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BORN APPROXIMATION IN THE PROBLEM OF RIGOROUS DERIVATION OF THE GROSS-PITAEVSKII EQUATION

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In this article the very recent problem of the rigorous derivation of the cubic nonlinear Schrödinger equation (the Gross-Pitaevskiĭ equation) is reviewed and discussed, with respect to the role of Born approximation that one ends up with in an appropriate scaling limit. The viewpoint is the one the Mathematical Physics community had at the very intermediate stage of years 2004-2005, when the program of the rigorous derivation of the cubic nonlinear Schrödinger equation was still to be completed, and some crucial points in its understanding were still missing. These are sketched in a conclusive extra section, added to the original version of this work.

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1 Introduction

The first experimental realization of Bose-Einstein condensation ten years ago has fired, inside the Mathematical Physics community, the problem of recovering rigorously a kind of cubic nonlinear Schrödinger equation from the dynamics of an N-particle system.

Bose-Einstein condensation (hereafter BEC) is a macroscopic *quantum* effect discovered by Einstein in 1924 (on the basis of ideas of Bose concerning photons) for thermal equilibrium states of an ideal Bose gas at sufficiently low temperature, and shortly after recognised by London to occur in superfluidity[1]. In 1995 BEC was realized, and the 2001 Nobel Prize in Physics was awarded to Cornell, Ketterle and Wieman "for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms"[2]. In a simplified picture, when a dilute gas of identical bosons is cooled to the point where the de Broglie wavelength is comparable to the distance between particles, the individual wavepackets start to overlap and the undistinguishability of particles becomes crucial: the system undergoes a phase transition and forms a Bose-Einstein condensate, where a macroscopic number of particles occupy the same lowest-energy quantum state[3].

One can model the gas of bosons as a system of N nonrelativistic spinless undistinguishable particles with mass m, interacting with a suitable spherically symmetric non negative (i.e. repulsive) pair potential V_N, and confined by a trapping potential $V_{\rm trap}$ with, say, unit characteristic length. The N-dependence in the pairing accounts for the desired *dilution*: V_N is such that when N is taken large enough its scattering length a_N is nonnegative and scales as N^{-1}, as in the Gross-Pitaevskiĭ scaling limit[A]. With the drastic simplification of zero temperature, one models the system to have the Hamiltonian

$$H_N = \sum_{i=1}^{N} \Big[-\frac{\hbar^2}{2m}\Delta_{x_i} + V_{\mathrm{trap}}(x_i)\Big] + \sum_{1\leqslant i<j\leqslant N} V_N(x_i - x_j) \tag{1}$$

$(x_i \in \mathbb{R}^3)$. At least *formally* one can derive (and experimentally observe) that each particle evolves in the same one-particle state $\phi_t(x)$ obeying the so called *Gross-Pitaevskiĭ equation* (GP from now on)

$$i\hbar\partial_t\phi_t(x) = \Big[-\frac{\hbar^2}{2m}\Delta_x + V_{\mathrm{trap}}(x)\Big]\phi_t(x) + \frac{4\pi\hbar^2 a_0}{m}\,|\phi_t(x)|^2\phi_t(x) \tag{2}$$

namely a *cubic* nonlinear Schrödinger equation (hereafter CNSE) where the constant a_0 appearing in the cubic coupling is the scattering length (assumed positive) of the two-body potential which the particles are assumed to interact with. Thus the 'condensate' is interpreted to have a N-particle wave function $\Psi_{N,t}(x_1,\dots,x_N)$ which is close in some sense to the tensor product of N copies of ϕ_t. In fact such a product $\phi_t(x_1)\cdots\phi(x_N)$ solves the *linear* Scrödinger equation for the N-particle wave function $\Psi_{N,t}$ with Hamiltonian (1) provided that one *formally* substitutes the pair potential $V_N(x)$ with the effective pseudopotential $(4\pi\hbar^2 a_0/m)\delta(x)$, i.e., coupling the particles with a three-dimensional delta interaction. The problem Mathematical Physicists have begun to face is then to recover rigorously (2) as $N\to\infty$.

