Some properties of generating functions obtained by Amann-Conley-Zehnder reduction

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Abstract

Consider an initial Lagrangian submanifold $\Lambda_0 \subset T^*\mathbb{R}^n$, that admits a global generating function, and a Hamiltonian isotopy Φ_H^t. Then, we provide a global generating function for the Lagrangian submanifold $\Lambda_t = \Phi_H^t(\Lambda_0)$ realized by applying the so-called Amann-Conley-Zehnder reduction. When Λ_0 is the zero-section, we study in some details the asymptotic behavior of such generating functions and we give an approximation result.

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Introduction

We assume that $\Lambda_0 \subset T^*\mathbb{R}^n$ is a Lagrangian submanifold that admits a global generating function, say F_0. Let us take the Hamiltonian isotopy $\Phi_H^s : T^*\mathbb{R}^n \longrightarrow T^*\mathbb{R}^n$ ($s \in [0,t]$), generated by a Hamiltonian function $H = H(s,x,p) \in \mathcal{C}^2$. We suppose that H has bounded second derivatives with respect to (x,p): $||\nabla^2 H||_\infty \leq C$ for some positive constant C. We are interested in describing the final Lagrangian submanifold Λ_t by means of a global parameterization, where $\Lambda_t = \Phi_H^t(\Lambda_0)$.

The subject was widely and successfully treated in the literature with different points of view and in different contexts (see for instance [1], [6], [16], [4]).

We recall that, in order to face such a problem, there exist at least two approaches. The first one is the "broken geodesics" method introduced by M. Chaperon in [5] (cf. also [12], [13] and [6] for further developments and references). The second one is obtained by the so-called Amann-Conley-Zehnder (briefly A-C-Z) reduction. Introduced by H. Amann, C. Conley and E. Zehnder in [2] and [8], it transforms an infinite dimensional variational problem involving the Action Functional into a finite dimensional one (cf. also [7]). This kind of reduction method was applied by C. Viterbo in order to find periodic solutions by studying critical points of the Action Functional; furthermore, he constructs a generating function for $\Lambda_t = \Phi_H^t(\Lambda_0)$ when Λ_0 is the zero-section of $T^*\mathbb{R}^n$ (see [15], [16] and cf. also the book [1]).

Successively, F. Cardin in [4] treats the case when Λ_0 is the graph of the gradient of a differentiable function and uses this approach to provide a global generating function for geometrical solutions of time-depending Hamilton-Jacobi equations (cf. also [3] for applications to some optimal control problems).

The first issue of the present paper is that A-C-Z machinery can be applied for any initial Lagrangian submanifold $\Lambda_0 \subset T^*\mathbb{R}^n$ (that admits a global generating function F_0), getting, in such a way, a global generating function F_t for $\Lambda_t = \Phi_H^t(\Lambda_0)$.

Moreover, letting Λ_0 be the zero-section, we analyze the asymptotic behavior of the generating function F_t obtained by the A-C-Z reduction. In particular we focus on the situation when F_t is quadratic at infinity, namely it behaves as a non degenerate quadratic form with respect to the auxiliary variables. In order to consider a bigger class of generating families we also introduce the notion of weakly quadratic at infinity generating function (that occurs when the associated quadratic form might be degenerate). Furthermore, we provide another proof of the so-called Sikorav-Chekanov Theorem (in the case Λ_0 is the zero-section). Finally, we give an approximation result.

The present paper is organized as follows. Section 1 is devoted to some preliminaries. In section 2 we treat the general construction of generating functions via A-C-Z reduction; while in section 3 we deal with their properties.

1 Preliminaries

By Λ we denote a Lagrangian submanifold of $T^*\mathbb{R}^n$. Let us consider a vector bundle over $\mathbb{R}^n$ $\pi : B \longrightarrow \mathbb{R}^n$ where $B = \mathbb{R}^n \times \mathbb{R}^k$ (we can restrict to the special case of product bundle):

$$\begin{array}{rrcl} \pi: & B = \mathbb{R}^n \times \mathbb{R}^k & \longrightarrow & \mathbb{R}^n \\ & (x,v) & \longmapsto & x. \end{array}$$

In such a case, x and v are called *principal* and *auxiliary* coordinate, respectively. Given $F : B \longrightarrow \mathbb{R}$ (regular enough), we can define the so-called "critical set" $\Sigma_F \subset B$ associated to the couple (π, F), or simply associated to F, that is determined by the points of vertical stationary of F:

$$\Sigma_F := \left\{ (x,v) \in B \mid \frac{\partial F}{\partial v}(x,v) = 0 \right\}.$$

The function F is called a (global) *generating function* for Λ if Λ is the image of Σ_F by means of the map:

$$i_F : (x,v) \longmapsto \left(x, \frac{\partial F}{\partial x}(x,v) \right) \qquad \forall (x,u) \in \Sigma_F.$$

If $F \in \mathcal{C}^2(B, \mathbb{R})$ is a function such that Σ_F is a submanifold of B, i.e., $d(\frac{\partial F}{\partial v})$ is injective, then we say F to be a Morse family and Σ_F to be the "critical manifold" of F.

We recall that a characterization of Lagrangian submanifolds of the cotangent bundle by means of local Morse families is provided by the Theorem of Maslov-Hörmander (see Weinstein [18]).

Now, we introduce some important notions concerning the asymptotic behavior of generating functions, that play a crucial role in studying critical points (cf. [14] or [6]).

Definition 1.1 *A generating function*

$$\begin{array}{rrcl} F: & \mathbb{R}^n \times \mathbb{R}^k & \longrightarrow & \mathbb{R} \\ & (x,v) & \longmapsto & F(x,v) \end{array}$$

is called quadratic at infinity (briefly GQI) if there exists a map $Q : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}$ such that Q is a non-degenerate quadratic form when restricted to the fibers of π and the function $(x,v) \mapsto \partial_v F(x,v) - \partial_v Q(x,v)$ is bounded.

The generating function F is called *special* when the associated quadratic form Q does not depend on the first (principal) coordinate x. Moreover, if $F = Q$ out of a compact set, then F is said to be *exactly* quadratic at infinity.

We recall that, in fact, any GQI F is equivalent to an exactly GQI by means of the following lemma (see for instance [14]).

Lemma 1.2 *Suppose that F is GQI, then there exists a fibered diffeomorphism $\Phi : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n \times \mathbb{R}^k$ such that*

i) Φ restricted to some compact subset is the identity;

ii) $F \circ \Phi$ is exactly quadratic at infinity.

In some cases, the generating function F_t obtained via A-C-Z reduction might fail to be GQI.

The following example shows how F_t could change property becoming degenerate or non-degenerate at some x_t, varying the final time t.

Example. (cf. also [3]) Let us take as starting Lagrangian submanifold the zero-section $\Lambda_0 = \mathbb{R}\times\{0\} \subset T^*\mathbb{R} \cong \mathbb{R}^2$ and consider the Hamiltonian $H(x,p) = \frac{1}{2}(x^2+p^2)$. For $t = \frac{\pi}{2}$, it is easy to see that at $x_t = 0$ F_t turns out to be (exactly) a degenerate quadratic form with respect to auxiliary variables. This is no more true, for instance, if $0 < t < \frac{\pi}{2}$: in such cases F_t becomes QIF at any x_t.

In order to consider a huger class of generating functions, we also introduce the following notion (cf. [3]).

Definition 1.3 *A generating function*

$$F: \begin{array}{ccc} \mathbb{R}^n \times \mathbb{R}^k & \longrightarrow & \mathbb{R} \\ (x,v) & \longmapsto & F(x,v) \end{array}$$

is called weakly quadratic at infinity (GWQI) if there exists a (any) quadratic form $Q : B \longrightarrow \mathbb{R}$ *such that the function* $(x,v) \mapsto \partial_v F(x,v) - \partial_v Q(x,v)$ *is bounded.*

2 Construction of generating functions via A-C-Z reduction

Let $F_0 = F_0(x,u)$ be a generating function for a Lagrangian submanifold $\Lambda_0 \subset T^*\mathbb{R}^n$ ($(x,u) \in \mathbb{R}^n \times \mathbb{R}^{k_0}$), namely,

$$\Lambda_0 = \left\{ (x_0,p_0) \in T^*\mathbb{R}^n : \; p_0 = \frac{\partial F_0}{\partial x}(x_0,u^*), \quad 0 = \frac{\partial F_0}{\partial u}(x_0,u^*) \quad \exists\ u^* \in \mathbb{R}^{k_0} \right\}.$$

The goal of this section is to give a detailed and auto consistent (as much as possible) construction of the generating function for $\Lambda_t = \Phi_H^t(\Lambda_0)$ obtained by the Amann-Conley-Zehnder reduction. We use a slight generalization of the well-known Viterbo's scheme to any initial Λ_0 (we refer the reader to [15], [16] and [1] for Λ_0 equal to zero-section in $T^*\mathbb{R}^n$ and to [4] for $\Lambda_0 = \mathrm{Graph}(\nabla\Psi)$ where Ψ is a differentiable function). Therefore, we write the necessary modifications (where they occur) from the original construction, while we refer the reader to the literature when the proofs are quite standard.

