A Deterministic Approach to the Skorokhod Problem

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Abstract

We prove an existence and uniqueness result for the solutions to the Skorokhod problem on uniformly prox-regular sets through a deterministic approach. This result can be applied in order to investigate some regularity properties of the value function for differential games with reflection on the boundary.

Key words. Skorokhod problem, reflection on the boundary, differential inclusions, differential games, value function.

AMS Mathematics Subject Classification 2000: 49J24, 49K24, 49N70.

SISSA Ref. 67/2005/M

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1 Introduction

The main purpose of the present article is to study the Skorokhod problem from a deterministic point of view and to get some properties on the value function for (zero-sum) differential games with reflection on the boundary.

Given $T > 0$ and the initial data $x \in K$, by solving the deterministic Skorokhod problem, we mean to find the solution to the following differential inclusion:

$$\begin{cases} \dot{y}(t) \in f(t, y(t)) - \partial_{-}\psi_K(y(t)) \\ y(t) \in K \qquad \forall\, t \in [0, T] \\ y(0) = x \in K \end{cases} \tag{1.1}$$

where $\partial_{-}\psi_K$ is the subdifferential of the indicator function of K, ψ_K.

More properly, the Skorokhod problem arises in the stochastic framework. It was studied and solved by Lions and Sznitman in [16] and [17] for smooth domains (cf. also the reference therein for a more complete literature scene). See also the article [12] where Frankowska used a viability approach.

In the deterministic frame the analogous problem is often called "reflecting boundary problem". Nevertheless, we prefer to keep the terminology used by Ishii in [14] talking about deterministic Skorokhod problem or (with a little abuse of language) simply Skorokhod problem.

We assume that $K \subset \mathbb{R}^n$ is a uniformly prox-regular set. This notion is equivalent to a very nice property: the projection to K is single-valued on the suitable ("tubular") neighborhood of K.

For a rather complete description of such sets we refer the reader to the article [19] and the reference therein (cf. also [7]). Since we deal with finite dimensional spaces, the notion of uniformly prox-normal sets coincides with the notion of Federer's sets of positive reach (for properties of these sets see [11], [9], [10]).

Examples of uniformly prox-regular sets are provided by $K \subset \mathbb{R}^n$ such that $\partial K \in \mathcal{C}^{1,1}$ or K convex.

Notice that system (1.1) without the term $\partial_{-}\psi_K(y(t))$ becomes an usual Cauchy problem that admits a unique global Carathéodory solution, and this is the case when $y(t) \in Int(K)$. On the other hand, when $y(t) \in \partial K$, the trajectory $y(\cdot)$ is pushed back along a direction that belongs to $\partial_{-}\psi_K(y(t))$, obtaining a sort of reflection on the boundary. In fact, we can clarify better this concept showing that system (1.1) really describes a reflection on the boundary of K. Indeed, for uniformly pro-regular sets the normal cone of K and the subdifferential of its indicator function coincide. Therefore, system (1.1) is equivalent to the viability problem

$$\begin{cases} \dot{y}(t) \in f(t, y(t)) - N_K(y(t)) \\ y(t) \in K \qquad \forall\, t \in [0, T] \\ y(0) = x \in K \end{cases} \tag{1.2}$$

that, in turns, has the same set of solutions of the following one

$$(1.3)\qquad \begin{cases} \dot{y}(t) = \Pi_{T_K(y(t))}(f(t, y(t))) \\ y(t) \in K \quad \forall\, t \in [0, T] \\ y(0) = x \in K. \end{cases}$$

The equivalence between (1.2) and (1.3) express quite well the reflecting role of the term $\partial_- \psi_K(y(t))$ in system (1.1) (or of the term $N_K(y(t))$ in (1.2)). Equivalence and existence results for these systems have been successfully studied in severals papers and in different context: see for instance Henry [13] who treats the case when K is convex and Cornet [8] who assumes a tangential regularity of K.

More recently, in the frame of sweeping process Thibault [21] provides existence results of viable solutions to (1.2) for K closed and with a reflection obtained by the Clarke cone. Furthermore, in [20] Serea gives an equivalence result dealing with a Mayer problem for controlled systems with reflection on the boundary of closed subsets K of $\mathbb{R}^n$; she formulates system (1.2) by using the strict normal cone $\hat{N}_K(y(t))$ providing an existence result for bounded sleek subsets and, moreover, also uniqueness when K is bounded and proximal retract.

In the present paper after introducing some preliminaries in section 2, we devote section 3 to a proof of an existence and uniqueness result using an adapted version of an approach due to Ishii in [14]: this approach yields not only a constructive proof of the trajectory that solves the deterministic Skorokhod problem (1.1), but also allows us to get existence and uniqueness also for not necessarily bounded prox-regular sets.

Finally, in the last section, we investigate the differential games with reflection on the boundary: once we fix the the controls of the players, the controlled Skorokhod problem becomes a system like (1.1). So, we show that the Lipschitz dependence of the solutions to the Skorokhod problem on the initial data provides a Hölder regularity for the value function. We also mention how this regularity could be useful to get important another property of the value function, as its limiting behaviour, namely the so-called ergodic problem.