A first step has been reached with the intensive work [4] mainly by Lieb, Seiringer and Yngvason, who proved around $1998-2001$ that as $N\to\infty$, and with a suitable *low density* scaling in V_N[a], the Hamiltonian (1) reproduces the expected features of the *ground state energy* and *density* of the condensate. In particular such features turn out to be recovered from the stationary solutions of (2) – which provides a rigorous derivation of the '*time-independent*' GP equation. For the following it has to be stressed here that the coupling constant $4\pi\hbar^2 a_0/m$ in front of the cubic term of (2) is the one that correctly fits with the ground state energy of the dilute Bose gas recovered in the low density limit investigated by such authors, according to the experimental evidence. Analogous results are obtained when one models the gas in a thermodynamic box, instead of a trap, (i.e., with periodic or Neumann boundary conditions) enlarging to infinity when $N\to\infty$ in such a way that the low density limit is performed, in the sense that a_N still scales with N. In this case H_N is without the confining term V_{trap} (in all the following discussion this setting will be assumed), and still one recovers rigorously the time independent GP equation as the one that correctly gives the ground state energy and density of the condensate.

To 'complete' this program one should rigorously derive (2) within a *time-dependent* scenario, i.e., proving that in some sense to be precised, the N-particle wave function of the condensate, $\Psi_{N,t}(x_1,\dots,x_N)$, when N is large enough *evolves* almost as the product $\phi_t(x_1)\cdots\phi(x_N)$ of N one-particle solutions of (2). In the following the formalization of this problem is stated, and a *taste* of the very latest results (2000 – 2005) is presented, including the emerging and the role of Born approximation.

[a] see, e.g., Ref. [*A*]

2 The 'time-dependent' scenario

The starting point is the N-particle Schrödinger equation for a Bose gas of N nonrelativistic spinless undistinguishable particles at temperature $T = 0$:

$$\begin{cases} i\partial_t \Psi_{N,t} = H_N \Psi_{N,t} \\ \Psi_{N,t}\Big|_{t=0} \equiv \Psi_N^{\text{in}} \end{cases} , \quad H_N = -\sum_{i=1}^{N} \Delta_{x_i} + \sum_{1\leqslant i<j\leqslant N} V_N(x_i - x_j) \tag{3}$$

(hereafter units $\hbar = 2m = 1$ are assumed). Here $x_i \in \mathbb{R}^3$ and H_N acts on the bosonic sector of $L^2(\mathbb{R}^{3N})$, namely the space $L^2_{\text{sym}}(\mathbb{R}^{3N})$ of measurable and square summable functions that are symmetric w.r.t. permutations of the variables. Analyticity and other conditions on the pair potential V_N are discussed later. Due to the permutation symmetry of H_N, undistinguishability is guaranteed at any time, provided that $\Psi_N^{\text{in}} \in L^2_{\text{sym}}(\mathbb{R}^{3N})$. In particular it is meaningful to assume the *factorized* initial condition

$$\Psi_N^{\text{in}}(x_1, \dots, x_N) = \prod_{i=1}^{N} \phi^{\text{in}}(x_i) \quad , \quad \|\phi^{\text{in}}\|_{L^2(\mathbb{R}^3)} = 1 \tag{4}$$

encoding *uncorrelated* condensation at $t = 0$. [b] Of course such a factorization is immediately destroyed as soon as $t > 0$ (although the undistinguishability is still preserved).

On the other side one has the one-particle problem (for which mathematical well-posedness, and existence and uniqueness of the solution are guaranteed in a suitable sense[6]) for the GP equation (2), rewritten in units $\hbar = 2m = 1$:

$$\begin{cases} i\partial_t \phi_t = -\Delta\phi_t + 8\pi a_0 |\phi_t|^2 \phi_t \\ \phi_{t=0} = \phi^{\text{in}} \,. \end{cases} \tag{5}$$

According to the interpretation 'condensation $\Leftrightarrow$ factorization', one would expect that in the limit $N \to \infty$, and with an appropriate scaling in the pair potential V_N, the N-particle wave function solving (3) with the initial condition (4) is indeed the same as the product of N one-particle solutions of the GP equation (5):

$$\Psi_{N,t}(x_1, \dots, x_N) \approx \prod_{i=1}^{N} \phi_t(x_i) \qquad \text{as } N \to \infty \,. \tag{6}$$