We recall that $H = H(s,x,p) \in \mathcal{C}^2$ and $||\nabla^2 H||_\infty \le C < +\infty$.

To this aim, let us consider the functional

$$(1) \qquad S_t(u,\gamma(\cdot)) = F_0(x(0),u) + \int_0^t [p(s)\dot{x}(s) - H(s,x(s),p(s))]\, ds$$

where $\gamma(\cdot) = (x(\cdot),p(\cdot)) : [0,t] \longrightarrow \mathbb{R}^{2n}$ is a curve in the Sobolev space $W := W^{1,2}([0,t],\mathbb{R}^{2n})$. First, notice that, thanks to Sobolev Inequalities, the curve $\gamma(\cdot) \in W$

is continuous. Moreover, we get a natural fibration over $\mathbb{R}^n$:

$$
\begin{array}{rlcl}
\pi: & W & \longrightarrow & \mathbb{R}^n \\
 & \gamma(\cdot) & \longmapsto & \pi(\gamma(\cdot)) = x_t
\end{array}
\tag{2}
$$

where x_t is the final point of the curve $x(\cdot)$. Indeed, a structure of vector space on the fibers $\pi^{-1}(x_t)$ with $x_t \in \mathbb{R}^n$ is provided by the space of derivatives of the curves $\gamma(\cdot) = (x(\cdot), p(\cdot))$ times the space of initial momenta p_0. This is expressed by means of the following bijection:

$$
\begin{array}{rlcl}
g: & \mathbb{R}^n \times \mathbb{R}^n \times L^2 & \longrightarrow & W \\
 & (x_t, p_0, \phi) & \longmapsto & g\big(x_t, p_0, (\phi_x, \phi_p)\big)(\cdot),
\end{array}
\tag{3}
$$

where

$$
\begin{array}{rlcl}
g\big(x_t, p_0, \phi\big): & [0,t] & \longrightarrow & \mathbb{R}^{2n} \\
 & s & \longmapsto & \Big(x_t - \int_s^t \phi_x(\tau)\, d\tau, p_0 + \int_0^s \phi_p(\tau)\, d\tau\Big),
\end{array}
\tag{4}
$$

$L^2 := L^2\big((0,t), \mathbb{R}^{2n}\big)$ and $\phi = (\phi_x, \phi_p) \in L^2$. One can immediately prove that g is injective and surjective. Indeed, roughly speaking, once we fix the final point x_t, and the initial momentum p_0, the (x,p)-components of $\gamma(\cdot)$, $(x(\cdot), p(\cdot))$, are provided by integrating the velocities $(\phi_x, \phi_p) \in L^2$ as in (4) above, obtaining, in such a way, a (only one) curve $\gamma(\cdot) \in W$. Vice versa, in order to get g^{-1}, given $\gamma(\cdot) = (x(\cdot), p(\cdot)) \in W$, we take $x_t = x(t)$, $p_0 = p(0)$ and $\phi(\cdot) = (\phi_x(\cdot), \phi_p(\cdot)) = (\dot{x}(\cdot), \dot{p}(\cdot))$.

Remark 2.1 *The second term on the right hand side of (1) is the so-called Action Functional related to the Hamiltonian H. We will denote it by $A_t : W \longrightarrow \mathbb{R}$:*

$$
A_t(\gamma(\cdot)) := \int_0^t [p(s)\dot{x}(s) - H(s, \gamma(s))]\, ds.
$$

The functional S_t

$$
\begin{array}{rlcl}
S_t: & \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{k_0} \times L^2 & \longrightarrow & \mathbb{R} \\
 & (x_t, p_0, u, \phi) & \longmapsto & S_t(x_t, p_0, u, \phi) = F_0(g_x(x_t, \phi)(0), u) + A_t(g(x_t, p_0, \phi))
\end{array}
\tag{5}
$$

can be considered as a "generating function" for Λ_t with a infinite dimensional space of auxiliary parameters $w = (p_0, u, \phi) \in \mathbb{R}^n \times \mathbb{R}^{k_0} \times L^2$; here, $g_x(x_t, \phi)(0)$ is the x-component of the curve $g(x_t, p_0, \phi)$ computed at time $s = 0$: namely, $g_x(x_t, \phi)(0) = x_t - \int_0^t \phi_x(\tau)\, d\tau$.

Proposition 2.2 *We have*

$$
\Lambda_t = \left\{ (x_t, p) \in T^*\mathbb{R}^n \quad \text{s. t. } \; p = \frac{\partial S_t}{\partial x_t}(x_t, w^*),\; \frac{DS_t}{Dw}(x_t, w^*) = 0 \right\}.
$$

Proof.

First, we recall the explicit expression of the functional S_t:

$$S_t(x_t, p_0, u, \phi) = F_0\left(x_t - \int_0^t \phi_x(s)\ ds, u\right) + \\ + \int_0^t \left[(p_0 + \int_0^s \phi_p(\tau)\ d\tau) \cdot \phi_x(s) - H\left(s, x_t - \int_s^t \phi_x(\tau)\ d\tau, p_0 + \int_0^s \phi_p(\tau)\ d\tau\right)\right]\ ds.$$

Now, let us compute the variation of S_t:

$$\begin{array}{rl} & dS_t(x_t, p_0, u, \phi)(\delta x_t, \delta p_0, \delta u, \delta\phi) = \\ = & D_{x_t}S_t\delta x_t + D_{p_0}S_t\delta p_0 + D_u S_t\delta u + D_\phi S_t\delta\phi = \\ = & \left\{\partial_x F_0(x_t - \int_0^t \phi_x(\tau)\ d\tau; u) + \int_0^t (-\frac{\partial H}{\partial x}(s, \gamma(s))\big|_{\gamma=g(x_t,p_0,\phi)})\ ds\right\}\delta x_t + \\ & + \left\{\int_0^t \left[\phi_x(s) - \frac{\partial H}{\partial p}(s, \gamma(s))\Big|_{\gamma=g(x_t,p_0,\phi)}\right]\ ds\right\}\delta p_0 + \\ & + \partial_u F_0(x_t - \int_0^t \phi_x(\tau)\ d\tau; u)\delta u + \\ & - \partial_x F_0(x_t - \int_0^t \phi_x(\tau)\ d\tau; u) \cdot \int_0^t \delta\phi_x(\tau)\ d\tau + \\ & + \int_0^t [(p_0 + \int_0^s \phi_p(\tau)\ d\tau) \cdot \delta\phi_x(s)]\ ds + \\ & + \int_0^t [\phi_x(s) \cdot (\int_0^s \delta\phi_p(\tau)\ d\tau)]\ ds + \\ & + \int_0^t (-\frac{\partial H}{\partial x}(s, \gamma(s))\big|_{\gamma=g(x_t,p_0,\phi)}) \cdot (-\int_s^t \delta\phi_x(\tau)\ d\tau)\ ds + \\ & + \int_0^t (-\frac{\partial H}{\partial p}(s, \gamma(s))\Big|_{\gamma=g(x_t,p_0,\phi)}) \cdot (\int_0^s \delta\phi_p(\tau)\ d\tau)\ ds. \end{array} \tag{6}$$

