}^{(1)} z^{2l} = \frac{1}{2}\left(f_\kappa^{(1)}(iz) + f_\kappa^{(1)}(-iz)\right) \tag{E.10}$$

Let us define

$$H(z) = G(z) - F(z) \tag{E.11}$$

So that $\mathfrak{c}(\kappa) = H(1)$. Notice that near 0, $H(z) = \frac{1}{z} + \ldots$.

Now

$$\frac{dH}{dz} = -\frac{1}{z^2}\cosh\left(\kappa \arctan(iz)\right) \tag{E.12}$$

If we try to integrate this function from $z = 0$ to $z = 1$ we will find a result involving $\Gamma(-1)$. Here we will try to fix that.

Since $V_2^{(1)}(\kappa) = \kappa$ we have

$$\sum_{n=1}^{\infty} C_{2,n} V_n^{(1)}(\kappa) = \mathfrak{c}(\kappa)\, V_2^{(1)}(\kappa) = \kappa\, \mathfrak{c}(\kappa) \tag{E.13}$$

On the other hand

$$C_{2,2l} = 2(-1)^l \left(\frac{1}{2l-3} - \frac{1}{2l-1}\right)$$

Therefore

$$\kappa \mathfrak{c}(\kappa) = 2\sum_{l=1}^{\infty}(-1)^l \left(\frac{1}{2l-3} - \frac{1}{2l-1}\right) V_{2l}^{(1)}(\kappa) = 2(F(1) - G(1)) \tag{E.14}$$

where

$$F(z) = \sum_{l=1}^{\infty}(-1)^l \frac{1}{2l-3} V_{2l}^{(1)}(\kappa) z^{2l-3} \tag{E.15}$$

$$G(z) = \sum_{l=1}^{\infty}(-1)^l \frac{1}{2l-1} V_{2l}^{(1)}(\kappa) z^{2l-1} \tag{E.16}$$

So

$$\frac{dF}{dz} = \sum_{l=1}^{\infty}(-1)^l V_{2l}^{(1)} z^{2l-4} = z^{-4}\frac{iz}{2}\left(f_\kappa^{(1)}(iz) - f_\kappa^{(1)}(-iz)\right) \tag{E.17}$$

$$\frac{dG}{dz} = \sum_{l=1}^{\infty}(-1)^l V_{2l}^{(1)} z^{2l-2} = z^{-2}\frac{iz}{2}\left(f_\kappa^{(1)}(iz) - f_\kappa^{(1)}(-iz)\right) \tag{E.18}$$

Let us define

$$H'(z) = 2(F(z) - G(z)) \tag{E.19}$$

We have $\kappa\mathfrak{c}(\kappa) = H'(1)$. Near 0 we have $H'(z) = 2\,\frac{\kappa}{z} + \ldots$. Therefore the combination $2\kappa H - H' = \hat{H}$ vanishes at 0.

Now

$$\frac{dH'}{dz} = 2\frac{i}{z^3}\sinh\left(\kappa\,\arctan(iz)\right)$$

Near $z = 0$ this has the same behavior as $\frac{dH'}{dz}$ in (E.12). We have

$$\hat{H}(0) = 0, \qquad \hat{H}(1) = \kappa\mathfrak{c}(\kappa) \tag{E.20}$$

Define

$$\begin{aligned}\mathfrak{c}(\kappa) &= \frac{1}{\kappa}(2\kappa H(1) - H'(1)) \\ &= \frac{1}{\kappa}\int_0^1 dz\frac{1}{z^3}\left[\sin\left(\frac{\kappa}{2}\,\mathrm{Log}\left(\frac{1+z}{1-z}\right)\right) - \kappa z\cos\left(\frac{\kappa}{2}\,\mathrm{Log}\left(\frac{1+z}{1-z}\right)\right)\right]\end{aligned} \tag{E.21}$$

One can verify that the numerical values of $\mathfrak{c}(\kappa)$ at any value of κ coincides exactly with the $\mathfrak{c}(\kappa)$ in (7.39) and it selects the plus sign.

In the same way we can compute

$$\begin{aligned}&\sum_{n=0}^{\infty} V_n^{(0)}(\kappa)(C^2 - A^2)_{nq} = \\ &= \frac{1}{2\pi i}\oint dz\, e^{\kappa\arctan(z)}\cdot\frac{\pi^2}{2}\sum_{n=0}^{\infty}\left(n^2\delta_{n,q} + \frac{1}{4}n(n+q)(\delta_{n,q+2} + \delta_{q,n+2})\frac{1}{z^{n+1}}\right) \\ &= \frac{\pi^2}{4}\frac{1}{2\pi i}\oint dz\, e^{\kappa\arctan(z)}\frac{d}{dz}\left((1+z^2)\frac{d}{dz}\left((1+z^2)\frac{1}{z^{q+1}}\right)\right) \\ &= \frac{\pi^2\kappa^2}{4}\,V_q^{(0)}(\kappa)\end{aligned} \tag{E.22}$$

References

[1] M. Schnabl, *Analytic solution for tachyon condensation in open string field theory,* Adv. Theor. Math. Phys. **10** (2006) 433 [arXiv:hep-th/0511286].