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[b] For realistic initial states, one should take into account *correlations* established in the many-body wave function due to interparticle interactions. For instance in the limit $N \to \infty$ the ground state of (1) has been proven by Lieb and Seiringer to undergo condensation [5], with a ground state wave function close to $W_N(x_1, \dots, x_N)\phi(x_1)\cdots\phi(x_N)$, where ϕ solves the time-independent GP equation with the minimum value of the corresponding energy functional, and the correlation factor W_N is essentially 1 but at a distance $|x_i - x_j| \lesssim N^{-1/3}$. Initial datum (4) deserves interest in this analysis to understand how correlations establish in time starting from an uncorrelated state. [B] Also, the two states $W_N\phi\cdots\phi$ and $\phi\cdots\phi$ of interest in this context can be proven to be close in norm but far in energy. [C]

This amounts to recover the factorization of the vector state of the system at $t > 0$, if the state is factorized at $t = 0$ and N is taken large enough (‘*propagation of chaos*’).[c]

Now, what kind of comparison among $\Psi_{N,t}$ and $(\phi_t)^{\otimes N}$ can be made as $N \to \infty$? Both these states are in $L^2(\mathbb{R}^{3N}) \cong L^2(\mathbb{R}^3)^{\otimes N}$, hence as $N \to \infty$ one is forced to deal with an *infinite* tensor product of Hilbert spaces. Instead of facing the many subtle problems related with such a kind of object, a strategy first developed by Spohn[7] is to fix any k (out of N) particles of the system and investigate the reduced density matrix of such a subsystem as $N \to \infty$, as follows.[d]

- First one equivalently describes the system in the vector state $|\Psi_{N,t}\rangle$ by its density matrix operator $\hat{\rho}_{N,t} = |\Psi_{N,t}\rangle\langle\Psi_{N,t}|$, namely the orthogonal projection onto the linear span of $|\Psi_{N,t}\rangle$. By *density matrix* one conventionally means a positive trace class operator acting on some Hilbert space: here the Hilbert space is $L^2(\mathbb{R}^{3N})$ and by $\mathcal{L}^1(L^2(\mathbb{R}^{3N}))$ one denotes the trace class operators on it[8]; also, $\hat{\rho}_{N,t}$ is positive being an orthogonal projection.

- Then one gets the k-particle *reduced* density matrix operator $\hat{\rho}^{(k)}_{N,t}$ of the subsystem of k particles, that is, a positive element in $\mathcal{L}^1(L^2(\mathbb{R}^{3k}))$, simply tracing out the remaining $N-k$ ones. This is performed as follows: let $\rho_{N,t}$ be the integral kernel of $\hat{\rho}_{N,t}$, i.e., the $L^2(\mathbb{R}^{3N} \times \mathbb{R}^{3N})$ function such that

$$(\hat{\rho}_{N,t}\Phi)(X_N) = \int_{\mathbb{R}^{3N}} \rho_{N,t}(X_N; Y_N)\, \Phi(Y_N)\, \mathrm{d}Y_N \quad , \quad \forall\, \Phi \in L^2(\mathbb{R}^{3N}) \tag{7}$$

  (hereafter $X_N \equiv (x_1, \dots, x_N) \in \mathbb{R}^{3N}$, $Z_N^k \equiv (z_{k+1}, \dots, z_N) \in \mathbb{R}^{3(N-k)}$, etc.); then the integral kernel of $\hat{\rho}^{(k)}_{N,t}$ is defined to be

$$\rho^{(k)}_{N,t}(X_k; Y_k) := \int_{\mathbb{R}^{3(N-k)}} \rho_{N,t}(X_k, Z_N^k; Y_k, Z_N^k)\, \mathrm{d}Z_N^k\,. \tag{8}$$

- This way the knowledge of $\Psi_{N,t}$ translates into the knowledge of the N reduced density matrices $\rho^{(1)}_{N,t}, \dots, \rho^{(N)}_{N,t}$, where of course $\rho^{(N)}_{N,t} = \rho_{N,t}$. The idea is then to fix any k and to study the sequence $\{\rho^{(k)}_{N,t}\}_N$ as $N \to \infty$. This is a sequence of density matrices all in $\mathcal{L}^1(L^2(\mathbb{R}^{3k}))$ and to recover the interpretation of the system as a condensate with each particle evolving in the same GP state, in the limit $N \to \infty$ $\rho^{(k)}_{N,t}$ is expected in some sense to take the form of the k-particle reduced density matrix of a system evolving in the factorized N-particle state $(\phi_t)^{\otimes N}$, ϕ_t being the solution of (5). Now, with the same procedure as above starting from the

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[c] Actually propagation of chaos, meant as persistence in time of totally factorized states, is a notion set at the level of many-body wave functions, whereas the notion of condensation is set at the level of marginals. A totally factorized many-body state has clearly condensation, but realistic condensed states build some correlations in, due to the interparticle interaction. What is factorized, in the limit, is the one-body reduced density matrix (see Ref. [*C*]).