Integrating by parts, we get the following equality:

$$\int_0^t \left[(p_0 + \int_0^s \phi_p(\tau)\ d\tau) \cdot \delta\phi_x(s)\right]\ ds = p_0 \int_0^t \delta\phi_x(\tau)\ d\tau + \int_0^t \phi_p(s) \cdot \left(\int_s^t \delta\phi_x(\tau)\ d\tau\right)\ ds;$$

therefore, substituting it into (6), we obtain

$$\begin{array}{rl} & dS_t(x_t, p_0, u, \phi)(\delta x_t, \delta p_0, \delta u, \delta\phi) = \\ = & \left\{\partial_x F_0(x_t - \int_0^t \phi_x(\tau)\ d\tau; u) + \int_0^t (-\frac{\partial H}{\partial x}(s, \gamma(s))\big|_{\gamma=g(x_t,p_0,\phi)})\ ds\right\}\delta x_t + \\ & + \left\{\int_0^t \left[\phi_x(s) - \frac{\partial H}{\partial p}(s, \gamma(s))\Big|_{\gamma=g(x_t,p_0,\phi)}\right]\ ds\right\}\delta p_0 + \\ & + \partial_u F_0(x_t - \int_0^t \phi_x(\tau)\ d\tau; u)\delta u + \\ & + [p_0 - \partial_x F_0(x_t - \int_0^t \phi_x(\tau)\ d\tau; u)] \cdot \int_0^t \delta\phi_x(\tau)\ d\tau + \\ & + \int_0^t \left[\phi_p(s) + \frac{\partial H}{\partial x}(s, \gamma(s))\big|_{\gamma=g(x_t,p_0,\phi)}\right] \cdot (\int_s^t \delta\phi_x(\tau)\ d\tau)\ ds + \\ & + \int_0^t \left[\phi_x(s) - \frac{\partial H}{\partial p}(s, \gamma(s))\Big|_{\gamma=g(x_t,p_0,\phi)}\right] \cdot (\int_0^s \delta\phi_p(\tau)\ d\tau)\ ds. \end{array} \tag{7}$$

Thus, if $D_w S_t(x_t, w^*) = 0$ with $w^* = (p_0^*, u^*, \phi^*) \in \mathbb{R}^n \times \mathbb{R}^{k_0} \times L^2$ at some x_t, namely,

$$\left\{\begin{array}{ll} \partial_u F_0(x^*(0), u^*) & = 0 \\ \partial_x F_0(x^*(0), u^*) & = p_0^* \\ (\phi_x^*(s), \phi_p^*(s)) & = (\partial_p H(s, \gamma^*(s)), -\partial_x H(s, \gamma^*(s))) \end{array}\right. \tag{8}$$

with $\gamma^*(\cdot) = (x^*(\cdot), p^*(\cdot)) = g(x_t, p_0^*, \phi^*)$, then we have $(x^*(0), p_0^*) \in \Lambda_0$ and

$$\partial_{x_t} S_t(x_t, w^*) = p_0^* + \int_0^t \phi^*(s)\ ds = p^*(t).$$

Therefore, we obtain $(x_t, p^*(t)) \in \Lambda_t$.

□

The next step is to apply the A-C-Z reduction method. In the space L^2 we consider the orthonormal basis $\{e_k(s) := e^{i\frac{2\pi k}{T}s}\}_{k\in\mathbb{Z}}$. Thus, for all $\phi \in L^2$, we have the Fourier expansion $\phi(s) = \sum_{k\in\mathbb{Z}} \phi_k\ e_k(s)$. For any $N \in \mathbb{N}$ fixed, we define the projection operator $\mathbb{P}_N$ on the $k(n,N)$ central components of ϕ, where $k(n,N) := 2n(2N+1)$,

$$\mathbb{P}_N\phi(s) := \sum_{|k|\le N} \phi_k\ e_k(s),$$

and the projection operator $\mathbb{Q}_N$ on the remaining infinite external components

$$\mathbb{Q}_N\phi(s) := \sum_{|k|>N} \phi_k\ e_k(s).$$

Taken an element $\phi \in L^2 = \mathbb{P}_N L^2 \oplus \mathbb{Q}_N L^2$, we denote the central and the external components of ϕ by $\mu := \mathbb{P}_N\phi$ and by $\eta := \mathbb{Q}_N\phi$, respectively.

By a fixed point argument, we show that for a suitable natural number N only the finite dimensional space $\mathbb{P}_N\phi$ is sufficient to find stationary points of S_t (and to construct a generating function for Λ_t).

We denote the symplectic matrix by $\mathbb{E}$, while ∇H is the gradient of H (with respect to x and p)

$$\mathbb{E} := \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ -\mathbb{I} & \mathbb{O} \end{pmatrix}; \qquad \nabla H(s,x,p) = \begin{pmatrix} \frac{\partial H}{\partial x}(s,x,p) \\ \frac{\partial H}{\partial p}(s,x,p) \end{pmatrix}.$$

The proof of the following Lemma is quite standard (cf. for instance [4] or [3]); so, we omit it.

Lemma 2.3 *Assume that $||\nabla^2 H|| \le C < +\infty$. Choose $N \in \mathbb{N}$ such that $\frac{Ct}{2\pi N}(1+\sqrt{2N}) < 1$. Then, the map*

$$\begin{array}{llll} \mathcal{G}: & \mathbb{Q}_N L^2 & \longrightarrow & \mathbb{Q}_N L^2 \\ & \eta & \longmapsto & \mathbb{Q}_N\mathbb{E}\nabla H\Big((\cdot), g\big(x_t, p_0, \mu+\eta\big)(\cdot)\Big); \end{array} \tag{9}$$

is a contraction map, whenever we fix $x_t \in \mathbb{R}^n$, $p_0 \in \mathbb{R}^n$ and $\mu \in \mathbb{P}_N L^2$.

Therefore, for a positive integer N large enough by the Banach-Caccioppoli contraction Lemma (see for example [11]) applied to $\mathcal{G}$ defined in (9), once we choose $x_t \in \mathbb{R}^n$, $p_0 \in \mathbb{R}^n$ and $\mu \in \mathbb{P}_N L^2$, we obtain one and only one fixed point of $\mathcal{G}$, denoted by $\mathfrak{q}(x_t, p_0, \mu)$, that satisfies the equation

$$\mathfrak{q}(x_t, p_0, \mu)(s) = \mathbb{Q}_N\mathbb{E}\nabla H(s, g(x_t, \mu + \mathfrak{q}(x_t, p_0, \mu))(s)). \tag{10}$$

Hereafter, we take $N \in \mathbb{N}$ according to the statement of Lemma 2.3.

Remark 2.4 *Thanks to the fact that the Hamiltonian function $H \in \mathcal{C}^2$ and by using the implicit function (Dini) Theorem, the fixed point map*

$$(11) \qquad \begin{array}{rlcl} q: & \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{P}_N L^2 & \longrightarrow & \mathbb{Q}_N L^2 \\ & (x_t, p_0, \mu) & \longmapsto & \mathfrak{q}(x_t, p_0, \mu), \end{array}$$

is continuously differentiable (see [1] or [3]).

Remark 2.5 *For each solution of the following equation*

$$\mu(\cdot) = \mathbb{P}_N \mathbb{E} \nabla H\Big((\cdot), g\big(x_t, p_0, \mu + \mathfrak{q}(x_t, p_0, \mu)\big)(\cdot)\Big),$$

in the finite dimensional space $\mathbb{P}_n L^2$ (we recall that $\dim(\mathbb{P}_N L^2((0,t), \mathbb{R}^{2n}) = k(n,N))$, the curve $\gamma = g(x_t, \mu + q(x_t, p_0, \mu))$ solves the Hamilton canonical differential equations related to H, that we can write in a short formulation as follows:

$$(12) \qquad \dot{\gamma}(\cdot) - \mathbb{E}\ \nabla H\big((\cdot), \gamma(\cdot)\big) = 0$$

or, equivalently,

$$(13) \qquad \mathbb{E}\ \dot{\gamma}(\cdot) + \nabla H\big((\cdot), \gamma(\cdot)\big) = 0,$$

with boundary conditions $x(t) = x_t$ and $p(0) = p_0$.

Let us define the finite dimensional space

$$E := \mathbb{R}^n \times \mathbb{R}^{k_0} \times \mathbb{P}_N L^2,$$

that will play the role of the space of "auxiliary variables" $v = (p_0, u, \mu) \in E$. Notice that $E \cong \mathbb{R}^k$ where $k := n + k_0 + k(n,N)$. Finally, let us construct a generating function for Λ_t (by reduction on S_t):

$$(14) \qquad \begin{array}{rlcl} F_t: & \mathbb{R}^n \times E & \longrightarrow & \mathbb{R} \\ & \big(x_t, v\big) & \longmapsto & F_t(x_t, v) := S_t\big(x_t; p_0, u, \mu + \mathfrak{q}(x_t, p_0, \mu)\big). \end{array}$$

In details F_t becomes

$$(15) \qquad \begin{aligned} F_t(x_t; p_0, u, \mu) = {} & F_0\left(x_t - \textstyle\int_0^t (\mu_x(s) + \mathfrak{q}_x(s))\ ds, u\right) + \\ & + \textstyle\int_0^t \Big[(p_0 + \int_0^s (\mu_p(\tau) + \mathfrak{q}_p(\tau)\ d\tau) \cdot (\mu_x(s) + \mathfrak{q}_x(s)) + \\ & \quad - H\left(s, x_t - \textstyle\int_s^t (\mu_x(\tau) + \mathfrak{q}_x(\tau))\ d\tau, p_0 + \int_0^s (\mu_p(\tau) + \mathfrak{q}_p(\tau))\ d\tau\right)\Big]\ ds, \end{aligned}$$

where $(\mathfrak{q}_x, \mathfrak{q}_p) = (\mathfrak{q}_x(x_t, p_0, \mu), \mathfrak{q}_p(x_t, p_0, \mu))$. In order to simplify notations we sometimes write only $\mathfrak{q}$ instead of $\mathfrak{q}(x_t, p_0, \mu)$ when the dependence by (x_t, p_0, μ) is made precise.