2 Basic notions and assumptions

Let $\mathbb{R}^n$ be endowed with the euclidean norm $|\cdot|$. Given $K \subset \mathbb{R}^n$, we denote the distance function from a point $x \in \mathbb{R}^n$ to K by $d_K(x) = \text{dist}(x; K) := \inf_{y \in K} |y - x|$. We recall that the projection on K $\Pi_K : \mathbb{R}^n \longrightarrow K$ is defined by

$$\Pi_K(x) := \{y \in K : |x - y| = d_K(x)\}.$$

We denote the closed ball of radius r centered in $x_0 \in \mathbb{R}^n$ by $B(x_0; r) := \{x \in X : |x_0 - x| \leq r\}$. For the closed unit ball in $\mathbb{R}^n$ we simply write B. A set $C \subset \mathbb{R}^n$ is called a cone if it is nonempty and for all $\lambda \geq 0$ and $v \in C$ we have $\lambda v \in C$. The negative polar cone of C is defined by

$$C^- = \{\xi \in \mathbb{R}^n \mid \langle \xi, v \rangle \leq 0, \quad \forall\, v \in C\}$$

where $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the canonical scalar product in $\mathbb{R}^n$.

Definition 2.1 *Let K be a subset of $\mathbb{R}^n$ and $x \in \overline{K}$. Then*

1. *the contingent cone to K at x, $T_K(x)$, is defined by*
$$T_K(x) := \left\{ v \mid \liminf_{h \to 0^+} \mathrm{dist}\left(\frac{x + hv}{h}; K \right) = 0 \right\};$$

2. *the Clarke tangent cone to K at x, $C_K(x)$, is defined by*
$$C_K(x) := \left\{ v \mid \lim_{h \to 0^+,\, y \to_K x} \mathrm{dist}\left(\frac{y + hv}{h}; K \right) = 0 \right\};$$

3. *the proximal normal cone to K at x, $N_K^P(x)$, is defined by*
$$N_K^P(x) := \left\{ \xi \ : \ \exists M > 0 \text{ such that } \langle \xi, y - x \rangle \le M|y - x|^2 \ \ \forall y \in K \right\};$$

4. *the strict normal cone to K at x, $\hat{N}_K(x)$, is defined by*
$$\hat{N}_K(x) := \left\{ \xi \ : \ \limsup_{y \to_K x} \frac{\langle \xi, y - x \rangle}{|y - x|} \le 0 \right\}.$$

5. *the limiting normal cone (also called general cone of normals) to K at x, $N_K(x)$, is defined by*
$$N_K(x) := \left\{ \xi \ : \ \exists x_n \to_K x \text{ and } \xi_n \to \xi \text{ with } \xi_n \in N_K^P(x_n) \right\}.$$

6. *the (Clarke) normal cone to K at x, $N_K^C(x)$, is defined by*
$$N_K^C(x) := T_K^-(x).$$

For many properties on tangent and normal cones we refer the reader for instance to the books [1], [2], [18] and [22].

Here, we recall that in general we have $C_K(x) \subset T_K(x)$ and $\{0\} \subset N_K^P(x) \subset \hat{N}_K(x) \subset N_K(x) \subset N_K^C(x)$. $N_K^P(x)$ is convex, $\hat{N}_K(x)$ is closed and convex, $N_K(x)$ is closed and $N_K^C(x) = \overline{\mathrm{co}} N_K(x)$ (if $Y \subset \mathbb{R}^n$, then $\overline{\mathrm{co}} Y$ denotes the closed convex hull of Y). Moreover, we have $N_K(x)^- = C_K(x)$ and $\hat{N}_K(x) = C_K^-(x)$. A closed set $K \subset \mathbb{R}^n$ is called *sleek* if the set-valued map
$$K \ni x \rightsquigarrow T_K(x)$$
is lower semicontinuous. A very important property of sleek subsets is that the contingent and the Clarke tangent cone to K at x coincide for every $x \in K$.

Here, we assume that for some $r_0 > 0$ fixed the closed set $K \subset \mathbb{R}^n$ is *uniformly prox-regular* with constant $\frac{1}{r}$ for any $r \in (0, r_0)$ (or also called proximal r_0-retract set), namely, whenever $x \in K$, $r \in (0, r_0)$ and $\xi \in N_K(x)$ with $|\xi| < 1$, then x is the unique nearest point of K to $x + \frac{1}{r}\xi$ (see for instance [19]).

Prox-regular sets are characterized in several interesting ways and have a lot of important properties. In the following Proposition we recall some of them referring the reader to the article [19] for the proofs (in particular see Theorem 4.1 and Corollary 4.6).