[d] See also Ref. [*C*] for a general treatment of this subject.

factorized $(\phi_t)^{\otimes N}$, the k-particle reduced density matrix turns out to be nothing but the tensor product of k orthogonal projections onto $|\phi_t\rangle$, namely it has integral kernel $\prod_{i=1}^{k} \phi_t(x_i)\overline{\phi_t}(y_i)$. Hence the goal is to prove $\forall\, k = 1, 2, \dots$ and $\forall\, t \geqslant 0$

$$\rho_{N,t}^{(k)}(X_k; Y_k) \xrightarrow[N\to\infty]{\text{in some sense}} \prod_{i=1}^{k} \phi_t(x_i)\overline{\phi_t}(y_i)\,. \tag{9}$$

A key point for the well-posedness, existence and uniqueness of the limit (9) is the choice of the scaling in the pair potential V_N. It is of interest to allow that its strength, namely its scattering length a_N, and its range r_N [9] scale possibly with different speeds. The choice

$$V_N(x) = N^{3\gamma-1} V(N^\gamma x) \quad , \quad 0 < \gamma \leqslant 1 \tag{10}$$

with a positive, smooth, compactly supported and radial $V : \mathbb{R}^3 \to \mathbb{R}^+$, guarantees indeed $a_N = \mathcal{O}(N^{-1})$, $r_N = \mathcal{O}(N^{-\gamma})$. The one-dimensional counterpart

$$V_N(x) = N^{\gamma-1} V(N^\gamma x) \quad , \quad V : \mathbb{R} \to \mathbb{R}^+ \tag{11}$$

is physically meaningful as well, in spite of its simplicity, being used to describe boson gases in elongated traps and the so called cigar-shaped BEC [10]. A further constraint is to choose the unscaled potential V in such a way that its *positive* scattering length is that constant a_0 appearing in the GP equation (2), (5): this is to agree with the scaling of the above mentioned time independent analysis.

Moreover, V is assumed to be non negative, smooth, compactly supported, spherically symmetric, and small enough when measured in terms of the dimensionless quantity $\frac{\alpha\hbar^2}{2m}$, with

$$\alpha = \sup_x |x|^2 V(x) + \int_{\mathbb{R}^3} \frac{V(x)}{|x|} \mathrm{d}x\,. \tag{12}$$

As $\Psi_{N,t}$ evolves according to the Schrödinger evolution (3), the $\rho_{N,t}^{(k)}$'s satisfy the evolutionary hierarchy (with $\mathcal{V}_N \equiv N\,V_N$ and $k = 1, \dots, N-1$)

$$\begin{aligned} i\partial_t \rho_{N,t}^{(k)}(X_k; Y_k) \;=&\; \sum_{i=1}^{k} \big(-\Delta_{x_i} + \Delta_{y_i} \big)\, \rho_{N,t}^{(k)}(X_k; Y_k) \\ &+ \frac{1}{N} \sum_{1\leqslant i<j\leqslant k} \big[\mathcal{V}_N(x_i - x_j) - \mathcal{V}_N(y_i - y_j) \big]\, \rho_{N,t}^{(k)}(X_k; Y_k) \\ &+ \Big(1 - \frac{k}{N}\Big) \sum_{i=1}^{k} \int_{\mathbb{R}^3} \big[\mathcal{V}_N(x_i - z) - \mathcal{V}_N(y_i - z) \big]\, \rho_{N,t}^{(k+1)}(X_k, z; Y_k, z)\, \mathrm{d}z \end{aligned} \tag{13}$$

usually called the *(finite) BBGKY hierarchy* – in fact it resembles the BBGKY hierarchy (Bogolubov-Born-Green-Kirkwood-Yvon) of equations satisfied by the k-particle probability densities in the classical Kinetic Theory as a consequence of the Liouville equations [11]. Now one has in mind a limit $N \to \infty$. Set $b_0 := \int \mathrm{d}x V(x) = \|V\|_{L^1}$, so that $\mathcal{V}_N(x) \to b_0 \delta(x)$ in the distributional sense as $N \to \infty$. Also, one can think that $\rho_{N,t}^{(k)}$ converges to an object $\rho_t^{(k)}$ that is expected to be the k-particle reduced density matrix of a factorized system with an infinite number of particles, so that the hierarchy (13) formally[e] takes the form of the ∞-*BBGKY* or *GP hierarchy*