Theorem 2.6 *Let Λ_t be the Lagrangian submanifold of $T^*\mathbb{R}^n$ connected to Λ_0 by the Hamiltonian isotopy generated by H. Suppose that F_0 is a global generating function for Λ_0 and $||\nabla^2 H||_\infty \leq C < +\infty$. Then the function $F_t = F_t(x_t; p_0, u, \mu)$, defined in (14), is a global generating function for $\Lambda_t = \Phi^t_H(\Lambda_0)$.*

Proof.
Let us compute the derivatives of function F_t with respect to all the variables:

$$\begin{aligned}\partial_u F_t(x_t, p_0, u, \mu) &= D_u S_t(x_t, p_0, u, \mu + \mathfrak{q}(x_t, p_0, \mu)) = \\ &= \partial_u F_0 \left(x_t - \int_0^t (\mu_x(\tau) + \mathfrak{q}_x(x_t, p_0, \mu)(\tau))\, d\tau, u \right);\end{aligned}$$

$$\begin{aligned}\partial_{p_0} F_t(x_t, p_0, u, \mu) &= D_{p_0} S_t(x_t, p_0, u, \mu + \mathfrak{q}(x_t, p_0, \mu)) = \tfrac{\partial S_t}{\partial p_0} + \tfrac{DS_t}{D\phi}\tfrac{D\phi}{D\eta}\tfrac{D\mathfrak{q}}{Dp_0} = \\ &= \int_0^t \left[(\mu_x + \mathfrak{q}_x) - \tfrac{\partial H}{\partial p}(s, \gamma(s))\right]\Big|_{\gamma = g(x_t, p_0, \mu + \mathfrak{q}(x_t, p_0, \mu))}\, ds + \\ &\quad + [p_0 - \partial_x F_0(x(0), u)] \cdot \int_0^t \tfrac{\partial \mathfrak{q}_x}{\partial p_0}\, ds + \\ &\quad + \int_0^t \mathbb{Q}_N[\mathbb{E}\dot\gamma(s) + \nabla H(s, \gamma(s))]|_{\gamma = g(x_t, p_0, \mu + \mathfrak{q}(x_t, p_0, \mu))} \cdot \left(\int_s^t \tfrac{\partial \mathfrak{q}_x}{\partial p_0}\, d\tau, -\int_0^s \tfrac{\partial \mathfrak{q}_p}{\partial p_0}\, d\tau\right) ds = \\ &= \int_0^t \mathbb{P}_N\left[\mu_x - \tfrac{\partial H}{\partial p}(s, \gamma(s))\right]\Big|_{\gamma = g(x_t, p_0, \mu + \mathfrak{q}(x_t, p_0, \mu))}\, ds + \\ &\quad + [p_0 - \partial_x F_0(x(0), u)] \cdot \int_0^t \tfrac{\partial \mathfrak{q}_x}{\partial p_0}\, ds\end{aligned}$$

since, by (10), we get $\mathbb{Q}_N\, [\mathbb{E}\dot\gamma(s) + \nabla H(s, \gamma(s))]|_{\gamma = g(x_t, p_0, \mu + \mathfrak{q}(x_t, p_0, \mu))} = 0$.
Similarly, we obtain:

$$\begin{aligned}\partial_\mu F_t(x_t, p_0, u, \mu) &= D_\mu S_t(x_t, p_0, u, \mu + \mathfrak{q}(x_t, p_0, \mu)) = \tfrac{\partial S_t}{\partial \phi}\left(\tfrac{D\phi}{D\mu} + \tfrac{D\phi}{D\eta}\tfrac{D\mathfrak{q}}{D\mu}\right) = \\ &= [p_0 - \partial_x F_0(x(0), u)] \cdot \int_0^t (1 + \tfrac{\partial \mathfrak{q}_x}{\partial \mu})\, ds + \\ &\quad + \int_0^t \mathbb{P}_N\, [\mathbb{E}\dot\gamma(s) + \nabla H(s, \gamma(s))]|_{\gamma = g(x_t, p_0, \mu + \mathfrak{q}(x_t, p_0, \mu))} \cdot \left(\int_s^t d\tau, -\int_0^s d\tau\right) ds + \\ &\quad + \int_0^t \mathbb{Q}_N[\mathbb{E}\dot\gamma(s) + \nabla H(s, \gamma(s))]|_{\gamma = g(x_t, p_0, \mu + \mathfrak{q}(x_t, p_0, \mu))} \cdot \left(\int_s^t \tfrac{\partial \mathfrak{q}_x}{\partial \mu}\, d\tau, -\int_0^s \tfrac{\partial \mathfrak{q}_p}{\partial \mu}\, d\tau\right) ds = \\ &= [p_0 - \partial_x F_0(x(0), u)] \cdot \int_0^t (1 + \tfrac{\partial \mathfrak{q}_x}{\partial \mu})\, ds + \\ &\quad + \int_0^t \mathbb{P}_N\, [\mathbb{E}\dot\gamma(s) + \nabla H(s, \gamma(s))]|_{\gamma = g(x_t, p_0, \mu + \mathfrak{q}(x_t, p_0, \mu))} \cdot \left(\int_s^t d\tau, -\int_0^s d\tau\right) ds,\end{aligned}$$

$$\begin{aligned}\partial_{x_t} F_t(x_t, p_0, u, \mu) &= D_{x_t} S_t(x_t, p_0, u, \mu + \mathfrak{q}(x_t, p_0, \mu)) = \tfrac{\partial S_t}{\partial x_t} + \tfrac{DS_t}{D\phi}\tfrac{D\phi}{D\eta}\tfrac{D\mathfrak{q}}{Dx_t} = \\ &= [p_0 - \partial_x F_0(x(0), u)] \cdot (-1 + \int_0^t \tfrac{\partial \mathfrak{q}_x}{\partial x_t}\, ds) + \\ &\quad - \int_0^t \left[(\mu_p + \mathfrak{q}_p)(s) + \tfrac{\partial H}{\partial x}(s, \gamma(s))\right]\Big|_{\gamma = g(x_t, p_0, \mu + \mathfrak{q}(x_t, p_0, \mu))}\, ds + \\ &\quad + \int_0^t \mathbb{Q}_N[\mathbb{E}\dot\gamma(s) + \nabla H(s, \gamma(s))]|_{\gamma = g(x_t, p_0, \mu + \mathfrak{q}(x_t, p_0, \mu))} \cdot \left(\int_s^t \tfrac{\partial \mathfrak{q}_x}{\partial x_t}\, d\tau, -\int_0^s \tfrac{\partial \mathfrak{q}_p}{\partial x_t}\, d\tau\right) ds \\ &\quad + p_0 + \int_0^t (\mu_p + \mathfrak{q}_p)\, ds = \\ &= [p_0 - \partial_x F_0(x(0), u)] \cdot (-1 + \int_0^t \tfrac{\partial \mathfrak{q}_x}{\partial x_t}\, ds) + \\ &\quad - \int_0^t \mathbb{P}_N\left[\mu_p(s) + \tfrac{\partial H}{\partial x}(s, \gamma(s))\right]\Big|_{\gamma = g(x_t, p_0, \mu + \mathfrak{q}(x_t, p_0, \mu))}\, ds + \\ &\quad + p_0 + \int_0^t (\mu_p + \mathfrak{q}_p)\, ds.\end{aligned}$$

Finally, we have

$$\partial_{x_t} F_t(x_t, p_0, u, \mu)|_{\partial_v F_t = 0} = p_0 + \int_0^t (\mu_p + \mathfrak{q}_p)\, ds = p(t).$$