Proposition 2.2 *Let K be a subset closed of $\mathbb{R}^n$ and $r_0 > 0$. Then, the following properties are equivalent:*

1. *K is uniformly prox-regular with constant $\frac{1}{r}$ for any $r \in (0, r_0)$;*
2. *Π_K is single-valued on the "tubular" neighborhood of K $U_K(r_0) := \{x \in \mathbb{R}^n : 0 < d_K(x) < r_0\}$;*
3. *every nonzero proximal normal to K at any point $x \in K$ can be realized by an r_0-ball, namely for any $x \in K$ and $0 \neq \xi \in N_K^P(x)$ one has*
$$K \cap Int \left\{ x + r_0 \left(\frac{\xi}{|\xi|} + B \right) \right\} = \emptyset;$$
4. *whenever $x_i \in K$ and $\xi_i \in N_K^{r_0}(x_i) = N_K(x_i) \cap Int(B(0, r_0))$ we have*
$$\langle \xi_1 - \xi_2, x_1 - x_2 \rangle \geq -|x_1 - x_2|^2.$$

If one of the above equivalent properties occurs, then
$$N_K^P(x) = \hat{N}_K(x) = N_K(x) = N_K^C(x).$$

Remark 2.3 *Property 3 of Proposition 2.2 can be rephrased in the useful following way: given $x \in K$, $0 \neq \xi \in N_K^P(x)$ if and only if*
$$\frac{1}{2r_0}|y - x|^2 \geq \left\langle \frac{\xi}{|\xi|}, y - x \right\rangle \quad \forall y \in K.$$
So, for all $x \in K$ $\xi \in N_K^P(x)$ we have
$$\xi \cdot (x - y) + C_0|\xi||y - x|^2 \geq 0 \quad \forall y \in K \tag{2.1}$$
where $C_0 = \frac{1}{2r_0}$. An immediate consequence of (2.1) is that for any $x_1, x_2 \in K$, $\xi_1 \in N_K^P(x_1)$ and $\xi_2 \in N_K^P(x_2)$
$$\langle \xi_1 - \xi_2, x - y \rangle + C_0(|\xi_1| + |\xi_2|)|x_1 - x_2|^2 \geq 0. \tag{2.2}$$

Remark 2.4 *Proposition 2.2 yields immediately that K is sleek. Property 4 of Proposition 2.2 says that the set-valued map $K \ni x \rightsquigarrow N_K^{r_0}(x) + x$ is monotone (i.e. if $\zeta_i \in N_K^{r_0}(x_i) + x_i$ for $i = 1, 2$, then $\langle \zeta_1 - \zeta_2, x_1 - x_2 \rangle \geq 0$).*

Thanks to Proposition 2.2, henceforth, we simply talk about normal cones $N_K(x)$ (and analogously for the tangent cones $T_K(x)$) without specifying the type, since all the notions coincide.

By $\psi_K : \mathbb{R}^n \longrightarrow [0, +\infty]$ we denote the indicator function of K, namely
$$\psi_K(x) = \begin{cases} 0 & \text{if} \quad x \in K \\ +\infty & \text{if} \quad x \notin K. \end{cases}$$
Note that ψ_K is lower semicontinuous on $\mathbb{R}^n$ because K is closed.

Now, we introduce the notion of (local) subdifferential.

Definition 2.5 *Given a function $\psi : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ and a point x_0 such that $\psi(x_0) \neq \pm\infty$, the (local) subdifferential of ψ at a point x_0 is so defined*

$$\partial_- \psi(x_0) := \left\{ \xi \in \mathbb{R}^n \; : \; \liminf_{x \to x_0} \frac{\psi(x) - \psi(x_0) - \langle \xi, x - x_0 \rangle}{|x - x_0|} \geq 0 \right\}.$$

We recall that by solving the deterministic Skorokhod problem we mean to find a solution to the following differential inclusion:

$$\text{(2.3)} \qquad \begin{cases} \dot{y}(t) \in f(t, y(t)) - \partial_- \psi_K(y(t)) \\ y(t) \in K \quad \forall\, t \in [0, T] \\ y(0) = x \in K. \end{cases}$$

In the present paper f is a function so defined

$$f : \begin{array}{ccc} \mathbb{R} \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ (t, x) & \longmapsto & f(t, x) \end{array}$$

and we assume that

$$\text{(2.4)} \qquad \begin{cases} (i) & f(t, \cdot) \text{ is continuous for all } t, \\ (ii) & f \text{ is (Lebesgue) measurable with respect to } t, \\ (iii) & |f| < M \text{ for some } M > 0; \\ (iv) & \exists L > 0 \text{ such that} \\ & \langle f(t, x_1) - f(t, x_2), x_1 - x_2 \rangle \leq L|x_1 - x_2|^2 \; \forall\, x_1, x_2 \in \mathbb{R}^n, \; t \in \mathbb{R}. \end{cases}$$

In general we have $\partial_- \psi_K(x) = \hat{N}_K(x)$. But, the regularity (in the Clarke sense) of the set K provides also a sort of regularity of the indicator function ψ_K, obtaining the following lemma (for the details of the proof, see for instance [18] Chapter 8 section B).

Lemma 2.6 *If $K \subset \mathbb{R}^n$ is uniformly prox-regular with constant $\frac{1}{r}$ for any $r \in (0, r_0)$ then we have*

$$\partial_- \psi_K(x) = N_K(x) \quad \forall x \in K.$$

Thus, thanks to Lemma 2.6, we can rewrite system (2.3) by means of the (viability) differential inclusion

$$\text{(2.5)} \qquad \begin{cases} \dot{y}(t) \in f(t, y(t)) - N_K(y(t)) \\ y(t) \in K \;\; \forall\, t \in [0, T] \\ y(0) = x \in K. \end{cases}$$

The set of solutions of system (2.5) is the same of the following one (see for instance [20] for the proof)

$$\text{(2.6)} \qquad \begin{cases} \dot{y}(t) = \Pi_{T_K(y(t))}(f(t, y(t))) \\ y(t) \in K \;\; \forall\, t \in [0, T] \\ y(0) = x \in K. \end{cases}$$