$$
\begin{aligned}
i\partial_t \rho_t^{(k)}(X_k;Y_k) \;&=\; \sum_{i=1}^{k} \big(-\Delta_{x_i}+\Delta_{y_i}\big)\, \rho_t^{(k)}(X_k;Y_k) \\
&\quad + b_0 \sum_{i=1}^{k} \int_{\mathbb{R}^3} \big[\,\delta(x_i-z)-\delta(y_i-z)\big]\; \rho_t^{(k+1)}(X_k,z;Y_k,z)\,\mathrm{d}z
\end{aligned}
\tag{14}
$$

for any $k=1,2,\dots$. It is straightforward to check that the family of factorized density matrices $\rho_t^{(k)}(X_k;Y_k)=\prod_{i=1}^{k}\phi_t(x_i)\overline{\phi_t}(y_i)$ is a solution of (14) if and only if ϕ_t is a solution of the CNSE

$$
i\partial_t\phi_t = -\Delta\phi_t + b_0|\phi_t|^2\phi_t\,. \tag{15}
$$

This is the GP equation (5) except for the 'wrong' coupling b_0, instead of $8\pi a_0$. Modulo such a wrong coupling, the program can be morally completed: N-body linear dynamics $\rightarrow$ k-particle reduced density matrix $\rightarrow$ finite hierarchy $\rightarrow$ infinite hierarchy $\rightarrow$ factorized solution $\rightarrow$ one-particle nonlinear Schrödinger equation.

Thus one ends up with a two-fold problem: the *convergence* of a solution $\rho_{N,t}^{(k)}$ of (13) to a solution $\rho_t^{(k)}$ of (14) as $N\to\infty$, and the *uniqueness* of such a solution, i.e., the factorized one, so that eventually $\rho_{N,t}^{(k)}(X_k;Y_k)\xrightarrow{N\to\infty}\rho_t^{(k)}(X_k;Y_k)=\prod_{i=1}^{k}\phi_t(x_i)\overline{\phi_t}(y_i)$. This issue was split into these two tasks first by Bardos, Golse and Mauser[12] in 2000, where in the current notations they worked in the pure mean-field case $\gamma=0$ (the program with $\gamma=0$ has been subsequently completed, even when V is the Coulomb potential, together with Erdős and Yau[13]; the emerging one-particle nonlinear Schrödinger equation is the Hartree one, with nonlinearity $V*|\phi_t|^2$).

The other issue within this scenario is to understand why this way one recovers the 'wrong' CNSE, for the coupling b_0 does not give the correct ground state energy and density asymptotics of the condensate.

The up-to-date results along these directions are collected and briefly discussed in the next section.

3 Results and discussion

REGIME $0<\gamma<1$.

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[e]The limit is formal in the following twofold sense. First, in plugging the ansatz $\rho_{N,t}^{(k)}\to\rho_t^{(k)}$ in. Then, in managing each of the three terms in the r.h.s. of (13). In particular, the second one, that is, the one accounting for the interaction among the first k particles, is assumed to be of order $\mathcal{O}(N^{-1})$ and hence to vanish. It is the formal treatment of the third term that we want to stress for its crucial relevance in the whole mainstream of the rigorous derivation of the GP equation. The emergence of the b_0 coupling in the limiting (14) is due to neglecting any short-scale correlation in the marginal $\rho_{N,t}^{(k)}$, as if it was $\rho_{N,t}^{(k)}\approx|\phi_t\rangle\langle\phi_t|^{\otimes k}$. Only when Erdős, Schlein and Yau realized that suitable short scale correlations had to be plugged into $\gamma_N^{(k)}$, limit $N\to\infty$ finally led to the correct infinite hierarchy, that is, the hierarchy with the right coupling constant (see Section 3 of Ref. [*D*]).