Recalling that $v = (p_0, u, \mu)$ and $w = (p_0, u, \phi)$, notice that if

$$\partial_v F_t(x_t, v^*) = 0 \qquad p(t) = \partial_{x_t} F_t(x_t, v^*),$$

then

$$D_w S_t(x_t, w^*) = 0 \qquad p(t) = \partial_{x_t} S_t(x_t, w^*),$$

where $\phi^* = \mu^* + \mathfrak{q}(x_t, p_0^*, \mu^*)$. Moreover, also the opposite implication holds true with $\mu^* = \mathbb{P}_N \phi^*$. Therefore, F_t is a global generating function for Λ_t. □

Remark 2.7 *In case Λ_0 is the zero-section, then we can eliminate the p_0-component in the auxiliary coordinates, taking $S_t = A_t$ (the Action Functional) and $v' = \mu \in E' = \mathbb{P}_N L^2$. This construction is very well known in literature (see [15], [1] and [4]).*

3 Properties of generating functions via A-C-Z reduction

In this section, for us Λ_0 will be the zero-section of $T^*\mathbb{R}^n$ and the time $t > 0$ is fixed. In fact, almost all the results hold true for any starting Lagrangian submanifold $\Lambda_0 \subset T^*\mathbb{R}^n$, generated by F_0 (which is GQI or GWQI if necessary). Here, we prefer to treat the case $\Lambda_0 = \mathbb{R}^n \times \{0\} \subset T^*\mathbb{R}^n$ in order to simplify notations. Moreover, in many applications we can reduce to consider the zero-section as starting Lagrangian submanifold: for instance, studying solutions of Hamilton-Jacobi equations, we can apply a suitable canonical transformation (cf. [4]); also in intersection theory the typical setting is given by Λ_0 equal to the zero-section (cf. [13], [15] and [16]).

Emphasizing the dependence on H, by $F_H : \mathbb{R}^n \times \mathbb{P}_N L^2 \longrightarrow \mathbb{R}$ we denote the generating function for $\Lambda_t = \Phi_H^t(\Lambda_0)$ provided by the A-C-Z reduction where (cf. Remark 2.7)

$$F_H(x_t, \mu) = \int_0^t [p(s)\dot{x}(s) - H(\gamma(s))]\, ds \Big|_{\gamma(s) = g(x_t;\mu + q^H(x_t,\mu))(s)}$$

and $\mathfrak{q}^H = \mathfrak{q}^H(x_t, \mu)$ is the corresponding fixed point map.

If the Hamiltonian $H = H(x, p)$ is a quadratic form, then, by linearity of the corresponding Hamiltonian system, it is straightforward to see that the mapping

$$(x_t, \mu) \mapsto F_H(x_t, \mu)$$

is a quadratic form. Now, we try to investigate the behavior of F_H on the fibers $\pi^{-1}(x_t)$, $\pi : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$, starting from the case in which the Hamiltonians are quadratic forms. First, we notice that the fixed point map turns out to be linear with respect to μ.

Lemma 3.1 *Assume that the Hamiltonian $H(x, p)$ is a quadratic form. Then for any fixed $x_t \in \mathbb{R}^n$ the map*

$$\mu \mapsto \mathfrak{q}^H(x_t, \mu)$$

is linear.

Proof. We recall that the fix point map is characterized by the following property (cf. (10)):

$$\mathfrak{q}^H(x_t,\mu) = \mathbb{Q}_N \mathbb{E}\nabla H\left(x_t - \int_s^t (\mu_x + \mathfrak{q}_x^H(x_t,\mu))d\tau, \int_0^s (\mu_p + \mathfrak{q}_p^H(x_t,\mu))d\tau\right), \tag{16}$$

where $(\mathfrak{q}_x^H, \mathfrak{q}_p^H) = \mathfrak{q}^H$ and $(\mu_x, \mu_p) = \mu$. By linearity of ∇H, equation (16) becomes

$$\mathfrak{q}^H(x_t,\mu) = \mathbb{Q}_N \mathbb{E}\nabla H\left(-\int_s^t (\mu_x + \mathfrak{q}_x^H(x_t,\mu))d\tau, \int_0^s (\mu_p + \mathfrak{q}_p^H(x_t,\mu))d\tau\right).$$

Thus, for any $\lambda \in \mathbb{R}$, $\mathfrak{q}^H(x_t,\lambda\mu)$ is the unique element that satisfies

$$\mathfrak{q}^H(x_t,\lambda\mu) = \mathbb{Q}_N \mathbb{E}\nabla H\left(-\int_s^t (\mu_x + \mathfrak{q}_x^H(x_t,\lambda\mu))d\tau, \int_0^s (\mu_p + \mathfrak{q}_p^H(x_t,\lambda\mu))d\tau\right). \tag{17}$$

Since $\lambda\mathfrak{q}^H(x_t,\mu)$ satisfies (17), then $\mathfrak{q}^H(x_t,\lambda\mu) = \lambda\mathfrak{q}^H(x_t,\mu)$. □

The following lemma states that when H is a quadratic form, we can have an estimate on the μ-derivatives of F_H

Lemma 3.2 *Assume that the Hamiltonian $H(x,p)$ is a quadratic form. Then, there exists a quadratic form $Q = Q(x_t,\mu)$ (possibly degenerate) such that for any $\mu \in \mathbb{P}_N L^2$:*

$$|\partial_\mu F_H(x_t,\mu) - \partial_\mu Q(x_t,\mu)| \leq M|x_t|$$

for some positive constant M.

Proof. Let us define the function

$$\begin{aligned} Q_H(x_t,\mu) := \quad & \textstyle\int_0^t \Big[(\mu_x + \mathfrak{q}_x^H(x_t,\mu))(s) \int_0^s (\mu_p + \mathfrak{q}_p^H(x_t,\mu))(\tau)\ d\tau + \\ & -H\left(-\textstyle\int_s^t (\mu_x + \mathfrak{q}_x^H(x_t,\mu))(\tau)\ d\tau, \int_0^s (\mu_p + \mathfrak{q}_p^H(x_t,\mu))(\tau)\ d\tau\right)\Big]\ ds. \end{aligned}$$

Notice that $Q_H(\mu) = A_H\big(0, \mu + \mathfrak{q}^H(x_t,\mu)\big)$ where A_H is the Action functional related to H. By linearity of the fixed point map with respect to μ-variables: $\mu \mapsto \mathfrak{q}^H(x_t,\mu)$ (see Lemma 3.1), we get that $\mu \mapsto Q_H(x_t,\mu)$ is a quadratic form. For $N \in \mathbb{N}$ large enough we have:

$$\begin{aligned} &\partial_\mu F_H(x_t,\mu) - \partial_\mu Q_H(x_t,\mu) = \\ &= \textstyle\int_0^t \Big[\mathbb{P}_N(\mathbb{E}\dot\gamma + \nabla H(\gamma))\Big|_{\gamma = g(x_t,\mu+\mathfrak{q}^H(x_t,\mu))} - \mathbb{P}_N(\mathbb{E}\dot\gamma + \nabla H(\gamma))\Big|_{\gamma = g(0,\mu+\mathfrak{q}^H(x_t,\mu))}\Big]\ ds. \end{aligned}$$

Therefore, by linearity of ∇H, we obtain

$$||\partial_\mu F_H(x_t,\mu) - \partial_\mu Q_H(x_t,\mu)|| = ||\int_0^t \mathbb{P}_N \nabla H\big(x_t, 0\big)\ ds|| \leq t||\nabla H||_0|x_t|$$

where $||\cdot||_0$ is the norm in the space of real matrices: if $(a_{ij}) = A \in M_{n\times n}(\mathbb{R})$, $||A||_0 = \max|a_{ij}|$. □

Remark 3.3 *If $H = 0$, by the proof of Lemma 3.2 we get $\partial_\mu F_H(x_t, \mu) = \partial_\mu Q(x_t, \mu)$.*