Let us recall the definition of the inf-convolution of the function $\psi_K(x)$ (cf. for instance [3] for this notion): for all $\varepsilon > 0$

$$\psi_\varepsilon(x) := \inf_{y\in K}\{\psi_K(y) + \frac{1}{2\varepsilon}|x-y|^2\}, \qquad \forall x \in \mathbb{R}^n.$$

Notice that the inf-convolution of ψ_K coincides with the Moreau envelope of ψ_K, denoted by $e_\varepsilon\psi_K$: indeed, $\forall x \in \mathbb{R}^n$ $\psi_\varepsilon(x) = \inf_{y\in\mathbb{R}^n}\{\psi_K(y) + \frac{1}{2\varepsilon}|x-y|^2\} =: e_\varepsilon\psi_K(x)$.

We introduce also the notion of proximal mapping associated to the indicator function ψ_K, we denote here by $P_\varepsilon\psi_K$ or simply by P_ε:

$$P_\varepsilon(x) = P_\varepsilon\psi_K(x) := \arg\min\left\{\psi_K(y) + \frac{1}{2\varepsilon}|x-y|^2 \mid y \in \mathbb{R}^n\right\}.$$

For general properties of Moreau envelopes and proximal mappings we refer the reader to the book [18].

By K_{r_0} we denote the set of the points of $\mathbb{R}^n$ whose distance from K is less than r_0:

$$K_{r_0} := \{x \in \mathbb{R}^n \ : \ \mathrm{dist}(x,K) < r_0\}.$$

Observe that P_ε is a single-valued map on K_{r_0}. Indeed, if $x \in K$ then obviously $P_\varepsilon(x) = x$. So, we assume that $y_1, y_2 \in P_\varepsilon(x)$ with $x \in U_K(r_0) = K_{r_0} \setminus K$. Then, clearly $y_i \in K$ and $|x-y_1| = |x-y_2| = d_K(x) < r_0$. Since Π_K is single-valued on $U(r_0)$ by Proposition 2.2, we have that $y_1 = y_2 = \Pi_K(x)$. This automatically implies also that if $x \in K_{r_0}$ then $\psi_\varepsilon(x) = \frac{1}{2\varepsilon}[\mathrm{dist}(x,K)]^2 = \frac{1}{2\varepsilon}|x - P_\varepsilon(x)|^2$.

Remark 2.7 *Notice that for all $x \in K_{r_0}$ $P_\varepsilon(x)$ has the property of being the closest point of K to x and it does not depend on ε. So, henceforth, in order to simplify the notations we drop the "ε", writing just $P(x)$.*

Now, if $x_1, x_2 \in K_{r_0}$, then we have

$$\frac{1}{\varepsilon}(x_i - P(x_i) \in N_K(P(x_i))) \qquad i = 1,2$$

and by (2.2)

$$\begin{aligned}&\langle (x_1 - P(x_1)) - (x_2 - P(x_2)), P(x_1) - P(x_2)\rangle + \\ &\quad + C_0(|x_1 - P(x_1)| - |x_2 - P(x_2)|)|P(x_1) - P(x_2)|^2 \geq 0.\end{aligned}$$

Therefore, we obtain

$$|P(x_1) - P(x_2)| \leq \frac{|x_1 - x_2|}{1 - C_0(d_K(x_1) + d_K(x_2))}. \tag{2.7}$$

Moreover, we get

(2.8)

$$\begin{aligned}\psi_\varepsilon(x_1) - \psi_\varepsilon(x_2) &= \tfrac{1}{2\varepsilon}(|x_1 - P(x_1)|^2 - |x_2 - P(x_2)|^2) = \\ &= \tfrac{1}{\varepsilon}\langle x_2 - P(x_2), x_1 - x_2 - (P(x_1) - P(x_2))\rangle + \tfrac{1}{2\varepsilon}|x_1 - x_2 - (P(x_1) - P(x_2))|^2 \geq \\ &\geq \tfrac{1}{\varepsilon}\langle x_2 - P(x_2), x_1 - x_2)\rangle + \tfrac{1}{\varepsilon}\langle x_2 - P(x_2), P(x_2) - P(x_1)\rangle \geq \\ &\geq \tfrac{1}{\varepsilon}\langle x_2 - P(x_2), x_1 - x_2)\rangle - \tfrac{C_0}{\varepsilon}|x_2 - P(x_2)|\ |P(x_2) - P(x_1)|^2\end{aligned}$$

where we use (2.1) since $x_i - P(x_i) \in N_K(P(x_i))$ and $P(x_i) \in K$. So, (2.8) and (2.7) yield

$$\psi_\varepsilon(x_1) - \psi_\varepsilon(x_2) \geq \frac{1}{\varepsilon}\langle x_2 - P(x_2), x_1 - x_2\rangle\rangle + O(|x_1 - x_2|^2) \qquad \text{as } x_1 \to x_2.$$

By a similar argument we have

$$\psi_\varepsilon(x_2) - \psi_\varepsilon(x_1) \geq \frac{1}{\varepsilon}\langle x_2 - P(x_2), x_2 - x_1\rangle\rangle + O(|x_1 - x_2|^2) \qquad \text{as } x_1 \to x_2$$

and, so

$$\psi_\varepsilon(x_1) = \psi_\varepsilon(x_2) + \frac{1}{\varepsilon}\langle x_2 - P(x_2), x_1 - x_2\rangle\rangle + O(|x_1 - x_2|^2) \qquad \text{as } x_1 \to x_2.$$

We summarize all the remarks above in the following lemma, whose proof is actually rather standard, but we decided to write here for the sake of the completeness and also because the introduced arguments will be successively used.