In this regime $a_N = \mathcal{O}(N^{-1}) \ll r_N = \mathcal{O}(N^{-\gamma}) \ll 1$: range and scattering length simultaneously shrink to zero, but the short-range limit is performed more slowly than the mean-field one (1 is by construction the spatial variation of the density of the gas). In the three-dimensional setting, Erdős, Schlein and Yau[14] proved in 2005 the following

Theorem. *Fix* $\phi^{\mathrm{in}} \in H^2(\mathbb{R}^3)$ *with* $\|\phi^{\mathrm{in}}\|_{L^2(\mathbb{R}^3)} = 1$*, and let* H_N *be the Hamiltonian (3) with pair potential (10),* but only *with* $0 < \gamma < 1/2$*. Let* $\Psi_{N,t}$ *be the* N*-particle vector state solving the Schrödinger equation (3) with the factorized initial condition (4) and let* $\rho^{(k)}_{N,t}$*'s be the (integral kernels of the) reduced density matrices associated to* $\Psi_{N,t}$*, as defined in (8). On the other side let* ϕ_t *be the solution of the CNSE (15) with initial condition* ϕ^{in}*. Then* $\forall\, t \in \mathbb{R}$ *and any integer* $k \geqslant 1$

$$\rho^{(k)}_{N,t} \xrightarrow[N\to\infty]{\text{weakly-}* \text{ in } \mathcal{L}^1(L^2(\mathbb{R}^{3k}))} \rho^{(k)}_t\,, \tag{16}$$

where

$$\rho^{(k)}_t(X_k;Y_k) = \prod_{i=1}^{k} \phi_t(x_i)\overline{\phi_t}(y_i)\,. \tag{17}$$

the weakly-$*$ *convergence in the trace-class sense*[8]*, that is, for every compact operator* K *acting on* $L^2(\mathbb{R}^{3k})$ *one has* $\mathrm{Tr}\big(\rho^{(k)}_{N,t}\,K\big) \xrightarrow{N\to\infty} \mathrm{Tr}\big(\rho^{(k)}_t K\big)$.

Actually this result essentially holds true in 1 and 2 dimensions as well. A key point in the proof is to get some a-priori bounds on $\rho^{(k)}_{N,t}$ of the form

$$\forall\, k = 1,2,\dots: \quad \text{a suitably regularized trace of } \rho^{(k)}_{N,t} \leqslant (\text{constant})^k \tag{18}$$

for sufficiently large N and uniformly in t, in order to bound similarly any limit point $\rho^{(k)}_t$. Because of the difficulty of extending such bounds beyond $\gamma = 1/2$, the case $\gamma \geqslant 1/2$ is still open, although the same result is expected to hold. In the one-dimensional setting it is easier to get (18) in the whole regime $0 < \gamma < 1$, and indeed Adami, Golse and Teta obtained in 2005 a similar result[15] (with an additional 'growth condition' on the expectation values of $(H_N)^k$ in the initial state, which is automatically satisfied in the theorem above), but at the price of a much weaker topology for the limit (9), namely the weakly-$*$ convergence in the Hilbert-Schmidt sense, averaged in time, instead of the trace-class sense (16).

What has to be stressed here is that in this regime one ends up with the one-particle CNSE (15) instead of the expected GP equation (5). In fact the associated infinite BBGKY hierarchy is (14), with the coupling b_0. If such coupling was $8\pi a_0$ then a trivial solution of the hierarchy would be $\rho^{(k)}_t(X_k;Y_k) = \prod_{i=1}^k \phi_t(x_i)\overline{\phi_t}(y_i)$ with ϕ_t solution of the GP equation (5). Then, to recover rigorously the GP equation, one needs to end up with an ∞-hierarchy with the coupling $8\pi a_0$, which is *not* the case, as long as $0 < \gamma < 1$.

REGIME $\gamma = 1$. MODIFIED DYNAMICS

Here $a_N \sim r_N = \mathcal{O}(N^{-1}) \ll 1$: the pair potential $N^2\,V(Nx)$ is so localized that its range and scattering length are comparable – as in the particular case of the hard-core interaction. Nevertheless the ∞-hierarchy one formally ends up with still has the coupling b_0.