[2] Y. Okawa, *Comments on Schnabl's analytic solution for tachyon condensation in Witten's open string field theory,* JHEP **0604** (2006) 055 [arXiv:hep-th/0603159].

[3] I. Ellwood and M. Schnabl, *Proof of vanishing cohomology at the tachyon vacuum,* JHEP **0702** (2007) 096 [arXiv:hep-th/0606142].

[4] L. Rastelli and B. Zwiebach, *Solving open string field theory with special projectors,* arXiv:hep-th/0606131.

[5] Y. Okawa, L. Rastelli and B. Zwiebach, *Analytic solutions for tachyon condensation with general projectors,* arXiv:hep-th/0611110.

[6] M. Schnabl, *Comments on marginal deformations in open string field theory,* arXiv:hep-th/0701248.

[7] M. Kiermaier, Y. Okawa, L. Rastelli and B. Zwiebach, *Analytic solutions for marginal deformations in open string field theory,* arXiv:hep-th/0701249.

[8] E. Fuchs and M. Kroyter, *Universal regularization for string field theory,* JHEP **0702** (2007) 038 [arXiv:hep-th/0610298].

[9] E. Fuchs, M. Kroyter and R. Potting, *Marginal deformations in string field theory,* arXiv:0704.2222 [hep-th].

[10] E. Fuchs and M. Kroyter, *On the validity of the solution of string field theory,* JHEP **0605** (2006) 006 [arXiv:hep-th/0603195].

[11] Y. Okawa, *Analytic solutions for marginal deformations in open superstring field theory,* arXiv:0704.0936 [hep-th].

[12] Y. Okawa, *Real analytic solutions for marginal deformations in open superstring field theory,* arXiv:0704.3612 [hep-th].

[13] S. Samuel, *The Ghost Vertex In E. Witten's String Field Theory*, Phys. Lett. B **181** (1986) 255.

[14] E.Cremmer,A.Schwimmer, C.Thorn, *"The vertex function in Witten's formulation of string field theory*, Phys.Lett. **179B** (1986) 57.

[15] D.J.Gross and A.Jevicki, *Operator Formulation of Interacting String Field Theory*, Nucl.Phys. **B283** (1987) 1.

[16] D.J.Gross and A.Jevicki, *Operator Formulation of Interacting String Field Theory, 2*, Nucl.Phys. **B287** (1987) 225.

[17] A.Leclair, M.E.Peskin, C.R.Preitschopf, *String Field Theory on the Conformal Plane. (I) Kinematical Principles*, Nucl.Phys. **B317** (1989) 411.

[18] L.Rastelli, A.Sen and B.Zwiebach, *Half-strings, Projectors, and Multiple D-branes in Vacuum String Field Theory*, JHEP **0111** (2001) 035 [hep-th/0105058].

[19] I.Ya.Aref'eva, D.M.Belov, A.A.Giryavets, A.S.Koshelev, P.B.Medvedev, *Noncommutative field theories and (super)string field theories*, [hep-th/0111208].

[20] L. Bonora, C. Maccaferri, D. Mamone and M. Salizzoni, *"Topics in string field theory,"* arXiv:hep-th/0304270.

[21] V.A.Kostelecky and R.Potting, *Analytical construction of a nonperturbative vacuum for the open bosonic string*, Phys. Rev. D **63** (2001) 046007 [hep-th/0008252].

[22] L.Rastelli, A.Sen and B.Zwiebach, *Star Algebra Spectroscopy*, [hep-th/0111281].

[23] D.Gaiotto, L.Rastelli, A.Sen and B.Zwiebach, *Ghost Structure and Closed Strings in Vacuum String Field Theory*, [hep-th/0111129].

[24] D. M. Belov, *Witten's ghost vertex made simple (bc and bosonized ghosts),* Phys. Rev. D **69** (2004) 126001 [arXiv:hep-th/0308147].

[25] D. M. Belov and C. Lovelace, *Star products made easy,* Phys. Rev. D **68** (2003) 066003 [arXiv:hep-th/0304158].

[26] K.Okuyama, *Ghost Kinetic Operator of Vacuum String Field Theory*, JHEP **0201** (2002) 027 [hep-th/0201015].

[27] D. M. Belov and C. Lovelace, *Unpublished*

[28] V.A.Kostelecky and R.Potting, *Analytical construction of a nonperturbative vacuum for the open bosonic string*, Phys. Rev. D **63** (2001) 046007 [hep-th/0008252].

[29] E.Fuchs, M.Kroyter and A.Marcus, *Squeezed States Projectors in String Field Theory*, JHEP **0209** (2002) 022 [hep-th/0207001].

[30] E. Fuchs and M. Kroyter, *Schnabl's L(0) operator in the continuous basis,* JHEP **0610** (2006) 067 [arXiv:hep-th/0605254].

[31] K. Furuuchi and K. Okuyama, *"Comma vertex and string field algebra,"* JHEP **0109** (2001) 035 [arXiv:hep-th/0107101].

[32] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher Trascendental Functions*, vol.I, McGraw-Hill, New York 1953.