Lemma 2.8 *Assume that K is uniformly prox-regular with constant $\frac{1}{r}$ for any $r \in (0, r_0)$. Then, we obtain the following properties (for any $\varepsilon > 0$):*

1. *the proximal mapping P_ε is single-valued on K_{r_0};*
2. *for all $x \in K_{r_0}$ $\psi_\varepsilon(x) = \frac{1}{2\varepsilon}[\mathrm{dist}(x, K)]^2$;*
3. *$\psi_\varepsilon \in \mathcal{C}^{1,1}(K_{r_0})$ and $\forall x \in K_{r_0}$ $\nabla\psi_\varepsilon(x) = \frac{1}{\varepsilon}(x - P_\varepsilon(x)) \in \partial_-\psi_K(P_\varepsilon(x))$.*

Next, we prove the existence and uniqueness result of the deterministic Skorokhod problem.

Theorem 2.9 *Assume that K is uniformly prox-regular with constant $\frac{1}{r}$ for any $r \in (0, r_0)$ and (2.4). Then, for any $T > 0$ and for any initial data $x \in K$ the deterministic Skorokhod problem (2.3) has a unique solution.*

Proof. First, we prove the existence. Let us fix $T > 0$ and chose a point $x \in K$. We start solving an approximated Skorokhod problem on $[0, T]$

$$\begin{cases} \dot{y}_\varepsilon(t) = f(t, y_\varepsilon(t)) - \nabla\psi_\varepsilon(y_\varepsilon(t)) \\ y_\varepsilon(0) = x \in K. \end{cases} \tag{2.9}$$

By the regularity of $\nabla\psi_\varepsilon(x)$ (cf. Lemma 2.8), the system (2.9) admits a unique absolute continuous solution $y_\varepsilon(t)$ as long as $y_\varepsilon(t) \in K_{r_0}$. Multiplying by $\dot{y}_\varepsilon$ the terms of the equation in (2.9) and integrating on $[0, t]$ with $0 < t \leq T$, we obtain

$$\int_0^t |\dot{y}_\varepsilon(s)|^2\, ds + \int_0^t \left\langle \nabla\psi_\varepsilon(y_\varepsilon(s))\dot{y}_\varepsilon(s)\right\rangle\, ds = \int_0^t \left\langle f(y_\varepsilon(s), a(s), b(s)), \dot{y}_\varepsilon(s)\right\rangle\, ds.$$

Recalling that $\psi_\varepsilon(x) = 0$ because $x \in K$, we obtain

$$\int_0^t |\dot{y}_\varepsilon(s)|^2 \, ds + \psi_\varepsilon(y_\varepsilon(t)) = \int_0^t \langle f(s, y_\varepsilon(s)), \dot{y}_\varepsilon(s) \rangle \, ds \leq M\sqrt{T} \left(\int_0^t |\dot{y}_\varepsilon(s)|^2 \, ds \right)^{\frac{1}{2}}.$$

Hence, we have

$$\int_0^t |\dot{y}_\varepsilon(s)|^2 \, ds + \psi_\varepsilon(y_\varepsilon(t)) \leq M\sqrt{T} \left(\int_0^t |\dot{y}_\varepsilon(s)|^2 \, ds \right)^{\frac{1}{2}},$$

that in turns yields the following important two estimates that hold true for any $t \in [0,T]$ as long as $y_\varepsilon(s) \in K_{r_0}$:

$$||\dot{y}_\varepsilon||_{L^2(0,t)} \leq M\sqrt{T} \tag{2.10}$$

and

$$\psi_\varepsilon(y_\varepsilon(t)) \leq M^2 T. \tag{2.11}$$

Therefore, as long as $y_\varepsilon(s) \in K_{r_0}$, since

$$\psi_\varepsilon(y_\varepsilon(t)) = \frac{1}{2\varepsilon} |y_\varepsilon(t) - P(y_\varepsilon(t))|^2 = \frac{1}{2\varepsilon} [\mathrm{dist}(y_\varepsilon(t), K)]^2$$

(see Lemma 2.8), the inequality (2.11) implies that

$$\mathrm{dist}(y_\varepsilon(t), K) \leq M\sqrt{2T\varepsilon}. \tag{2.12}$$

Thus, $y_\varepsilon(t)$ belongs to K_{r_0} for all $t \in [0,T]$ if

$$0 < \varepsilon < \frac{r_0^2}{2TM^2}. \tag{2.13}$$

We summarize what we obtain so far: if (2.13) is satisfied, then system (2.9) admits a unique solution y_ε on $[0,T]$ and $y_\varepsilon(t) \in K_{r_0}$ $\forall t \in [0,T]$. Moreover, thanks to (2.10) there exists a sequence $\{\varepsilon_j\} \subset \left(0, \frac{r^2}{2TM^2}\right)$ which converges to zero such that

$$\begin{cases} y_{\varepsilon_j} \to y \text{ in } \mathcal{C}([0,T]) \\ y_{\varepsilon_j} \rightharpoonup y \text{ in } H^1(0,T) \end{cases}$$

for some $y \in \mathcal{C}([0,T]) \cap H^1(0,T)$.