It is recognised that $b_0/8\pi$ is the first *Born approximation* to the scattering length a_0 of the unscaled potential V, and

$$8\pi a_0 \leqslant b_0 = \int_{\mathbb{R}^3} \mathrm{d}x\, V(x)\;. \tag{19}$$

Furthermore [14]

$$\lim_{N\to\infty} N a_N = \begin{cases} b_0/8\pi & \text{if } 0<\gamma<1\,,\\ a_0 & \text{if } \gamma = 1\,. \end{cases} \tag{20}$$

To go beyond the Born approximation, an understanding of the *short scale correlation of the ground state* is needed: Erdős, Schlein and Yau investigated this in a work [16] of 2004, and exploited the possibility to recover the GP equation in the scaling limit $\gamma = 1$, from a dynamics conveniently modified to take into account these short scale correlations. In this framework H_N is replaced by a modified Hamiltonian $\widetilde{H}_N$ where the pair interaction is suppressed whenever three (or any greater fixed number of) particles come close together in a very short distance. The modified $\widetilde{H}_N$ is *approximately* of the form

$$\widetilde{H}_N = -\sum_{i=1}^{N}\Delta_{x_i} + \sum_{1\leqslant i<j\leqslant N}\Big[N^2 V\big(N(x_i-x_j)\big)\cdot\prod_{i\neq j,k}\mathbf{1}\big(|x_i-x_k|\geqslant \ell\big)\Big]\,. \tag{21}$$

Here $x\mapsto \mathbf{1}(|x|\geqslant\ell)$ is the characteristic function of the set $\{x\in\mathbb{R}^3\,:\,|x|\geqslant\ell\}$ and ℓ denotes a distance much smaller than the typical interparticle distance, say $\ell = N^{-1/3-\eta}$ for some $\eta > 0$ (details in section 2 of Ref. [16]). After the precise definition, the difference $H_N - \widetilde{H}_N$ turns out to be exponentially small unless three (or any greater fixed number of) particles are closer than ℓ to each other – although the expected little effect of this modification on the dynamics has not been controlled rigorously. This eventually allows to prove that the ∞-hierarchy one ends up with in the formal limit $N\to\infty$ takes the form (14) with $8\pi a_0$ instead of its Born approximation b_0.

In 2004 Erdős, Schlein and Yau's work[16] the convergence task is rigorously established: however the apriori estimate obtained there is not strong enough to apply the uniqueness theorem of 2005 Erdős, Schlein and Yau's work[14], so uniqueness is recognised to be a major conceptual open problem. Furthermore to complete the project for $\gamma = 1$ one still has to remove the cutoff (21).

In conclusion, in the up-to-date knowledge *either* one can recover rigorously a CNSE in a mean-field + a slower short-range limit ($0<\gamma<1$), but with the price of missing the expected scattering length, recovering only its Born approximation, *or* one can prove a partial result towards the correct GP equation (convergence, not uniqueness yet), still without being able to remove the cutoff of the dynamics when many particles are in a very small physically unlikely region. The last one being the major conceptual open problem and challenge for the next future.

4 Extras. How things went on.

A satisfactory and organic comprehension of the problem has come only in 2006, when Erdős, Schlein, and Yau[D] completed the program of deriving the expected time-dependent

CNSE within the aforementioned technique of a joined control of ”convergence + uniqueness” (see also Ref. [*B*,*E*]).

The correct limiting hierarchy (i.e., with the coupling in agreement with experimental data) has been shown to emerge by making an appropriate ansatz on the short-scale correlations at the level of marginals *before* the limit $N \to \infty$.

For scalings with $0 < \gamma < 1$, the interaction potential is eventually a δ-function on a much larger scale than a_N, and the outcoming CNSE (15) encodes the many-body effects into a nonlinear on-site self-interaction of a complex order parameter ϕ_t, with a coupling that accounts only for the semiclassical approximation of the two-body process.

For a true GP scaling $\gamma = 1$, one ends up with the GP equation (5) containing the full scattering length of the two-body process, being the interaction potential eventually a δ-function on the same scale as a_N.

In this second case, two classes of many-body condensed states at time $t = 0$ are proven to be good initial data for the *time-stability* of condensation: the totally factorized one $\phi^{\otimes N}$, and the state $W_N\, \phi \cdots \phi$ with ground-state-like correlations (see Ref. [*B*,*D*]).

For a review of the physical and mathematical meaning of the scalings as $N \to \infty$ see Ref. [*A*]. For a general discussion of the scenario in terms of reduced density matrices see Ref. [*C*]. For a unified and strengthened version of the topology under which convergence $\rho^{(k)}_{N,t} \to |\phi_t\rangle\langle\phi_t|^{\otimes k}$ is controlled, both in the analysis of Erdős, Schlein, and Yau and of Adami, Golse, and Teta, see Ref. [*F*].

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