Theorem 3.4 *Consider two Hamiltonian functions $H(x, p)$ and $K(x, p)$ such that*
i) $||\nabla H(x, p) - \nabla K(x, p)||_\infty \le C$,
ii) $||\nabla^2 H||_\infty,\ ||\nabla^2 K||_\infty \le C$
for some constant $C > 0$. Then there exists a positive real number M_0 such that

$$||\partial_\mu[F_H(x_t, \mu) - F_K(x_t, \mu)]||_\infty \le M_0.$$

Proof. Take $N \in \mathbb{N}$ suitably large such that (cf. Lemma 2.3)

$$\frac{Ct}{2\pi N}(1 + \sqrt{2N}) < 1. \tag{18}$$

Let us denote by $\mathfrak{q}^H$ and by $\mathfrak{q}^K$ the fixed point map related to H and K, respectively. By hypothesis i) and ii), we get

$$\begin{aligned}
&\partial_\mu F_K(x_t, \mu) - \partial_\mu F_H(x_t, \mu) = \\
&= \int_0^t \Big[\mathbb{P}_N(\mathbb{E}\dot\gamma + \nabla K(\gamma))\Big|_{\gamma=g(x_t,\mu+\mathfrak{q}^K(x_t,\mu))} - \mathbb{P}_N(\mathbb{E}\dot\gamma + \nabla H(\gamma))\Big|_{\gamma=g(x_t,\mu+\mathfrak{q}^H(x_t,\mu))}\Big]\, ds = \\
&= \int_0^t \mathbb{P}_N\Big[\nabla K(\gamma))\Big|_{\gamma=g(x_t,\mu+\mathfrak{q}^K(x_t,\mu))} - \nabla H(\gamma))\Big|_{\gamma=g(x_t,\mu+\mathfrak{q}^H(x_t,\mu))}\Big]\, ds = \\
&= \int_0^t \mathbb{P}_N\Big[\nabla K(\gamma))\Big|_{\gamma=g(x_t,\mu+\mathfrak{q}^K(x_t,\mu))} - \nabla K(\gamma))\Big|_{\gamma=g(x_t,\mu+\mathfrak{q}^H(x_t,\mu))}\Big]\, ds + \\
&\quad + \int_0^t \mathbb{P}_N\Big[\nabla K(\gamma))\Big|_{\gamma=g(x_t,\mu+\mathfrak{q}^H(x_t,\mu))} - \nabla H(\gamma))\Big|_{\gamma=g(x_t,\mu+\mathfrak{q}^H(x_t,\mu))}\Big]\, ds \le \\
&\le C\int_0^t |g(x_t, \mu + \mathfrak{q}^K(x_t, \mu)) - g(x_t, \mu + \mathfrak{q}^H(x_t, \mu))|\, ds + Ct.
\end{aligned}$$

Thanks to Hölder inequality we obtain

$$\begin{aligned}
&\int_0^t |g(x_t, \mu + \mathfrak{q}^K(x_t, \mu)) - g(x_t, \mu + \mathfrak{q}^H(x_t, \mu))|\, ds \le \\
&\quad\le \sqrt{t}||g(x_t, \mu + \mathfrak{q}^K(x_t, \mu)) - g(x_t, \mu + \mathfrak{q}^H(x_t, \mu))||_{L_2}.
\end{aligned} \tag{19}$$

Notice that

(20)
$$\begin{aligned}
&||\mathfrak{q}^K(x_t, \mu) - \mathfrak{q}^H(x_t, \mu)||_{L^2} = \\
&= \Big|\Big|\mathbb{Q}_N\mathbb{E}\Big[\nabla K(g(x_t, \mu + \mathfrak{q}^K(x_t, \mu))) - \nabla H(g(x_t, \mu + \mathfrak{q}^H(x_t, \mu)))\Big]\Big|\Big|_{L^2} \le \\
&\le \Big|\Big|\mathbb{Q}_N\mathbb{E}\Big[\nabla K(g(x_t, \mu + \mathfrak{q}^K(x_t, \mu))) - \nabla K(g(x_t, \mu + \mathfrak{q}^H(x_t, \mu)))\Big]\Big|\Big|_{L^2} + \\
&\quad + \Big|\Big|\mathbb{Q}_N\mathbb{E}\Big[\nabla K(g(x_t, \mu + \mathfrak{q}^H(x_t, \mu))) - \nabla H(g(x_t, \mu + \mathfrak{q}^H(x_t, \mu)))\Big]\Big|\Big|_{L^2} \le \\
&\le C\Big|\Big|g(x_t, \mu + \mathfrak{q}^K(x_t, \mu)) - g(x_t, \mu + \mathfrak{q}^H(x_t, \mu))\Big|\Big|_{L^2} + Ct.
\end{aligned}$$

Thus, by Lipschitz property of function $\eta \mapsto g(x_t, \mu + \eta)$ and by (20) we have

$$\begin{aligned}
&||g(x_t, \mu + \mathfrak{q}^K(x_t, \mu)) - g(x_t, \mathfrak{q}^H(x_t, \mu))||_{L_2} \le \\
&\quad\le \tfrac{t}{2\pi N}(1 + \sqrt{2N})||\mathfrak{q}^K(x_t, \mu) - \mathfrak{q}^H(x_t, \mu)||_{L^2} \le \\
&\quad\le \tfrac{Ct}{2\pi N}(1 + \sqrt{2N})\Big(||g(x_t, \mu + \mathfrak{q}^K(x_t, \mu)) - g(x_t, \mathfrak{q}^H(x_t, \mu))||_{L^2} + t\Big).
\end{aligned} \tag{21}$$

So, by the choice of N (such that (18) holds true) and by (21), we get

$$||g(x_t, \mu + \mathfrak{q}^K(x_t, \mu)) - g(x_t, \mathfrak{q}^H(x_t, \mu))||_{L_2} \leq \frac{Ct^2(1+\sqrt{2N})}{2\pi N - Ct(1+\sqrt{2N})}.$$

Hence, we conclude

$$|\partial_\mu F_H(x_t, \mu) - \partial_\mu F_K(x_t, \mu)| \leq M_0$$

for some positive constant M_0. □

Remark 3.5 *Under the hypothesis of the previous theorem, if F_H is GWQI (resp. GQI), then also F_K is GWQI (resp. GQI).*

The following corollary might be seen as a simple case (when Λ_0 is equal to the zero-section in $T^*\mathbb{R}^n$) of the well known Theorem of Sikorav-Chekanov (cf. for instance [6] and [12]).

Corollary 3.6 *Assume that $K(x,p)$ is a compactly supported Hamiltonian function. Then F_K is GQI.*

Proof. By Remark 3.5 with $H = 0$, we could immediately say that F_H is GWQI. But, notice that the quadratic form constructed in the proof of Lemma 3.2 (cf. also Remark 3.3), namely

$$Q_H(\mu) = \int_0^t \left[(\mu_x + \mathfrak{q}_x^H)(s) \int_0^s (\mu_p + \mathfrak{q}_p^H)(\tau)\ d\tau \right] ds \tag{22}$$

actually is non-degenerate. Therefore, F_H ($H = 0$) turns out to be GQI and we conclude by applying Theorem 3.4. □

Remark 3.7 *By Viterbo's Uniqueness Theorem (see [17] and also [14]) the generating functions for Λ_t obtained by A-C-Z reduction and by broken geodesics method are equivalent (in the sense of [14]).*

We conclude with an approximation theorem.

Theorem 3.8 *Consider the sequence of Hamiltonian functions $H_n(x,p)$ and $H(x,p)$ such that*

i) $H_n(x,p) \to H(x,p)$ uniformly,

ii) $\nabla H_n(x,p) \to \nabla H(x,p)$ uniformly,

iii) $||\nabla^2 H_n(x,p)||_\infty \leq C$

for some positive constant C. Then, called F_n the generating functions of the Lagrangian submanifolds Λ_n associated to H_n, the limit

$$F := \lim_{n \mapsto +\infty} F_n$$

generates $\Lambda := \lim_n \Phi^t_{H_n}(\Lambda_0)$. Furthermore, $\Lambda = \lim_n \Lambda_n$ (in the Hausdorff distance sense).

Before starting with the proof of the Theorem 3.8 we need some technical Lemmas.