Now, we have to check that y is really a good candidate, namely

$$\dot{y}(t) \in f(t, y(t)) - \partial_- \psi_K(y(t)) \text{ for a.e. } t \in [0,T].$$

Since

$$f(t, y_\varepsilon(t)) - \dot{y}_\varepsilon(t) = \nabla \psi_\varepsilon(y_\varepsilon(t)) \in \partial_- \psi_K(P(y_\varepsilon(t))),$$

by (2.1) for all $z \in K$ we obtain

$$\big\langle f(t,y_\varepsilon(t)) - \dot y_\varepsilon(t), P(y_\varepsilon(t)) - z\big\rangle + C_0\big|f(t,y_\varepsilon(t)) - \dot y_\varepsilon(t)\big|\big|P(y_\varepsilon(t)) - z\big|^2 \geq 0$$

and, so, for any $\varphi \in \mathcal{C}([0,T])$ such that $\varphi \geq 0$

$$\int_0^T \Big\{\big\langle f(t,y_\varepsilon(t)) - \dot y_\varepsilon(t), P(y_\varepsilon(t)) - z\big\rangle + \\ C_0\big|f(t,y_\varepsilon(t)) - \dot y_\varepsilon(t)\big|\big|P(y_\varepsilon(t)) - z\big|^2\Big\}\varphi(t)\ dt \geq 0.$$

In order to simplify notations we set $\xi_\varepsilon(t) := f(t,y_\varepsilon(t)) - \dot y_\varepsilon(t)$ and, splitting the above expression into two integrals, we get

$$\int_0^T \big\langle \xi_\varepsilon(t), P(y_\varepsilon(t)) - z\big\rangle\varphi(t)\ dt + C_0\int_0^T \Big\{\big|\xi_\varepsilon(t)\big|\big|P(y_\varepsilon(t)) - z\big|^2\Big\}\varphi(t)\ dt \geq 0.$$

Letting $\varepsilon \to 0^+$ (taking a subsequence if necessary), by (2.10) the measure $|\xi_\varepsilon|\ dt$ weakly converges to some measure $d\mu(t)$ and we obtain

$$\int_0^T \big\langle \xi(t), \big(P(y(t)) - z\big\rangle\varphi(t)\ dt + C_0\int_0^T \Big\{\big|P(y_\varepsilon(t)) - z\big|^2\Big\}\varphi(t)\ d\mu(t) \geq 0,$$

where ξ is such that $\xi_{\varepsilon_j} \rightharpoonup \xi$ in $L^1(0,T)$. If we define the measure $d\lambda(t) := |\xi(t)|dt$, it turns out that $d\lambda << d\mu$; in other words, by the theorem of Radon-Nikodym, there exists a function $h \in L^1(\mu)$ such that $d\lambda = hd\mu$. Finally, the following inequality holds

$$\int_0^T \left\{\left\langle h(t)\frac{\xi(t)}{|\xi(t)|}, P(y(t)) - z\right\rangle + C_0\big|P(y_\varepsilon(t)) - z\big|^2\right\}\varphi(t)\ d\mu(t) \geq 0.$$

Hence, for any countably dense set $D \subset K$, we have

$$\left\langle h(t)\frac{\xi(t)}{|\xi(t)|}, P(y(t)) - z\right\rangle + C_0\big|P(y(t)) - z\big|^2 \geq 0 \tag{2.14}$$

for each $t \in [0,T] \setminus J_z$ with $\mu(J_z) = 0$ and $z \in D$: the inequality (2.14) holds true for all z fixed, so the set of zero measure could depend on z, say J_z; setting Z as the union of J_z such that z is in a countably dense subset D, then Z has zero measure:

$$\mu(Z) = \mu(\bigcup_{z\in D} J_z) = 0.$$

Inequality (2.12) implies that $y(t) \in K$ and, hence, $P(y(t)) = y(t)$ $\Big(P(y(t))$ is the closest point of K to $y(t)\Big)$. So, we get

$$\left\langle h(t)\frac{\xi(t)}{|\xi(t)|}, P(y(t)) - z\right\rangle + C_0\big|y(t) - z\big|^2 \geq 0 \tag{2.15}$$

for all $t \in [0,T] \setminus Z$ and, moreover, for each $z \in K$: indeed, for any $z \in K$ we can consider a sequence $z_k \in D$ which converges to z and such that the above estimate (2.15) is satisfied. By Remark 2.3

$$h(t)\frac{\xi(t)}{|\xi(t)|} \in \partial_{-}\psi_K(y(t)) \quad \text{for a.e. } t \in [0,T]$$

and, hence, we obtain

$$\dot{y}(t) \in f(t, y(t)) - \partial_{-}\psi_K(y(t)) \text{ for a.e. } t \in [0,T].$$

Notice that if $\dot{y}(t) \in \Pi_{T_K(x(t))}(f(t, y(t)))$ then

$$|\dot{y}(t)| = |\Pi_{T_K(x(t))}(f(t, y(t)))| \leq |f(t, y(t))| < M.$$

So, actually, in order to solve the deterministic Skorokhod problem, it is not necessary to consider the whole $\partial_{-}\psi_K(y(t))$ in system (2.3), but it is sufficient to consider only the set

$$\partial_{-}^{b}\psi_K(y(t)) := \{\xi(t) \in \partial_{-}\psi_K(y(t)) : |\xi(s)| < 2M\},$$

with $M > 0$ is the constant defined in (2.4). Indeed, since systems (2.5) and (2.6) are equivalent, if $\xi(t) = f(t, y(t)) - \dot{y}(t) \in \partial_{-}\psi_K(y(t))$ then $|\xi(t)| < 2M$, namely, $\xi(t) \in \partial_{-}^{b}\psi_K(y(t))$.

Thus, once the initial data x is fixed, we found a solution, say $y_x(\cdot)$, to the following system with reflection on the boundary

$$\begin{cases} \dot{y}(t) \in f(y(t), a(t), b(t)) - \partial_{-}^{b}\psi_K(y(t)) \\ y(t) \in K \quad \forall t \in [0,T] \\ y(0) = x \in K. \end{cases} \tag{2.16}$$

The uniqueness is an immediate consequence of the following Lemma.

Lemma 2.10 *Under the assumptions of Theorem 2.9, there exists $\lambda_0 > 0$ such that if $y_1(\cdot)$ and $y_2(\cdot)$ are solutions to (2.16) with initial condition $y_1(0) = x_1$ and $y_2(0) = x_2$ respectively, then we have*

$$|y_1(t) - y_2(t)| \leq e^{\lambda_0 t}|x_1 - x_2| \quad \forall t \in [0,T].$$

Proof. The proof is an easy consequence of Proposition 2.2 and Gronwall's inequality. Indeed, let us consider two starting points $x_1, x_2 \in K$ and two corresponding solution of the Skorokhod problem (2.16), say $y_1(\cdot)$ and $y_2(\cdot)$. So there exist elements $\xi_i(t) \in N_K(y_i(t)) \cap Int(B(0; 2M))$ such that

$$y_i(t) = f(t, y_i(t)) - \xi_i(t) \qquad \forall t \in [0,T].$$

By property 4 of Proposition 2.2 we obtain

$$\frac{r_0}{2M}\langle \xi_1(t) - \xi_2(t), y_1(t) - y_2(t)\rangle \geq -|y_1(t) - y_2(t)|^2.$$

Therefore, we have

$$\begin{aligned}
&|y_1(t)-y_2(t)|\tfrac{d}{dt}|y_1(t)-y_2(t)| \le \\
&\qquad \le \langle y_1(t)-y_2(t), f(t,y_1(t))-f(t,y_2(t))\rangle - \langle \xi_1(t)-\xi_2(t), y_1(t)-y_2(t)\rangle \le \\
&\qquad \le L|y_1(t)-y_2(t)|^2 + \tfrac{2M}{r_0}|y_1(t)-y_2(t)|^2 = \left(L+\tfrac{2M}{r_0}\right)|y_1(t)-y_2(t)|^2
\end{aligned}$$

and, so,

$$\frac{d}{dt}|y_1(t)-y_2(t)| \le \left(L+\frac{2M}{r_0}\right)|y_1(t)-y_2(t)|.$$

By applying Gronwall's inequality we deduce that

$$|y_1(t)-y_2(t)| \le e^{\left(L+\frac{2M}{r_0}\right)t}|x_1-x_2| \qquad \forall t\in[0,T].$$

Thus, the statement follows for $\lambda_0 = L+\frac{2M}{r_0}$.

□

Now, the proof of Theorem 2.9 is complete.

□

Remark 2.11 *It is possible to prove that the above result still holds true when we consider the deterministic Skorokhod problem with oblique reflection:*

$$\begin{cases} \dot y(t) \in f(t,y(t)) - A(y(t))\partial_-\psi_K(y(t)) \\ y(0)=x\in K \end{cases}$$

where for each $x\in\mathbb{R}^n$ $A(x)$ is a symmetric $n\times n$-matrix such that $A:\mathbb{R}^n\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is of class $\mathcal{C}^1$ with bounded differential and, moreover, there exists $\nu>0$ such that

$$\nu I \le A(x) \le \nu^{-1}I \qquad \forall x\in\mathbb{R}^n.$$

For the details of the proof see [14] (cf. also Appendix B in [5]).

3 The Skorokhod control problem and some applications

In this section we deal with the following differential game with reflection on the boundary.