Lemma 3.9 *Under the hypothesis of Theorem 3.8, the fixed point map* $\mathfrak{q}^n$ *associated to* H_n *uniformly converges (in* L^2*) to a map* $\mathfrak{q}$ *that satisfies:*

$$\mathfrak{q}(x_t, \mu) = \mathbb{QE}\nabla H(g(x_t, \mu + \mathfrak{q}(x_t, \mu))).$$

Proof. Fix $(x_t, \mu) \in \mathbb{R}^n \times \mathbb{P}_N L^2$. For any $\varepsilon > 0$, take $n, m > N_\varepsilon$ such that $||\nabla H_n - \nabla H_m||_\infty < \varepsilon$ and so:

$$\begin{aligned}
&||\mathfrak{q}^n(x_t,\mu) - \mathfrak{q}^m(x_t,\mu)||_{L^2} = \\
&= \Big|\Big|\mathbb{Q}_N\mathbb{E}\Big[\nabla H_n(g(x_t,\mu+\mathfrak{q}^n(x_t,\mu))) - \nabla H_m(g(x_t,\mu+\mathfrak{q}^m(x_t,\mu)))\Big]\Big|\Big|_{L_2} \le \\
&\le \Big|\Big|\mathbb{Q}_N\mathbb{E}\Big[\nabla H_n(g(x_t,\mu+\mathfrak{q}^n(x_t,\mu))) - \nabla H_n(g(x_t,\mu+\mathfrak{q}^m(x_t,\mu)))\Big]\Big|\Big|_{L_2} + \\
&\quad + \Big|\Big|\mathbb{Q}_N\mathbb{E}\Big[\nabla H_n(g(x_t,\mu+\mathfrak{q}^m(x_t,\mu))) - \nabla H_m(g(x_t,\mu+\mathfrak{q}^m(x_t,\mu)))\Big]\Big|\Big|_{L_2} \le \\
&\le C||g(x_t,\mu+\mathfrak{q}^n(x_t,\mu)) - g(x_t,\mu+\mathfrak{q}^m(x_t,\mu))||_{L_2} + \varepsilon t \le \\
&\le \tfrac{Ct}{2\pi N}(1+\sqrt{2N})||\mathfrak{q}^n(x_t,\mu) - \mathfrak{q}^m(x_t,\mu)||_{L^2} + \varepsilon t.
\end{aligned}$$

Since N is such that $\frac{Ct}{2\pi N}(1+\sqrt{2N}) < 1$, therefore

$$||\mathfrak{q}^n(x_t,\mu) - \mathfrak{q}^m(x_t,\mu)||_{L^2} \le \frac{\varepsilon t}{1 - \frac{Ct}{2\pi N}(1+\sqrt{2N})},$$

which implies that $\mathfrak{q}^n$ is a Cauchy sequence.

□

Let us fix $x_t \in \mathbb{R}^n$ and $\mu \in P_N L^2$; for any $n \in \mathbb{N}$ we set $\gamma_n(\cdot) := g(x_t, \mu + \mathfrak{q}^n(x_t,\mu))(\cdot)$.

Lemma 3.10 *Assume that the hypothesis of Theorem 3.8 are satisfied. Then, we have*

i) γ_n *converges to* $\gamma = g(x_t, \mu + \mathfrak{q}(x_t,\mu))$ *(in the sup-norm) uniformly in* x_t *and* μ

ii) $H_n(\gamma_n)$ *converges to* $H(\gamma)$ *(in the sup-norm).*

Proof. i) For any $\varepsilon > 0$ there exists N_ε such that if $n \ge N_\varepsilon$ then $||\mathfrak{q}^n - \mathfrak{q}||_{L^2} \le \varepsilon$. Thus, we have

$$\begin{aligned}
&||\gamma_n(\cdot) - \gamma(\cdot)||_\infty = \\
&= ||\Big(\textstyle\int_\cdot^t(\mathfrak{q}_x^n - \mathfrak{q}_x)\,d\tau, \int_0^\cdot(\mathfrak{q}_p^n - \mathfrak{q}_p)\,d\tau\Big)||_\infty = \\
&= ||t(\textstyle\sum_{|k|>N} \frac{e_k(\cdot)}{i2\pi k}\eta_k - \sum_{|k|>N}\frac{1}{i2\pi k}\eta_k)||_\infty,
\end{aligned}$$

where $\eta := \mathfrak{q}^n - \mathfrak{q}$ and $\eta_k := \mathfrak{q}_k^n - \mathfrak{q}_k$.

But, on one hand we have

$$||\sum_{|k|>N}\frac{1}{i2\pi k}\eta_k||_\infty \le |\langle(\eta_k), (\frac{1}{i2\pi k})\rangle_{\ell^2}| \le ||(\eta_k)||_{\ell^2}||(\frac{1}{i2\pi k})||_{\ell^2} \le C_1||\eta||_{L^2}$$

for some constant $C_1 > 0$, and, on the other hand, we get

$$||\sum_{|k|>N} \frac{e_k(\cdot)}{i2\pi k}\eta_k||_\infty \leq ||\langle(\eta_k), (\frac{e_k(\cdot)}{i2\pi k})\rangle_{\ell^2}||_\infty \leq C_1||\eta||_{L^2}.$$

Therefore, we obtain

$$||\gamma_n(\cdot) - \gamma(\cdot)||_\infty \leq 2C_1||\eta||_{L^2} \leq 2C_1\varepsilon.$$

ii) Notice that for n large enough we have $||H_n(\cdot)-H(\cdot)||_\infty \leq \varepsilon$ and, by i), both γ_n and γ are contained in a ball of radius R, B_R, for some $R > 0$ and $||\gamma_n(\cdot)-\gamma(\cdot)||_\infty \leq \varepsilon$. Therefore, we get

$$\begin{aligned}&||H_n(\gamma_n) - H(\gamma)||_\infty \leq ||H_n(\gamma_n) - H(\gamma_n)||_\infty + ||H(\gamma_n) - H(\gamma)||_\infty \leq \\ &\leq \varepsilon + \sup_{(x,p)\in B_R} |\nabla H(x,p)|||\gamma_n(\cdot) - \gamma(\cdot)||_\infty \leq (\sup_{(x,p)\in B_R} |\nabla H(x,p)| + 1)\varepsilon.\end{aligned}$$

□

Lemma 3.11 *Assume that the hypothesis of Theorem 3.8 are satisfied. Then, the limit* $\lim_{n\mapsto+\infty} F_n$ *exists and we have*

$$\lim_{n\mapsto+\infty} F_n(x_t, \mu) = F(x_t, \mu) \tag{23}$$

where $F(x_t, \mu) = \int_0^t [p(s)\dot{x}(s) - H(\gamma(s))]\ ds\Big|_{\gamma(s)=g(x_t;\mu+q(x_t,\mu))(s)}$.

Proof. Let us take $n \geq N_\varepsilon$ such that $||\mathfrak{q}^n - \mathfrak{q}|| < \varepsilon$, $||\gamma_n - \gamma|| < \varepsilon$ and $||H_n(\gamma_n) - H(\gamma)|| < \varepsilon$. For any $(x_t, \mu) \in \mathbb{R}^n \times \mathbb{P}_N L^2$ we have

$$\begin{aligned}&F_n(x_t, \mu) - F(x_t, \mu) = \\ &= \int_0^t \Big[\ p(s)\dot{x}(s)|_{\gamma_n} - p(s)\dot{x}(s)|_\gamma\ \Big]\ ds + \int_0^t [H(\gamma(s)) - H_n(\gamma_n(s))]\ ds\end{aligned} \tag{24}$$

where $\gamma_n = g(x_t; \mu + q_n(x_t, \mu))$ and $\gamma = g(x_t; \mu + q(x_t, \mu))$. Thanks to Lemma 3.10, the last term in (24) is bounded from above by $t\varepsilon$. On the other hand, we get

(25)

$$\begin{aligned}&\int_0^t \Big[\ p(s)\dot{x}(s)|_{\gamma_n(s)} - p(s)\dot{x}(s)|_{\gamma(s)}\ \Big]\ ds = \\ &= \int_0^t \Big[(\mu_x(s) + \mathfrak{q}_x^n(s)) \int_0^s(\mu_p(\tau) + \mathfrak{q}_p^n(\tau))\ d\tau - (\mu_x(s) + \mathfrak{q}_x(s)) \int_0^s(\mu_p(\tau) + \mathfrak{q}_p(\tau))\ d\tau\Big]\ ds = \\ &= \int_0^t \Big[\mu_x(s) \int_0^s(\mathfrak{q}_p^n(\tau) - \mathfrak{q}_p(\tau))\ d\tau + (\mathfrak{q}_x^n(s) - \mathfrak{q}_x(s)) \int_0^s \mu_p(\tau)\ d\tau + \\ &\quad +\mathfrak{q}_x^n(s) \int_0^s \mathfrak{q}_p^n(\tau)\ d\tau - \mu_x(s) \int_0^s \mathfrak{q}_p(\tau)\ d\tau\Big]\ ds.\end{aligned}$$