Let us consider the functions $g:\mathbb{R}^n\times U\times V\longrightarrow\mathbb{R}^n$ and $\ell:\mathbb{R}^n\times U\times V\longrightarrow\mathbb{R}$ where U and V are compact subsets of $\mathbb{R}^m$. We assume that there exists $M>0$ such that for all $x_1,x_2\in\mathbb{R}^n,\ u_1,u_2\in U,\ v_1,v_2\in V$

$$(3.1)\qquad \begin{cases} \|g(x,u,v)\|_\infty, \|\ell(x,u,v)\|_\infty \le M; \\ |g(x_1,u_1,v_1) - g(x_2,u_2,v_2)| \le M(|x_1-x_2| + |u_1-u_2| + |v_1-v_2|); \\ |\ell(x_1,u_1,v_1) - \ell(x_2,u_2,v_2)| \le M(|x_1-x_2| + |u_1-u_2| + |v_1-v_2|). \end{cases}$$

Let us denote the set of the controls of the first player, called Ursula, and of the second player, called Victor, by $\mathcal{U} := \{u : [0,+\infty[\longrightarrow U \text{ measurable}\}$ and $\mathcal{V} := \{v : [0,+\infty[\longrightarrow V \text{ measurable}\}$, respectively. As usual, we denote an element of $\mathcal{U}$ by $u(\cdot)$ or simply by u when it is understood we are using measurable and bounded functions (and similarly for elements in $\mathcal{V}$).

Once the controls $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$ are fixed, for any $T > 0$ the following deterministic Skorokhod control problem

$$(3.2)\qquad \begin{cases} \dot y(t) \in g(y(t), u(t), v(t)) - \partial_- \psi_K(y(t)) \\ y(t) \in K \quad \forall t \in [0,T] \\ y(0) = x \in K \end{cases}$$

reduces to (2.3) simply by taking $f(t,x) = g(x, u(t), v(t))$ So, by Theorem 2.9 for any $T > 0$, once we fix $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, we get a unique solution to the differential inclusions (3.2) starting from the initial data $x \in K$, say $y_x(\cdot)$.

Take a discount factor $\lambda > 0$. In our differential game, Ursula wants to minimize the cost functional

$$\int_0^\infty \ell(y_x(s), u(s), v(s)) e^{-\lambda s}\, ds,$$

while Victor has to maximize it, where $y_x(\cdot) \in W^{1,1}_{loc}(0,+\infty)$ is the solution to the controlled Skorokhod problem associated to u and v.

For any $\lambda > 0$, the (lower) value functions w_λ is so defined

$$w_\lambda(x) := \inf_{\alpha \in S_U} \sup_{v \in \mathcal{V}} \int_0^\infty \ell(y_x(s), \alpha[v](s), v(s)) e^{-\lambda s}\, ds \qquad \text{for all } x \in K,$$

where S_U denotes the nonanticipative strategies set of the first player, namely the set of the maps $\alpha : \mathcal{V} \longrightarrow \mathcal{U}$ such that if, for any $t > 0$ and $v(\cdot), v'(\cdot) \in \mathcal{V}$, $v(s) = v'(s)\ \forall s \le t$ implies $\alpha[v](s) = \alpha[v'](s)\ \forall s \le t$.

Let us consider the constant $\lambda_0 > 0$ defined in the proof of Lemma 2.1. We get the following Hölder estimate on the value function w_λ.

Proposition 3.1 *Assume that for some $r_0 > 0$ fixed the closed set $K \subset \mathbb{R}^n$ is uniformly prox-regular with constant $\frac{1}{r}$ for any $r \in (0, r_0)$ and (3.1). Take any $x_0 \in K$ and $R > 0$. For all $\lambda > 0$ such that $\lambda < \lambda_0$, we have the following estimate*

$$(3.3)\qquad |\lambda w_\lambda(x_1) - \lambda w_\lambda(x_2)| \le C|x_1 - x_2|^{\frac{\lambda}{\lambda_0}} \qquad \forall\, x_1, x_2 \in K \cap B(x_0, R)$$

where $C = \frac{2M}{R+1}$ (it does not depend on λ).

Proof. The proof follows by applying a standard argument based on the Lipschitz dependence of the trajectories on the initial data provided by Lemma 2.1 (see for instance [6]).

□

It is well known, through the classical theory, that the value function w_λ provides the viscosity solutions of the following stationary Hamilton-Jacobi-Isaacs equation (see the books [3] and [4] for the general theory and [16] for the Neumann type boundary conditions):

$$\lambda w_\lambda + H(x, Dw_\lambda) = 0,$$

coupled with the Neumann boundary conditions:

$$\frac{Dw_\lambda}{D\gamma} = 0 \qquad \text{on } \partial K,$$

where we suppose that K is uniformly prox-regular with $\partial K \in \mathcal{C}^1$ and $\gamma = \gamma(x)$ is a smooth vector field on ∂K pointing outwards i.e.

$$\exists \nu > 0 \text{ s. t. } \forall x \in \partial K \quad \langle n(x), \gamma(x) \rangle \geq \nu.$$

The Hamiltonian function H is given by

$$H(x,p) := \min_{v \in V} \max_{u \in U} \{-g(x,u,v) \cdot p - \ell(x,u,v)\}.$$

Remark 3.2 *The Hölder regularity expressed in (3.3), not only is interesting by itself, but is quite useful for other important properties of the value function. Indeed, if for instance the domain K is not only uniformly prox-regular with $\partial K \in \mathcal{C}^1$ but is also connected and bounded, then we can study the asymptotic behaviour of the term λw_λ as $\lambda \to 0^+$: it is the so-called ergodic problem (for a general introduction to this problem we refer the reader to [3]). Under a uniform approximate controllability assumption of one player it turns out that there exists a unique $\chi_0 \in \mathbb{R}$ such that for all $x \in K$, we have (see [6] for the proof):*

$$\lim_{\lambda \to 0^+} \lambda w_\lambda(x) = \chi_0.$$

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