We have:

$$\int_0^t \Big[\mu_x(s) \int_0^s (\mathfrak{q}_p^n(\tau) - \mathfrak{q}_p(\tau))\ d\tau\Big]\ ds \leq \sqrt{t}||\mathfrak{q}_p^n - \mathfrak{q}_p||_{L^2}||\mu_x||_{L^1} \leq t\varepsilon||\mu_x||_{L^2};$$

$$\int_0^t \left[(\mathfrak{q}_x^n(s)-\mathfrak{q}_x(s))\int_0^s \mu_p(\tau)\, d\tau\right] ds \le ||\mathfrak{q}_x^n-\mathfrak{q}_x||_{L^2}||\int_0^s \mu_p(\tau)\, d\tau||_{L^2} \le \varepsilon||\int_0^s \mu_p(\tau)\, d\tau||_{L^2};$$

$$\begin{aligned}
&\int_0^t \left[\mathfrak{q}_x^n(s)\int_0^s \mathfrak{q}_p^n(\tau)\, d\tau - \mu_x(s)\int_0^s \mathfrak{q}_p(\tau)\, d\tau\right] ds = \\
&\quad = \int_0^t \left[(\mathfrak{q}_x^n(s)-\mathfrak{q}_x(s))\int_0^s (\mathfrak{q}_p^n(\tau)-\mathfrak{q}_p(\tau))\, d\tau\right] ds + \int_0^t \left[(\mathfrak{q}_x^n(s)-\mathfrak{q}_x(s))\int_0^s \mathfrak{q}_p(\tau)\, d\tau\right] ds + \\
&\qquad + \int_0^t \left[\mathfrak{q}_x(s)\int_0^s (\mathfrak{q}_p^n(\tau)-\mathfrak{q}_p(\tau))\, d\tau\right] ds \le \\
&\quad \le t||\mathfrak{q}_x^n-\mathfrak{q}_x||_{L^2}||\mathfrak{q}_p^n-\mathfrak{q}_p||_{L^2} + ||\mathfrak{q}_x^n-\mathfrak{q}_x||_{L^2}||\int_0^s \mathfrak{q}_p(\tau)\, d\tau||_{L^2} + t||\mathfrak{q}_p^n-\mathfrak{q}_p||_{L^2}||\mathfrak{q}_x||_{L^2} \le \\
&\quad \le t\varepsilon^2 + \varepsilon||\int_0^s \mathfrak{q}_p(\tau)\, d\tau||_{L^2} + t\varepsilon||\mathfrak{q}_x||_{L^2}.
\end{aligned}$$

Thus, we obtain

$$|F_n(x_t,\mu)-F(x_t,\mu)| \le t\varepsilon||\mu_x||_{L^2}+\varepsilon||\int_0^s \mu_p(\tau)\, d\tau||_{L^2}+t\varepsilon^2+\varepsilon||\int_0^s \mathfrak{q}_p(\tau)\, d\tau||_{L^2}+t\varepsilon||\mathfrak{q}_x||_{L^2}.$$

□

Proof of Theorem 3.8.

First, we notice that Λ_n converges in the Hausdorff distance to $\Lambda = \lim_n \Phi^t_{H_n}(\Lambda_0)$. This is due to the following fact: if by $s \mapsto \Phi_n^s(x_0,p_0)$ we denote the flow such that

$$\begin{cases} \frac{d}{ds}\Phi_n^s(x_0,p_0) &= \mathbb{E}\nabla H_n(\Phi_n^s(x_0,p_0)) \\ \Phi_n^0(x_0,p_0) &= (x_0,p_0) \end{cases} \tag{26}$$

then for any starting point $(x_0,p_0) \in \Lambda_0$, $\Phi_n^t(x_0,p_0)$ is a Cauchy sequence. Indeed, for any $\varepsilon > 0$ there exist N_ε such that for all $n, m > N_\varepsilon$, we get

$$\begin{aligned}
&|\Phi_n^t(x_0,p_0) - \Phi_m^t(x_0,p_0)| \le \\
&\quad \le \int_0^t |\mathbb{E}\nabla H_n(\Phi_n^s(x_0,p_0)) - \mathbb{E}\nabla H_m(\Phi_n^s(x_0,p_0))|\, ds \le \\
&\quad \le \int_0^t |\mathbb{E}\nabla H_n(\Phi_n^s(x_0,p_0)) - \mathbb{E}\nabla H_m(\Phi_n^s(x_0,p_0))|\, ds \le + \\
&\qquad + \int_0^t |\mathbb{E}\nabla H_m(\Phi_n^s(x_0,p_0)) - \mathbb{E}\nabla H_m(\Phi_n^s(x_0,p_0))|\, ds \le \\
&\quad \le \varepsilon + C\int_0^t |\Phi_n^s(x_0,p_0) - \Phi_m^s(x_0,p_0)|\, ds.
\end{aligned} \tag{27}$$

Applying Gronwall Lemma, we have:

$$|\Phi_n^t(x_0,p_0) - \Phi_m^t(x_0,p_0)| \le \varepsilon t(\frac{Cte^{Ct}}{2}+1).$$

Recall that

$$\partial_\mu F_n(x_t,\mu) = \frac{\partial S_n}{\partial \phi}\left(\frac{\partial \phi}{\partial \mu} + \frac{\partial \phi}{\partial \eta}\frac{\partial \mathfrak{q}^n}{\partial \mu}\right), \qquad \partial_{x_t} F_n(x_t,\mu) = \frac{\partial S_n}{\partial x_t} + \frac{\partial S_t}{\partial \phi}\frac{\partial \phi}{\partial \eta}\frac{\partial \mathfrak{q}^n}{\partial x_t}$$

where S_n is the functional defined in (5) with $F_0 = 0$ (see Remark 2.7).

Moreover, we recall that (see for instance [3])

$$(28)\qquad \begin{pmatrix} \frac{\partial \mathfrak{q}^n}{\partial x_t} \\ \frac{\partial \mathfrak{q}^n}{\partial \mu} \end{pmatrix} = \left(\mathbb{I} - \frac{\partial \mathcal{G}}{\partial \eta}\right)^{-1} \begin{pmatrix} \mathbb{Q}_N \mathbb{E}\nabla^2 H_n\Big(g(x_t, \mu + \mathfrak{q}^n(x_t,\mu))\Big)\frac{\partial g}{\partial x_t} \\ \mathbb{Q}_N \mathbb{E}\nabla^2 H_n\Big(g(x_t, \mu + \mathfrak{q}^n(x_t,\mu))\Big)\frac{\partial g}{\partial \phi}\frac{\partial \phi}{\partial \mu} \end{pmatrix}.$$

Since $||\nabla^2 H_n||_\infty \leq C$ and since the operator $\left(\mathbb{I} - \frac{\partial \mathcal{G}}{\partial \eta}\right)^{-1}$ is bounded (due to the fact that $\mathcal{G}$ is a contraction map, cf. [1] or [3]), then the terms $\frac{\partial \mathfrak{q}^n}{\partial x_t}$ and $\frac{\partial \mathfrak{q}^n}{\partial \mu}$ are uniformly bounded. Therefore, the terms $\frac{\partial S_n}{\partial \phi}\frac{\partial \phi}{\partial \eta}\frac{\partial \mathfrak{q}^n}{\partial \mu}$ and $\frac{\partial S_t}{\partial \phi}\frac{\partial \phi}{\partial \eta}\frac{\partial \mathfrak{q}^n}{\partial x_t}$ not only vanish but also tend to zero as n goes to ∞.

So, we have

$$(29)\qquad \begin{aligned} &|\partial_{x_t} F_n(x_t,\mu) - \partial_{x_t} F(x_t,\mu)| = \\ &= \left|\int_0^t \Big[\mathbb{P}_N(\mathbb{E}\dot{\gamma}_n + \nabla H_n(\gamma_n)) - \mathbb{P}_N(\mathbb{E}\dot{\gamma} + \nabla H(\gamma))\Big]\, ds\right| = \\ &= \left|\int_0^t \mathbb{P}_N\Big[\nabla H(\gamma) - \nabla H_n(\gamma_n)\Big]\, ds\right|. \end{aligned}$$

and, by Lemmas (3.9)-(3.10), it is straightforward to see that

$$\lim_{n\mapsto+\infty} \partial_{x_t} F_n(x_t,\mu) = \partial_{x_t} F(x_t,\mu).$$

Similarly, we get

$$\lim_{n\mapsto+\infty} \partial_{\mu} F_n(x_t,\mu) = \partial_{\mu} F(x_t,\mu)$$

and, so, finally

$$\Lambda = \left\{(x_t,p_t) \in T^*\mathbb{R}^n : \ p_t = \frac{\partial F}{\partial x_t}(x_t,\mu^*), \quad \frac{\partial F}{\partial \mu}(x_t,\mu^*) = 0 \quad \exists\, \mu^* \in \mathbb{P}_N L^2\right\}.$$

□

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